Structural approximation and quantum mechanics

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Abstract

We develop a notion of approximation and a model of finite quantum mechanics. In particular, we calculate the Feynman propagator for quantum harmonic oscillator via the path integral method, where according to our model the space of paths is (pseudo)finite.

[1]

Contents

[1] A preliminary version of the paper has been on the author web-page since 2017

1 Introduction

1.1 The paper develops ideas, notions and techniques suggested and indicated in [1], [5] and [6].

Section 2 of the paper is mostly model theory. We revise and develop further the general notion of structural approximation introduced in [1].

We give a few examples of finitely approximated structures over which the new notions are well-defined. Then we concentrate on our main example, the approximation of the compactified real line $\bar{\mathbb{R}}$ by finite structures of the form $\mathbb{Z}/N$. This example comes from [6] but there its formalism was not determined and studied properly. We note that the language of the structures on $\bar{\mathbb{R}}$ and respectively on $\mathbb{Z}/N$ can not be the full ring language. The relation $x_1 \cdot x_2 = y$ on $\mathbb{R}$ has to be replaced by the relation $x_1 \cdot x_2 \equiv y_1 \cdot y_2 \mod \mathbb{Z}$ (a *weak ring*).

In section 4-5 we study in detail weak rings of degree 2 and the respective approximation of $\bar{\mathbb{R}}$.

Sections 6-8 largely recast the quantum mechanical formalism sketchily presented in [6], in a rigorous way. Section 9 is completely new. We calculate the Feynman propagator for quantum harmonic oscillator via the path integral method, where according to our model the space of paths is pseudo-finite. This calculation leads to an expected result but unlike the conventional calculation we get the result right without the usual renormalisation or a special summation method, see e.g. [9], 7.7.4 for the latter.

Our further plans are to extend the approach to higher order oscillating integrals and into the setting of quantum field theory.

In the rest of the introduction we present a more logic perspective on the study.

2 Structural approximation

This section mainly presents basic notions of *postivie model theory* in the form close to one in [1]. The theory together with *continuous model theory* at present is a quite developed part of general model theory. Our aim here is not in developing the theory but rather to understand in its context a very special example, which comes from quantum mechanics.

2.1 Definitions. Given a topological[2] language $\mathcal{C}$ and a $\mathcal{C}$-structure $\mathbf{M}$

Very often we use the language $\mathcal{C}(M)$ in place of $\mathcal{C}$, the extension by names of elements of a $\mathcal{C}$-structure $\mathbf{M}$.

Let $\{\mathrm{R}_i : i \in I\}$ be a family of $\mathcal{C}$-structures. We associate to it the family $\{\mathrm{R}^\sharp_j : j \in I^\sharp\}$ of all expansions of the R_i to the language $\mathcal{C}(M)$.

$\mathbf{M}^\sharp$ will always stand for the natural expansion of $\mathbf{M}$ to $\mathcal{C}(M)$.

Given a $\mathcal{C}$-structure $\mathbf{M}$ and $\mathcal{C}$- structures $\{\mathrm{R}_i : i \in I\}$ we say that $\mathbf{M}$ is **syntactically approximated** by $\{\mathrm{R}_i : i \in I\}$ if

for every h-universal $\mathcal{C}(M)$ sentence φ such that $\mathbf{M}^\sharp \vDash \varphi$ there is $\mathrm{R}^\sharp_j$ in the family of $\mathcal{C}(M)$-expansions of the R_i such that $\mathrm{R}^\sharp_j \vDash \varphi$.

We say that $\mathbf{M}$ is **semantically approximated**[3] by $\{\mathrm{R}_i : i \in I\}$ if there is an ultrafilter $\mathcal{D}$ on I such that for some $\mathrm{R} \succ \prod_i \mathrm{R}_i/\mathcal{D}$ there is a surjective homomorphism

$$\lim : \mathrm{R} \to \mathbf{M}.$$

2.2 In accordance with [1] we say that $\mathbf{M}$ is **quasi-compact** if any filter of subsets of M^n defined by existential positive $\mathcal{C}(\mathbf{M})$-formulas has a non-empty intersection.

2.3 Lemma. Suppose R_1 and R_2 are $\mathcal{C}(M)$ structures and $L : \mathrm{R}_1 \to \mathrm{R}_2$ is a total homomorphism (with domain equal to N_1). Then for every h-universal formula σ

$$\mathrm{R}_2 \vDash \sigma \Rightarrow \mathrm{R}_1 \vDash \sigma.$$

Proof. By definitions. $\square$

2.4 Lemma *Suppose* $\mathbf{M}$ *is a quasi-compact topological* $\mathcal{C}$*-structure,* $\mathbf{M}^\sharp$ *its natural expansion to* $\mathcal{C}(M)$ *and* $\mathbf{N}$ *is a* $\mathcal{C}(M)$*-structure such that for every h-universal sentence* σ

$$\mathbf{M}^\sharp \vDash \sigma \Rightarrow \mathrm{R} \vDash \sigma.$$

[2]In the sense of [1]. That is all positive quantifier-free formulas define subsets which are closed by definition.

[3]The same as just “approximated” in the terminology of [1].

Then there is a surjective homomorphism $\lim : \mathbf{N} \to \mathbf{M}$.

Proof. Given $A \subseteq N$, a partial *strong homomorphism* $\lim_A : A \to \mathbf{M}$ is a map defined on A such that for every $a \in A^k$, $\hat{a} := \lim_A a$ and $S(x, y) \in \mathcal{C}$ such that $\mathbf{N} \vDash \exists y S(a, y)$, we have $\mathbf{M} \vDash \exists y\, S(\hat{a}, y)$.

When $A = \emptyset$ the map is assumed empty but the condition still holds, for any sentence of the form $\exists y\, S(y)$. So it follows from our assumptions that $\lim_\emptyset$ does exist.

Now for a proof by induction suppose for some $A \subseteq N$ there is a partial strong homomorphism $\lim_A : A \to \mathbf{M}$, and $b \in N$. Then $\lim_A$ can be extended to a partial strong homomorphism $\lim_{Ab} : Ab \to \mathbf{M}$.

Indeed, let $\mathbf{N} \vDash \exists z\, S(a, b, z)$, for $S(x, y, z)$ a positive formula and a a tuple in $\mathbf{N}$. Then $\mathbf{N} \vDash \exists yz\, S(a, y, z))$ and hence $\mathbf{M} \vDash \exists yz S(\hat{a}, y, z))$.

We claim that the family of closed sets in $\mathbf{M}$ of the form

$$\{c \in M : \mathbf{M} \vDash \exists z\, S(\hat{a}, c, z) \text{ and } \mathrm{R} \vDash \exists z\, S(a, b, z)\}$$

is a filter, for otherwise for some $S_1, \ldots, S_k$ we have

$$\mathbf{M} \vDash \neg \exists y \bigwedge_{i=1}^{k} \exists z_i S_i(\hat{a}, y, z_i) \text{ and } \mathrm{R} \vDash \exists z_1 \ldots z_k \bigwedge_{i=1}^{k} S_i(a, b, z_i)$$

Since $\bigwedge_{i=1}^{k} S_i(a, b, z_i)$ is equivalent to an atomic formula, we get a contradiction with our assumtion.

By quasi-compactness of $\mathbf{M}$ and the claim there is a point, say $\hat{b}$ in the intersection of the filter. Clearly, letting $\lim_{Ab} : b \to \hat{b}$, we preserve formulas of the form $\exists z\, S(x, y, z)$.

Thus we have proved that lim can be extended to the whole of N.

Since the element of R named by a constant symbol c must go to the element of $\mathbf{M}$ named by a constant symbol c the image of lim is M. $\square$

2.5 Theorem. *Suppose* $\mathbf{M}$ *is a quasi-compact topological* $\mathcal{C}$*-structure. Then the following are equivalent:*

(i) $\mathbf{M}$ *is semantically approximated by* $\{\mathrm{R}_i : i \in I\}$ *along an ultrafilter* $\mathcal{D}$*;*

(ii) $\{\mathrm{R}_i : i \in I\}$ *syntactically approximates* $\mathbf{M}$.

Proof. Suppose $\mathbf{M}$ is semantically approximated. Then we have $\mathrm{R} \succ \prod_{i \in I} \mathrm{R}_i/\mathcal{D}$ and $\lim : \mathrm{R} \to \mathbf{M}$, the surjective homomorphism. By choosing a section

$$s : M \to N, \quad \lim \circ s = \mathrm{id}$$

we can name respective elements in $s(M) \subset N$ by elements of M and so consider R as a $\mathcal{C}(M)$-structure.

Using the fact that a homomorphism preserves positive formulas we can claim that for every h-universal formula σ

$$\mathbf{M}^\sharp \vDash \sigma \Rightarrow \mathrm{R} \vDash \sigma \tag{1}$$

Consider the family $\{\mathrm{R}_j^\sharp : j \in J\}$ of all the $\mathcal{C}(M)$-structures obtained by expanding structures $\mathrm{R}_i, i \in I$ by constants.

It follows from (1) that for each h-universal formula σ such that $\mathbf{M}^\sharp \vDash \sigma$,

$$S_\sigma := \{j \in J : \mathrm{R}_j^\sharp \vDash \sigma\} \neq \emptyset.$$

Since the conjunction of any two h-universal formulas is equivalent to a h-universal formula, the family of subsets of J,

$$D' := \{S_\sigma : \ \mathbf{M}^\sharp \vDash \sigma\}$$

is closed under intersections and so is contained in an ultrafilter, call it $\mathcal{D}'$. This proves (i)⇒(ii).

To prove the opposite we may start with a family $\{\mathrm{R}_j^\sharp : j \in J\}$ and an ultrafilter $\mathcal{D}'$ as in (ii). Consider the family $\{\mathrm{R}_j : j \in J\}$ of $\mathcal{C}$-reducts of structures $\mathrm{R}_j^\sharp$, $j \in J$. The ultraproduct $\mathrm{R} := \prod_{j \in J} \mathrm{R}_j/\mathcal{D}'$ is by definitions a $\mathcal{C}$-reduct of $\mathrm{R}^\sharp := \prod_{j \in J} \mathrm{R}_j^\sharp/\mathcal{D}'$. By 2.4 there is a surjective homomorphism $\lim : \mathrm{R} \to \mathbf{M}$. □

2.6 Lemma. *Let* $\mathbf{M}$ *be a* $\mathcal{C}$*-structure,* $\mathbf{M}^\sharp$ *its natural expansion to* $\mathcal{C}(M)$ *and* $\mathbf{N}$ *is a* $\mathcal{C}(M)$*-structure such that for every h-universal formula* σ

$$\mathbf{M}^\sharp \vDash \sigma \Rightarrow \mathrm{R} \vDash \sigma.$$

Then, for any $|M|$*- saturated extension* $\mathbf{M}^* \succ \mathbf{M}^\sharp$ *there is a totally defined homomorphism* $\lim : \mathbf{N} \to \mathbf{M}^*$. *such that* $\lim(N) \supseteq M$.

Proof. Just follow the proof of 2.4, replacing $\mathbf{M}$ by $\mathbf{M}^*$ to construct a totally defined homomorphism. Since the image of constants in R are respective constants in $\mathbf{M}^*$, the image of lim contains M. □

2.7 Theorem. *Suppose* $\{\mathrm{R}_i : i \in I\}$ *is a family of* $\mathcal{C}$*-structures. Then the following are equivalent:*

(i) *For some ultrafilter* $\mathcal{D}$ *on* I, *some* $\mathrm{R} \succ \prod_{i\in I} \mathrm{R}_i/\mathcal{D}$ *and some* $\mathbf{M}^* \succ \mathbf{M}$ *there is a totally defined homomorphism*

$$\lim : \mathrm{R} \to \mathbf{M}^*$$

such that $\lim(N) \supseteq M$.

(ii) $\{\mathrm{R}_i : i \in I\}$ *syntactically approximates* $\mathbf{M}$.

Proof. We use the same argument as in the proof of 2.5.

Assume (i). By choosing a section

$$s : M \to N$$

we can name respective elements in $s(M) \subset N$ by elements of M and so consider R as a $\mathcal{C}(M)$-structure.

Using the fact that a homomorphism preserves positive formulas we can claim that for every h-universal formula σ

$$\mathbf{M}^\sharp \vDash \sigma \Rightarrow \mathrm{R} \vDash \sigma \tag{2}$$

Consider the family $\{\mathrm{R}^\sharp_j : j \in J\}$ of all the $\mathcal{C}(M)$-structures obtained by expanding structures $\mathrm{R}_i, i \in I$ by constants.

It follows from (2) that for each h-universal formula σ such that $\mathbf{M}^\sharp \vDash \sigma$,

$$S_\sigma := \{j \in J : \mathrm{R}^\sharp_j \vDash \sigma\} \neq \emptyset.$$

Since the conjunction of any two h-universal formulas is equivalent to a h-universal formula, the family of subsets of J,

$$D' := \{S_\sigma : \ \mathbf{M}^\sharp \vDash \sigma\}$$

is closed under intersections and so is contained in an ultrafilter, call it $\mathcal{D}'$. This proves (i)⇒(ii).

To prove the opposite we may start with a family $\{\mathrm{R}^\sharp_j : j \in J\}$ and an ultrafilter $\mathcal{D}'$ as in (ii). Consider the family $\{\mathrm{R}_j : j \in J\}$ of $\mathcal{C}$-reducts of structures $\mathrm{R}^\sharp_j$, $j \in J$. The ultraproduct $\mathrm{R} := \prod_{j\in J} \mathrm{R}_j/\mathcal{D}'$ is by definitions a $\mathcal{C}$-reduct of $\mathrm{R}^\sharp := \prod_{j\in J} \mathrm{R}^\sharp_j/\mathcal{D}'$. By 2.6 there is a totally defined homomorphism $\lim : \mathrm{R} \to \mathbf{M}$, $\lim(N) \supseteq M$.

□

A structure $\mathbf{K}$ is positively $\exists$-definable (interpretable) in $\mathbf{M}$ if the domain N (the set U and the equivalence relation E such that $N = U/E$) and the basic predicates $P \subset N^k$ of $\mathbf{K}$ are defined by positive $\exists$-formulas.

2.8 Proposition. *Let* $\mathbf{M}$ *be quasi-compact and* $\mathbf{K}$ *is positively* $\exists$*-interpretable in* $\mathbf{M}$. *Then* $\mathbf{K}$ *is quasi-compact.*

Proof. By definition any positive $\exists$-formula on $\mathbf{K}$ can be reinterpreted as a positive $\exists$-formula on $\mathbf{M}$, and a filter of positive $\exists$-definable subsets of N^m can be interpreted as a filter of positive $\exists$-definable subsets of M^{km} for some k.

Thus such a filter on $\mathbf{K}$ has a non-empty intersection. $\square$

2.9 Proposition. *Suppose* $\mathbf{M}$ *is syntactically approximated by* $\{\mathrm{R}_i : i \in I\}$ *and suppose there is a uniform positive* $\exists$*-interpretation* $DefK(\mathrm{R}_i)$ *of* $\mathbf{K}_i$ *in* R_i. *Then* $\{\mathbf{K}_i : i \in I\}$ *syntactically approximates* $\mathbf{K} := DefK(\mathbf{M})$.

Proof. Pick a string c of elements in $\mathbf{K}$ and an h-universal formula $\neg\exists y S(c, y)$ which holds in $\mathbf{K}$. The quantifier-free positive formula S in $\mathbb{N}$ can be rewritten as a positive $\exists$-formula $\tilde{S}$ in $\mathbf{M}$. Hence $\mathbf{K} \vDash \neg\exists y S(c, y)$ iff $\mathbf{M} \vDash \neg\exists y\ y \in U \& \tilde{S}(c, y)$, the latter formula being an h-formula. It follows that $\mathbf{K}_i$ approximate $\mathbf{K}$. $\square$

2.10 Corollary. *Suppose a quasi-compact* $\mathbf{M}$ *is semantically approximated by* $\{\mathrm{R}_i : i \in I\}$ *and suppose there is a uniform positive* $\exists$*-interpretation* $DefK(\mathrm{R}_i)$ *of* $\mathbf{K}_i$ *in* R_i. *Then* $\{\mathbf{K}_i : i \in I\}$ *semantically approximates* $\mathbf{K} := DefK(\mathbf{M})$.

This follows from the above, 2.5 and 2.8.

2.11 Amalgamation. We consider the following properties that $\mathcal{N}$ may have (all R in $\mathcal{N}$) :

(jcp) joint cover property : for any R_m and R_l there is R_n and surjective homomorphisms

$$f_{n:m} : \mathrm{R}_n \twoheadrightarrow \mathrm{R}_m \text{ and } f_{n:l} : \mathrm{R}_n \twoheadrightarrow \mathrm{R}_l$$

(acp) amalgamating cover property: given surjective homomorphisms

$$f_{m:k} : \mathrm{R}_m \twoheadrightarrow \mathrm{R}_k \text{ and } f_{l:k} : \mathrm{R}_l \twoheadrightarrow \mathrm{R}_k$$

there is R_n and surjective homomorphisms

$$f_{n:m} : R_n \twoheadrightarrow R_m \text{ and } f_{n:l} : R_n \twoheadrightarrow R_l$$

such that

$$f_{n:m} f_{m:k} = f_{n:l} f_{l:k}.$$

(pub) projective upper bounds: given a set $\mathcal{I} \subset \mathcal{N}$ of finite structures linearly order by the surjective homomorphisms $g_{ij} : R_i \twoheadrightarrow R_j$, for $i \geq j$, there is an $R \in \mathcal{N}$ and surjections $f_i : R \twoheadrightarrow R_i$, for each $i \in \mathcal{I}$, such that $f_i g_{ij} = f_j$.

We call a L-structure **K a projective object for finite $\mathcal{N}$-structures** (a projective Fraisse limit) if

(a) given a finite $R \in \mathcal{N}$ there is a surjective homomorphism $f : \mathbf{K} \twoheadrightarrow R$

(b) given finite $R \in \mathcal{N}$ and surjective homomorphisms $f_1 : \mathbf{K} \twoheadrightarrow R$ and $f_2 : \mathbf{K} \twoheadrightarrow R$, there is an automorphism $\phi : \mathbf{K} \to \mathbf{K}$ such that $f_2 = f_1 \phi$;

Remark. This is essentially the definition 2.2 of [11], properties (L1) and (L3). The property (L2) is automatic in our case.

Note that $\mathbf{K}$ is not assumed to be in $\mathcal{N}$.

2.12 We say that $R^* \in \mathcal{N}$ is **weakly projective object in $\mathcal{N}$** if there exists $\mathbf{K}$, a projective object for finite $\mathcal{N}$-structures, and a surjective homomorphism

$$\psi : R^* \twoheadrightarrow \mathbf{K}.$$

2.13 Lemma ([11], Lemma 2.3). *Suppose* $\mathbf{K}$ *is a projective object for finite $\mathcal{N}$-structures. Then, given finite* $R, R' \in \mathcal{N}$ *and surjective homomorphisms* $f : \mathbf{K} \twoheadrightarrow R$ *and* $g : R' \twoheadrightarrow R$, *there is a surjective homomorphisms* $h : \mathbf{K} \twoheadrightarrow R'$ *such that* $f = hg$.

2.14 Lemma. *The property (pub) is satisfied in any elementary class $\mathcal{N}$ with properties (jcp) and (acp).*

Proof. In the language of set theory we can write down the theory T_I describing a multisorted structure with sorts isomorphic to R_i, $i \in I$ and homomorphisms g_{ij} between the sorts, plus a sort R hosting an L-structure which belongs to $\mathcal{N}$, together with surjective homomorphism $f_i : \mathrm{R} \to \mathrm{R}_i$ such that $f_i g_{ij} = f_j$.

Such a theory is consistent since any finite collection of its sentences which mention $\mathrm{R}_i, g_{ij}, f_i$ $i, j \in \{1, \ldots, m\}$, only, has a model in which $\mathrm{R} := \mathrm{R}_m$, the maximal among the R_i, and $f_i := g_{mi}$.

By compactness there is a model of T_I. The L-structure as required. □

The following is essentially a corollary of Lemma 2.14 and the theorem of Irwin and Solecki, [11], Theorem 2.4.

2.15 Theorem. *Assume there are only countably many, up to isomorphism, finite structures in $\mathcal{N}$ and assume also jcp and acp for $\mathcal{N}$. Then*

(i) *There are projective objects $\mathbf{K}$ for finite structures of $\mathcal{N}$.*

(ii) *Any two projective objects are isomorphic*

(iii) *There exist weakly projective objects R^* in $\mathcal{N}$*

(iv) *The positive theory of any weakly projective object in $\mathcal{N}$ coincides with the positive theory of projective objects $\mathbf{K}$ for finite structures of $\mathcal{N}$. That is given any positive sentence σ we have*

$$\mathbf{K} \vDash \sigma \textit{ iff } \mathrm{R}^* \vDash \sigma$$

(v) *For a positive sentence σ*

$$\mathbf{K} \vDash \sigma \textit{ iff } \mathcal{N} \vDash \sigma$$

Proof. (i) and (ii) are from [11], Theorem 2.4.

(iii). By construction,

$$\mathbf{K} = \lim_{\leftarrow} \mathrm{R}_i$$

is a projective limit of linearly ordered by surjections $g_{ij} : \mathrm{R}_i \to \mathrm{R}_j$ sequence $\{\mathrm{R}_i\}$ of finite structures. By Lemma 2.14 there is $\mathrm{R}^* \in \mathcal{N}$ and surjective homomorphisms $f_i : \mathrm{R}^* \to \mathrm{R}_i$ commuting with g_{ij}. We choose such an R^* with extra assumption of $\aleph_1$-saturatedness. Now we can construct a surjective homomorphism $\psi : \mathrm{R}^* \to \mathbf{K}$ as follows:

for any $a \in \mathrm{R}^*$ set

$$\psi(a) = \{f_i(a) : i \in I\}$$

where on the right we have a string, an element of $\prod_{i\in I} \mathrm{R}_i$, belonging to the inverse limit of the system. In other words this defines a homomorphism into $\mathbf{K}$. This homomorphism is surjective because by saturatedness for any string $\alpha \in \mathbf{K}$ we can realise in R^* the type $\{f_i(x) = \alpha(i) : i \in I\}$.

(iv) Since there exists surjective homomorphism $\mathrm{R}^* \twoheadrightarrow \mathbf{K}$ every positive sentence σ which holds in R^* also holds in $\mathbf{K}$. For the converse suppose $\mathrm{R}^* \vDash \neg\sigma$ for some positive σ. Since R^* is pseudo-finite, there is a finite $\mathrm{R} \in \mathcal{N}$ satisfying $\neg\sigma$. By definition R is a surjective image of $\mathbf{K}$, and since positive sentences are preserved by surjective homomorphisms $\mathbf{K} \vDash \neg\sigma$. As required.

(v) If σ holds in $\mathbf{K}$ then it holds in all finite structures from $\mathcal{N}$ and hence in all structures from $\mathcal{N}$.

If, on the other hand, all finite structures satisfy σ then $\mathcal{N} \vDash \sigma$ and hence $\mathrm{R}^* \vDash \sigma$, consequently $\mathbf{K} \vDash \sigma$. $\square$

We denote

$$\mathrm{ThNeg}(\mathrm{R}) = \{\neg\sigma : \mathrm{R} \vDash \neg\sigma,\ \sigma \text{ positive sentence}\}.$$

2.16 Corollary. *Under the assumptions of the theorem the positive theory of* R^* *is equal to the positive theory of* $\mathbf{K}$ *and to the positive theory of* $\mathcal{N}$:

$$\mathrm{ThPos}(\mathrm{R}^*) = \mathrm{ThPos}(\mathbf{K}) = \mathrm{ThPos}(\mathcal{N})$$

and the class of weakly projective structures of $\mathcal{N}$ *is axiomatisable by the set of sentences*

$$T_{proj}(\mathcal{N}) := \mathrm{ThPos}(\mathbf{K}) \cup \mathrm{ThNeg}(\mathbf{K}) \cup \mathrm{Th}(\mathcal{N}).$$

2.17 Theorem. *Under assumptions of 2.15 the class of weakly projective structures in* $\mathcal{N}$ *is* **positively model complete***, that is for any weakly projective* $\mathrm{R}_1, \mathrm{R}_2$ *and an embedding* $\mathrm{R}_1 \hookrightarrow \mathrm{R}_2$*, for any positive formula* $\psi(x_1, \ldots, x_n)$ *and* $a_1, \ldots, a_n \in \mathrm{R}_1$,

$$\mathrm{R}_1 \vDash \psi(a_1, \ldots, a_n) \text{ iff } \mathrm{R}_2 \vDash \psi(a_1, \ldots, a_n).$$

Proof. To do.

2.18 A stronger version. *Under assumptions of 2.15 the class of weakly projective structures in $\mathcal{N}$ is* **inverse model complete***, that is for any weakly projective* $\mathrm{R}_1^*, \mathrm{R}_2^*$ *and a surjective homomorphism* $f : \mathrm{R}_2^* \twoheadrightarrow \mathrm{R}_1^*$ *there is a section* $f^\dagger : \mathrm{R}_1^* \hookrightarrow \mathrm{R}_2^*$ *such that for any positive formula* $\psi(x_1, \ldots, x_n)$ *and* $a_1, \ldots, a_n \in \mathrm{R}_1^*$,

$$\mathrm{R}_1^* \vDash \psi(a_1, \ldots, a_n) \textit{ iff } \mathrm{R}_2^* \vDash \psi(f^\dagger(a_1), \ldots, f^\dagger(a_n)).$$

Proof. Suppose first

$$\mathrm{R}_1^* = \prod_{I/D} \mathrm{R}_k,$$

ultraproduct of finite structures for some Freché ultrafilter. So, each element of R_1^* can be represented as an $\alpha : I \to \bigcup \mathrm{R}_k$, $\alpha(k) \in \mathrm{R}_k$, and for $g_{km} : \mathrm{R}_k \to \mathrm{R}_m$ it holds $g_{km} : \alpha(k) \mapsto \alpha(m)$.

Extend the language L to $L(\mathrm{R}_1^*)$ which has names for all elements α of R_1^*. The diagram of R_1^* in language $L(\mathrm{R}_1^*)$ has a realisation in a large saturated model R_3^* of $T_{proj}(\mathcal{N})$...

2.19 Examples. (i) The theory T_{proj} for the class of residue rings

$$\mathcal{N}_\mu = \{\mathbb{Z}/m\mathbb{Z} : m \in \mathbb{Z}_{>0}\}$$

is decidable. This follows from the statement in section 17 of [12] (answering a question of J.Ax) that $\mathrm{Th}(\mathcal{N}_\mu)$, the elementary theory of $\mathcal{N}$ is decidable.

(ii) Moreover, T_{proj} for the class of residue rings of the form

$$\mathcal{N}_{\mu^2} = \{\mathbb{Z}/m^2\mathbb{Z} : m \in \mathbb{Z}_{>0}\}$$

is decidable.

This follows from the same result by noting that one can axiomatise $\mathcal{N}_{\mu^2}$ inside $\mathcal{N}_\mu$ by the sentence

$$\exists m\ m^2 = 0\ \&\ \forall x\ (mx = 0 \to \exists y\ x = my)$$

3 Weak residue rings

3.1 We define a structure of a weak ring on $\mathbb{Z}/n$ by using the addition of the residue ring on $\mathbb{Z}/n$ and defining, for $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{Z}/n$,

$$P^4(\alpha_1, \beta_1, \alpha_2, \beta_2) \quad :\equiv \quad \alpha_1\beta_1 - \alpha_2\beta_2 = 0.$$

Each element α of $\mathbb{Z}/n$ can be uniquely represented in the form $\alpha = \bar{a}$, for $a \in [-\frac{1}{2}n, \frac{1}{2}n)$ (an interval in $\mathbb{Z}$).

As for $\mathbb{R}_w$ we can re-interpret the structure by introducing a second sort

$$\mathbb{C}_n := \{\mathrm{e}^{2\pi i \frac{z}{n}} : \ z = \{0, 1, \ldots, n-1\}$$

and define the structure R_n as consisting of two sorts $(\mathbb{Z}/n, +, 0, \bar{\lesssim})$ and $(\mathbb{C}_n, \cdot, 1)$ and the map between the sorts:

$$\mathrm{e} : \mathbb{Z}/n \times \mathbb{Z}/n \to \mathbb{C}_n, \quad \mathrm{e}(z_1 z_2) = \mathrm{e}^{2\pi i \frac{z_1 z_2}{n}}.$$

Clearly,

$$P^4(\alpha_1, \beta_1, \alpha_2, \beta_2) \quad \equiv \mathrm{e}(\alpha_1\beta_1) = \mathrm{e}(\alpha_2\beta_2)$$

so the weak ring structure on $\mathbb{Z}/n$ is definable in R_n.

Conversely, one can interpret $(\mathbb{C}_n, \cdot, 1)$ in the weak residue ring as a quotient $\mathbb{Z}/n \times \mathbb{Z}/n$ by the equivalence relation E^4 defined as:

$$\langle \alpha_1, \beta_1 \rangle E^4 \langle \alpha_2, \beta_2 \rangle \quad \equiv \quad P^4(\alpha_1, \beta_1, \alpha_2, \beta_2).$$

The additive group structure on $\mathbb{Z}/n \times \mathbb{Z}/n$ induces respectively the multiplicative group structure on the quotient. The canonical quotient-map becomes $\mathrm{e} : \mathbb{Z}/n \times \mathbb{Z}/n \to \mathbb{C}_n$.

3.2 The classe $\mathcal{N}$.

The finite structures R_n are two-sorted structures with sorts:

$$\mathbb{Z}/n, , \text{ the main sort,}$$

and

$$\mathbb{C}_n, \quad \text{the evaluating sort.}$$

On $\mathbb{Z}/n$ there is an additive group structure $(+, 0)$, constants named μ and ν, and a collection of binary relations $\{y = s \cdot x)_{s \in \mathbb{Q}}$. The interpretation of $y = s \cdot x$, for $s = \frac{l}{k}$ is defined for $x, y \in \mathbb{Z}/n$ as:

$$\mathbb{Z}/n \vDash y = \frac{l}{k}x \quad \text{iff} \quad ky = lx.$$

Note that since h is a rational number, $y = \mathrm{h} \cdot x$ is one of the binary predicates of the structure.

The structure on the evaluating sort is that of a multiplicative group $(\mathbb{C}_n; \cdot, 1)$ of complex roots of unity of order 1.

Finally, the two sorts are connected by a map

$$\mathrm{e} : \mathbb{Z}/n \times \mathbb{Z}/n \to \mathbb{C}_n, \quad \mathrm{e}(z_1 z_2) = \mathrm{e}^{2\pi i \frac{z_1 \cdot z_2}{n}}$$

where $z_1, z_2 \in \mathbb{Z}$ are treated also as representing elements of $\mathbb{Z}/n$.

Define the positive rational number

$$\mathrm{h} = \frac{\nu}{\mu} \tag{3}$$

We define $\mathcal{N}$ to be the axiomatic closure of the class of all finite R_n such that

$$\mathrm{e}(\mu, \nu) = 1 \tag{4}$$

3.3 Any R_n satisfies the following positive sentences:

$$(\mathrm{R}_n, +, 0) \text{ is an abelian group} \tag{5}$$

$$\forall x_1, x_2, y \ \mathrm{e}(x_1 + x_2, y) = \mathrm{e}(y, x_1 + x_2) = \mathrm{e}(x_1, y) + \mathrm{e}(x_2, y), \tag{6}$$

symmetric bi-linearity, and

$$\exists z_1 \forall x \in \mathbb{C}_n \exists z_2 \in \mathbb{Z}/n \ \mathrm{e}(z_1, z_2) = x \tag{7}$$

stating that e is surjective.

Also, the group structure of R_n, for each positive integer m, satisfies the sentence

$$\forall y_1, \dots y_m \exists x \bigvee_{i=0}^{m-1} \bigvee_{k=0}^{m-1} y_i \equiv kx \mod m \tag{8}$$

The sentence implies for an abelian group A that A/mA is cyclic (including the case $A = mA$). In particular, for finite A of order n, and m co-prime with n, the sentence implies that the group is cyclic of order n.

Call Σ the system of axioms (4)-(8), and $\Sigma_{\mu\nu}(\mathrm{h})$ its extension by (3).

3.4 Lemma. *Any finite model $\mathbf{M}_n$ of Σ of size n is isomorphic to an R_n such that*

$$\mu\nu | n.$$

Proof. As noted above, $(\mathbf{M}_n, +)$ is a cyclic group. Let p be a generator of $\mathbf{M}_n$. Since e is a surjective bilinear map, C_n is a cyclic group of some order k dividing n. and

$$\mathrm{e}^{\frac{2\pi i}{n}} = \mathrm{e}(1,1).$$

This determines the bilinear map e uniquely as the one defined in 3.2. Thus the choice of the generator 1 determines unique isomorphism type of models of Σ of size n.

Any other generator of the additive group is of the form $p \cdot 1$ and choosing this generator we get an isomorphism $z \mapsto pz$, $\mathrm{e}^t \mapsto \mathrm{e}^{p^2 t}$.
$\square$

3.5 Lemma. *A finite* $\mathbf{M}$ *is a model of* $\Sigma_{\mu\nu}(\mathrm{h})$ *if and only if* $\mathbf{M}$ *is a homomorphic image of* R_{mn} *as a ring, for some positive integers* m, n *such that* $n = \mathrm{h} \cdot m$, *and the images of* m, n *interpret* μ, ν *respectively.*

Proof. It is clear that R_{mn} with the interpretation of μ, ν is a model of (3)-(4). Since the axioms have a positive form this is preserved by any homomorphic image.

Conversely, let R_K be a model of $\Sigma_{\mu\nu}(\mathrm{h})$ and $m, n \in \mathbb{Z} \cap (0, K]$ interpret μ, ν in the structure. Then axioms (3)-(4) are equivalent to

$$m \cdot n \equiv 0 \,\mathrm{mod} K \text{ and } kn \equiv lm \,\mathrm{mod} K, \text{ for } \frac{k}{l} = \mathrm{h}, \quad (k,l) = 1.$$

We may assume $(m, n) = 1$ otherwise just dividing m, n and K by (m, n). Then $K = m' \cdot n'$, for some $m'|m$, $n'|n$. Since $m'n'|(kn - lm)$ we have

3.6 Proposition. *Class* $\mathcal{N}$ *satisfies properties jcp, acp and pub.*

Proof. Follows from the above. $\square$

4 A structure on the real line and its approximation

4.1 Compactification of the field of reals. Define $\bar{\mathbb{R}}_{\text{field}}$ to be a $\mathbf{P}^1(\mathbb{R})$, the projective real line in the Zariski language:

the universe of $\bar{\mathbb{R}}$ is $\mathbb{R} \cup \{\infty\}$, the addition:

$$S(x,y,z) \equiv x+y = z \vee (x = \infty \& z = \infty) \vee (y = \infty \& z = \infty) \vee (x = \infty \& y = \infty),$$

the multiplication:

$$P(x,y,z) \equiv x{\cdot}y = z \vee (x = \infty \& z = \infty) \vee (y = \infty \& z = \infty) \vee (x = \infty \& y = 0) \vee (x = 0 \& y = \infty).$$

The following is an unpublished result by M.Lau. A weaker version, non-approximability by finite fields is proved in [1].

4.2 Theorem. $\bar{\mathbb{R}}_{\text{field}}$ *is not approximable by the class of finite rings.*

4.3 Let $\mathbb{R}_w(\mathrm{h})$, $\mathrm{h} \in \mathbb{R}_+$, be the reals with the **weak ring structure** defined in [6], 7.5: $\mathcal{C}$ consists of all positive quantifier-free formulas generated by the ternary relation $S^3(x,y,z)$ meaning $x+y=z$ and 4-ary relation

$$P^4(x_1,y_1,x_2,z_2) \equiv \ e^{2\pi i \frac{x_1 y_1}{\mathrm{h}}} = e^{2\pi i \frac{x_2 y_2}{\mathrm{h}}}$$

Equivalently, the right-hand side says

$$x_1 y_1 \equiv x_2 y_2 \operatorname{mod} \mathrm{h}\,\mathbb{Z}.$$

The **compactified weak ring** structure $\bar{\mathbb{R}}_w$ on the compactified reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is given by $S(x,y,z)$ above and the extension of P^4 to $\bar{\mathbb{R}}$ defined by the condition

$$\infty \in \{x_1,y_1,x_2,y_2\} \Rightarrow \bar{\mathbb{R}}_w \vDash P^4(x_1,y_1,x_2,y_2).$$

4.4 Proposition. *For any choice of* $\mathrm{h} \in \mathbb{R}_+$, *the compactified weak ring* $\bar{\mathbb{R}}_w(\mathrm{h})$ *is approximable by finite weak residue rings.*

Proof. It is enough to construct a $\mathcal{C}$-homomorphism lim from a pseudo-finite weak ring of the form ${}^*\mathbb{Z}/\eta$, ${}^*\mathbb{Z}$ the non-standard integers, η a non-standard positive integer, into the non-standard reals ${}^*\mathbb{R}$ such that the image contains $\mathbb{R}$. We follow [6] and choose η of the form $\eta = \mu\nu$, where μ and ν are infinite non-standard integer related by a special (non-standard rational) parameter h satisfying

$$\frac{\nu}{\mu} = \mathrm{h} = \frac{\kappa_h}{\rho_h}, \quad 0 \le \kappa_h << \mu, \quad 0 \le \rho_h << \mu. \tag{9}$$

Later we will also use the real parameter

$$\hbar := \mathrm{st}(\frac{\mathrm{h}}{2\pi})$$

Here, the notation are chosen so that h will eventually be read as (approximately) the Planck constant, and $\hbar$ as the (exact) reduced Planck constant.

Denote

$$\mathrm{R}_{\mu\nu} := {}^*\mathbb{Z}/\mu\nu, \quad \mathrm{R}^{\leq}_{\mu\nu} := {}^*\mathbb{Z} \cap [-\frac{\mu\nu}{2}, \frac{\mu\nu}{2}).$$

Note that for $a \in \mathrm{R}^{\leq}_{\mu\nu}$, $\frac{a}{\mu}$ are elements of ${}^*\mathbb{Q}$.

st be the standard part map

$$\mathrm{st} : {}^*\mathbb{Q} \to \bar{\mathbb{R}}.$$

Set

$$\lim_\mu : \bar{a} \mapsto \mathrm{st}(\frac{a}{\mu}). \tag{10}$$

Now we prove an intermediate statement.

4.5 Lemma.

(i) $\lim_\mu$ *preserves* S^3 :

$$\alpha + \beta = \gamma \Rightarrow S^3(\lim_\mu\alpha, \lim_\mu\beta, \lim_\mu\gamma).$$

(ii) assuming $\lim_\mu(a_1), \lim_\mu(b_1), \lim_\mu(a_2)$ *and* $\lim_\mu(b_2)$ *are finite,*

$$\lim_\mu a_1 \cdot \lim_\mu b_1 - \lim_\mu a_2 \cdot \lim_\mu b_2 = \mathrm{st}(\frac{a_1 b_1}{\mu^2}) - \mathrm{st}(\frac{a_1 b_1}{\mu^2}) = \mathrm{st}(\frac{a_1 b_1 - a_2 b_2}{\mu^2})$$

(iii) assuming $\lim_\mu(a_1), \lim_\mu(b_1), \lim_\mu(a_2)$ *and* $\lim_\mu(b_2)$ *are finite,*

$$\bar{a}_1\bar{b}_1 = \bar{a}_2\bar{b}_2 \Rightarrow (\lim_\mu a_1 \cdot \lim_\mu b_1 - \lim_\mu a_2 \cdot \lim_\mu b_2) \in \mathrm{h}^{-1}\mathbb{Z}$$

(iv) assuming μ *is highly divisible (divisible by all positive* $\kappa << \mu$*), for any* $x_1, y_1, x_2, y_2 \in \mathbb{R}$ *satisfying* $x_1 y_1 - x_2 y_2 = k\mathrm{h}^{-1}$, $k \in \mathbb{Z}$ *there are* $a_1, b_1, a_2, b_2 \in {}^*\mathbb{Z}$ *such that*

$$\lim_\mu(a_1) = x_1, \ \lim_\mu(b_1) = y_1, \ \lim_\mu(a_2) = x_2, \ \lim_\mu(b_2) = y_2$$

and

$$\bar{a}_1\bar{b}_1 = \bar{a}_2\bar{b}_2$$

Proof. (i) The cases $\lim_\mu(\alpha), \lim_\mu(\beta) \in \mathbb{R}$ or $\lim_\mu(\alpha) = \infty = \lim_\mu(\beta)$ are straightforward.

The remaining case can be quickly reduced to the case $\alpha = \bar{a}, \beta = \bar{b}$,

$$\frac{\mu^2}{2} > a > \mu\mathbb{Z}, \quad 0 < b < m\mu, \ m \in \mathbb{Z},$$

in which case,

$$\lim_\mu \alpha = \infty, \ \lim_\mu \beta \in \mathbb{R}.$$

But now either $\frac{\mu^2}{2} > a + b > \mu\mathbb{Z}$ and so $\lim_\mu(\alpha + \beta) = \infty$, as required, or

$$\frac{\mu^2}{2} \leq a + b \equiv -\frac{\mu^2}{2} + (a + b - \frac{\mu^2}{2}) \text{mod} \mu^2.$$

Here

$$0 \leq a + b - \frac{\mu^2}{2} = b - (\frac{\mu^2}{2} - a) \leq b,$$

which proves that

$$-\frac{\mu^2}{2} \geq -\frac{\mu^2}{2} + (a + b - \frac{\mu^2}{2}) > \mu\mathbb{Z}$$

and so

$$\lim_\mu(\alpha + \beta) = \infty$$

as required.

(ii). Just note that the products on left-hand side can be written as

$$\text{st}(\frac{a_i}{\mu}) \cdot \text{st}(\frac{b_i}{\mu})$$

and the standard part map is a ring homomorphism on finite non-standard rationals.

(iii). $\bar{a}_1\bar{b}_1 = \bar{a}_2\bar{b}_2$ is equivalent, under the assumptions, to $a_1b_1 - a_2b_2 = k\mu\nu = k\mu^2\frac{\nu}{\mu}$ for some $k \in \mathbb{Z}$. This is equivalent to

$$\frac{a_1b_1}{\mu^2} - \frac{a_2b_2}{\mu^2} = k\frac{\nu}{\mu}.$$

By (i) and (iii) we then have

$$\lim_\mu a_1 \cdot \lim_\mu b_1 - \lim_\mu a_2 \cdot \lim_\mu b_2 = k\mathrm{h}^{-1}.$$

(iv). We may assume that x_1, y_1, x_2, y_2 are non-negative. Choose $\kappa_1, \rho_1, \kappa_2, \lambda \in {}^*\mathbb{Z}_+$ so that $0 < \kappa_1, \rho_1, \kappa_2, \lambda << \mu$ (which also means the numbers divide μ) and

$$\mathrm{st}(\frac{\kappa_1}{\lambda}) = x_1,\ \mathrm{st}(\frac{\rho_1}{\lambda}) = y_1,\ \mathrm{st}(\frac{\kappa_2}{\lambda}) = x_2 \tag{11}$$

Now we look for a non-standard rational number $q := \frac{\rho_2}{\lambda'}$ for $\rho_2, \lambda' \in {}^*\mathbb{Z}$ so that

$$\frac{\kappa_1\rho_1}{\lambda^2} - \frac{\kappa_2}{\lambda}q = k\mathrm{h}^{-1},\ \text{that is}\ \frac{\kappa_1\rho_1}{\lambda^2} - \frac{\kappa_2\rho_2}{\lambda\lambda'} = k\mathrm{h}^{-1},$$

$$q = \frac{\lambda}{\kappa_2}(\frac{\kappa_1\rho_1}{\lambda^2} - k\mathrm{h}^{-1})$$

and $|\rho_2|, |\lambda'| << \mu$. [4]

Applying st to the latter by (11) we have

$$x_1y_1 - x_2\,\mathrm{st}(q) = k.$$

Hence $\mathrm{st}(q) = y_2$.

Define

$$\mathrm{st}(\frac{\kappa_1}{\lambda}) = x_1,\ \mathrm{st}(\frac{\rho_1}{\lambda}) = y_1,\ \mathrm{st}(\frac{\kappa_2}{\lambda}) = x_2,\ \mathrm{st}(\frac{\rho_2}{\lambda'}) = y_2.$$

Set

$$a_1 := \mu\frac{\kappa_1}{\lambda},\ b_1 := \mu\frac{\rho_1}{\lambda},\ a_2 := \mu\frac{\kappa_2}{\lambda},\ b_2 := \mu\frac{\rho_2}{\lambda'}.$$

By divisibility and the construction, these are elements of ${}^*\mathbb{Z}$ that satisfy the required. □

4.6 It is useful also to consider $R_{\mu\nu}$ and $\bar{\mathbb{R}}$ with respect to a stronger language with the constant symbols $\frac{\mathbf{m}}{n}$, , $m, n \in \mathbb{Z}_+$, which we interpret as the reals $\frac{m}{n}$ in $\bar{R}$ and as $\mu \cdot \frac{m}{n}$ in $R_{\mu\nu}$. It is clear that $\lim_\mu$ is still a homomorphism with respect to this language.

[4]Here we use the assumptions (definition) that the set of

$$\mu\text{−Small} := \{\kappa \in {}^*\mathbb{Z} : \kappa << \mu\}$$

is closed under arithmetic operations. This can be achieved by choosing μ-Small to be a submodel $\mathbb{Z}' \prec {}^*\mathbb{Z}$ and $\mu \in {}^*\mathbb{Z} \setminus \mathbb{Z}'$.

5 Linear structure on rings of highly divisible type

5.1 A pseudo-finite weak ring $\mathrm{R}_{\mu\nu} = {}^*\mathbb{Z}/\mu\nu$ of 4.3 will be said to be **of highly divisible type** if there is a non-standard submodel of arithmetic $\mathbb{Z}' \prec {}^*\mathbb{Z}$ such that both μ and ν are divisible by any non-zero $\lambda \in \mathbb{Z}'$.

Note that by the compactness theorem there is ${}^*\mathbb{Z}$ and elements $\mu, \nu \in {}^*\mathbb{Z}$ such that $\mathrm{R}_{\mu\nu}$ is of highly divisible type.

5.2 Let R be an additive group. A **linear partial transformation** F **of** R is given by

- subgroups I_0, I_1, I_2 of R, $I_0 = I_1 \cap I_2$;
- a pair of surjective homomorphisms

$$f_1 : I_1 \to I_0,\ \ f_2 : I_2 \to I_0;$$

- the binary relation $F \subset R \times R$ is defined as:

$$\langle x, y\rangle \in F \Leftrightarrow x \in I_1 \ \&\ y \in I_2 \ \&\ f_1(x) = f_2(y).$$

Note that the restriction of F to $I_1 \times I_2$ is a subgroup.

We will also consider R with a **collection of linear partial transformations**.

5.3 The group $(\mathbb{R}, +)$ has pseudo-transformations of the form

$$F(x, y) \equiv\ \ y = sx,$$

for $s \in \mathbb{R}_+$. $I_1 = I_2 = \mathbb{R}$.

This extends uniquely to $s : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, $s \cdot \infty = \infty$, which will be our definition of a linear pseudo-transformation s on $\bar{\mathbb{R}}$.

5.4 Proposition. *The ordered semigroup $\bar{\mathbb{R}}$ with a linear transformation $x \mapsto s\,x$ is finitely approximable by finite groups with linear partial transformations, more precisely on any pseudo-finite weak ring R of highly divisible type there is partial linear relation F such that*

$$\langle a, b\rangle \in F \Rightarrow \lim\nolimits_\mu(b) = s \lim\nolimits_\mu(a).$$

Moreover, the homomorphism $\lim_\mu$ *maps* F *surjectively on the relation* $y = s \cdot x$, *that is given* $u, v \in \bar{\mathbb{R}}$ *satisfying* $v = s \cdot u$, *there are* $\langle a, b \rangle \in F$ *such that*

$$u = \lim_\mu(a) \text{ and } v = \lim_\mu(b).$$

Proof. Choose positive co-prime (non-standard) integers κ and ρ such that $\kappa\rho << \mu$, $\kappa\rho$ divides both μ and ν and

$$\mathrm{st}(\frac{\kappa}{\rho}) = s.$$

Take for R the residue ring ${}^*\mathbb{Z}/\mu\nu$ and

$$I_1 = \rho\, {}^*\mathbb{Z}/\mu\nu,\ I_2 = \kappa\, {}^*\mathbb{Z}/\mu\nu,\ I_0 = \kappa\rho\, {}^*\mathbb{Z}/\mu\nu.$$

Note that I_0, I_1, I_2 are actually subrings.

Define natural homomorphisms

$$f_1 : I_1 \twoheadrightarrow I_0, \quad f_2 : I_2 \twoheadrightarrow I_0$$

$$f_1 : a \mapsto a\kappa, \quad f_2 : b \mapsto b\rho.$$

Claim. Given $a \in I_1$ and $b \in I_2$, we have $a = \alpha\rho$ and $b = \beta\kappa$ for unique $\alpha \in [-\frac{\mu\nu}{2\rho}, \frac{\mu\nu}{2\rho})$ and $\beta \in [-\frac{\mu\nu}{2\kappa}, \frac{\mu\nu}{2\kappa})$. And

$$f_1(a) = f_2(b) \text{ iff } \exists \gamma \in [-\frac{\mu\nu}{2\kappa\rho}, \frac{\mu\nu}{2\kappa\rho})\ \alpha = \gamma\kappa \ \&\ \beta = \gamma\rho \text{ iff } \ b = \frac{\kappa}{\rho} a$$

Indeed, the left-hand side is equivalent to

$$\alpha\rho \equiv \beta\kappa \mod \frac{\mu\nu}{\kappa\rho}$$

equivalently, $\alpha = \gamma_1\kappa + \sigma_1,\ \beta = \gamma_2\rho + \sigma_2,\ 0 \le \sigma_1 < \kappa,\ 0 \le \sigma_2 < \rho$ and

$$\sigma_1\rho - \sigma_2\kappa \equiv 0 \mod \frac{\mu\nu}{\kappa\rho}.$$

Since $\mu\nu$ is divisible by $(\kappa\rho)^2$, we have $\sigma_1 = \sigma_2 = 0$ and $\gamma_1 = \gamma_2 = \gamma$. Claim proved.

Let $b \in I_2$. Then

$$\lim_\mu(b) = \mathrm{st}(\frac{b}{\mu}) = \mathrm{st}(\frac{\kappa}{\rho}) \cdot \mathrm{st}(\frac{a}{\mu}) = s \lim_\mu(a).$$

Hence for all $a, b \in R$

$$\langle a, b\rangle \in F \Rightarrow \lim_\mu(b) = s \lim_\mu(a).$$

Conversely, assume $u, v \in \bar{\mathbb{R}}$ and $v = s \cdot u$.
Since $\lim_\mu$ is a surjection there is $a \in \mathrm{R}_{\mu\nu}$ such that $u = \lim_\mu(a)$.
Dividing a by $\kappa\rho$ find γ and ϵ in ${}^*\mathbb{Z}$

$$a = \gamma\kappa\rho + \epsilon, \quad 0 \le \epsilon < \kappa\rho.$$

Then $a' := \gamma\kappa\rho \in I_1$ and $\lim_\mu(a') = \lim_\mu(a) = u$, so we may assume $a = \gamma\kappa\rho \in I_1$.
Set $b = \frac{\kappa}{\rho}a$. Then $b \in I_2$ and

$$\lim_\mu(b) = s \lim_\mu(a) = s \cdot u \text{ and } v = \lim_\mu(b).$$

□

5.5 Remarks. (1) One sees from the construction that the approximation in 5.4 is consistent with the approximation of weak ring structure.

(2) The pseudo-transformation F is the graph of a multivalued map $f : R \to R$. However, the differences between the values $f(x)$ in "finite" points x are "infinitesimal", that is disappear after the application of lim .

(3) The restriction of the multivalued map f to the smaller subgroup

$$I^0 = \{\gamma\kappa\rho : \gamma \in [-\frac{\mu\nu}{2\kappa^2\rho^2}, \frac{\mu\nu}{2\kappa^2\rho^2})\}, \quad f : \gamma\kappa\rho \mapsto \gamma\kappa^2$$

is a well-defined injective map. It is essential that I^0 is also big enough, we say **dense in the nonstandard group** ${}^*\mathbb{Z}/\mu\nu$:

$$\lim_\mu(I^0) = \bar{\mathbb{R}}.$$

In this circumstances we say that f **is well-defined on a dense subgroup and extended from it by continuity**.

5.6 Proposition. *The linear structure on the compactified ordered group with linear transformations*

$$\bar{\mathbb{R}}_{\mathrm{lin}} = (\bar{\mathbb{R}}; +, \leq, s\cdot\,)_{s\in\mathbb{R}_+^\times}$$

is strongly approximable by respective linear finite structures on $\mathbb{Z}/N$.

Proof. The elementary theory of densely ordered abelian groups (without linear transformations) is well understood, see [8].

Consider a finite part of the quantifier-free diagram of the structure. It is easy to see that such a finite diagram will coincide with a finite diagram of a such group with all the finitely many s involved in it being rationals. But then such a diagram can be replaced by an equivalent diagram of an ordered group without linear transformations. Then, the finite diagram can be realised in any cyclically ordered group which contains a densely ordered abelian group, that is realised in ${}^*\mathbb{Z}/\eta$ with highly divisible η. This is enough to prove the existence of colim. □

6 A line bundle with connection.

6.1 Recall the notation $N = \mu\nu$, $q = \exp\frac{2\pi i}{N}$.

We also set the value of the special parameter h of 4.3 and (9) so that

$$\mathrm{st}(\mathrm{h}) = \frac{\hbar}{2\pi} \tag{12}$$

where $\hbar$ is the Planck constant of physics expressed in some units.

An **algebraic Hilbert space with unitary operators U and V satisfying $UV = qVU$** is an N-dimensional vector space over the field k of characteristic 0with finite family of canonical orthonormal bases. More precisely it is given by axioms in the following language.

- k $= \mathbb{Q}[q]$ is with usual $+$, $\cdot$ and 1 and also with extra symbol q for a root of unity of order N. Thus any element of k is named. We also have an involution, "complex conjugation", $x \mapsto x^*$ defined as follows element-wise:
$$f(q)^* := f(q^{-1})$$
for any polynomial f over $\mathbb{Q}$.

- The universe of the structure is $\mathbf{V}$, which has a structure of an N-dimensional k-vector space.
- U and V are linear operators on $\mathbf{V}$ satisfying the above commutation relation.

  U and V have N distinct eigenvectors with distinct eigenvalues $1, q, \dots q^{N-1}$.
- There is a unary predicate $E^{U,V}$ satisfying following properties:

  $E^{U,V}$(V) has N^2 distinct elements.

$$\mathbf{u} \in E^{U,V} \Rightarrow U\mathbf{u}, V\mathbf{u} \in E^{U,V}.$$

- A canonical U-base (with respect to V) is an N-element sequence $\{\mathbf{u}(1), \mathbf{u}(q), \dots, \mathbf{u}(q^{N-1})\} \subset E^{U,V}$ satisfying

$$U\mathbf{u}(q^k) = q^k\,\mathbf{u}(q^k), \quad V\mathbf{u}(q^k) = \mathbf{u}(q^{k+1}).$$

- We define an inner product by assuming that each element of $E^{U,V}$ is of norm 1 (that is $\langle u, u\rangle = 1$) and that eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ with distinct eigenvalues are orthogonal (that is $\langle u_1, u_2\rangle = 0$). Hence a canonical basis is an orthonormal basis and for arbitrary

$$\mathbf{x}_1 := \sum_{i=0}^{N-1} k_{i1}\mathbf{u}(q^i), \;\; \mathbf{x}_2 := \sum_{i=0}^{N-1} k_{i2}\mathbf{u}(q^i)$$

  we have

$$\langle \mathbf{x}_1, \mathbf{x}_2\rangle = \sum k_{i1}k_{i2}^*.$$

6.2 We define

$$\mathrm{Reals}(\mathrm{k}) = \{a \in \mathrm{k} : a^* = a\}, \quad \mathrm{Ims}(\mathrm{k}) = \{a \in \mathrm{k} : a^* = -a\}$$

Reals(k) is a subfield of k which consists of a of the form

$$a = f(q) + f(q^{-1}), \quad f \in \mathbb{Q}[X]$$

and Ims(k) consists of

$$b = f(q) - f(q^{-1}).$$

Clearly,

$$\mathrm{k} = \mathrm{Reals}(\mathrm{k}) + \mathrm{Ims}(\mathrm{k}).$$

6.3 Lemma. *The field* $\mathbb{R}$ *is approximable by the field* Reals(k).

Proof. Clearly $\mathrm{k} \subset \mathbb{C}$ and $\mathrm{Reals(k)} \subset \mathbb{R}$.

Since k is dense in $\mathbb{C}$ and Reals(k) dense in $\mathbb{R}$ for the standard part map

$$\mathrm{st} : {}^*\mathbb{C} \to \mathbb{C} \cup \{\infty\} = \bar{\mathbb{C}}$$

we have $\mathrm{st(k)} = \bar{\mathbb{C}}$ and $\mathrm{st(Reals(k))} = \bar{\mathbb{R}}$. $\Box$

Clearly, st is also an approximation in the language with norm topology, defining the unary predicates $|x| \geq r$ (or $\leq r$) for each rational r. In this language $\mathrm{st} : {}^*\mathrm{k} \to \bar{\mathbb{C}}$ is a strong approximation, since $\mathrm{st}(\beta) = \infty$ iff $|\beta| \geq r$ for all rational r.

6.4 A trivial line bundle with connection over $\bar{\mathbb{R}}$. A trivial $\mathbb{C}$-line bundle E is isomorphic to $\mathbb{C} \times \bar{\mathbb{R}}$. Let $\Gamma(E)$ be the space of smooth sections $\varphi : \mathbb{R} \to E$, (which is isomorphic to the space of smooth functions $f : \mathbb{R} \to \mathbb{C}$)

A connection

$$P : \Gamma(E) \to \Gamma(E) \otimes T^*\mathbb{R}$$

is a map satisfying Leibnitz rule: for any $\varphi \in \Gamma(E)$, any smooth $f : \mathbb{R} \to \mathbb{C}$

$$P(f\varphi) = fP(\varphi) + df \otimes \varphi.$$

We will also consider the case when there is an action of a cyclic group generated by Q acting on the bundle E. We will require that

$$QP - PQ = i\hbar I$$

(I the identity operator on E).

We similarly define a k-line bundle over the finite ring $R = {}^*\mathbb{Z}/\mu\nu$, where $\mathrm{k} = \mathbb{Q}(q)$, $q = \exp \frac{2\pi i}{\mu\nu}$.

We set the fibre over $a \in {}^*\mathbb{Z}/\mu\nu$ to be the vector space $\mathrm{k}\,\mathbf{u}(q^a)$ with the action of operators U and V as described in 6.1.

Ket-notation. Choose a canonical basis $\mathbf{u}(q^a)$ in $\mathbf{V}$ and a section $x \mapsto |x\rangle$ of the line bundle over $\bar{\mathbb{R}}$ such that $|x\rangle$ is an eigenvector of Q with eigenvalue x.

Similarly, $|p\rangle$ will stand for a section $p \mapsto |p\rangle$ $p \in \bar{\mathbb{R}}$, $|p\rangle = \mathbf{v}(q^b)$ such that $|p\rangle$ is an eigenvector of P with eigenvalue p.

6.5 Proposition. *A trivial line bundle with connection over $\bar{\mathbb{R}}$ is approximable by* k*-line bundle over finite weak rings R with action of operators U and V.*

Proof. For simplicity of calculations below we introduce more convenient notation:

$\epsilon := \frac{1}{\mu}$, $\delta := \frac{1}{\nu}$.

On the line bundle over R define a linear operator Q on the fibres and on $\mathbf{V}$ as

$$Q = \frac{U-1}{i\delta}; \quad \text{thus} \quad Q : \mathbf{u}(q^a) \mapsto \frac{q^a - 1}{i\delta}\mathbf{u}(q^a).$$

Define P as an operator on $\mathbf{V}$

$$P = \frac{V-1}{i\delta}; \quad \text{thus} \quad P : \mathbf{u}(q^a) \mapsto \frac{1}{i\delta}\left(\mathbf{u}(q^{a-1}) - \mathbf{u}(q^a)\right).$$

We define the limit on the line bundle:

$$\lim{}_\mu : \mathbf{u}(q^a) \mapsto |\lim{}_\mu a\rangle = |\mathrm{st}(\frac{a}{\mu})\rangle,$$

that is

$$\lim{}_\mu(a) = x \in \mathbb{R} \Leftrightarrow \lim{}_\mu \mathbf{u}(q^a) = |x\rangle. \tag{13}$$

In other words

$$Q|x\rangle = x|x\rangle. \tag{14}$$

By definitions we will have on the bundle over R :

$$\begin{array}{l} QP : \mathbf{u}(q^a) \mapsto \frac{q^a-1}{\delta^2}\mathbf{u}(q^a) - \frac{q^{a-1}-1}{\delta^2}\mathbf{u}(q^{a-1}) \\ PQ : \mathbf{u}(q^a) \mapsto \frac{1-q^a}{\delta^2}\left(\mathbf{u}(q^{a-1}) - \mathbf{u}(q^a)\right) \end{array}$$

Hence

$$QP - PQ : \mathbf{u}(q^a) \mapsto q^{a-1}\frac{q-1}{\delta^2}\mathbf{u}(q^{a-1}) \tag{15}$$

Now recall that

$$q = \exp\frac{2\pi i}{\mu\nu} = \exp\{2\pi i\delta^2\frac{\mu}{\nu}\}, \quad q^{a-1} = \exp\{2\pi i(a-1)\delta^2\frac{\mu}{\nu}\}$$

so, when $\lim_\mu(a) = x \in \mathbb{R}$, then $a\delta$ is finite and so

$$\mathrm{st}\left((a-1)\delta^2\right) = 0, \ \mathrm{st}(q^{a-1}) = 1.$$

For the other factor in (15),

$$\mathrm{st}(\frac{q-1}{\delta^2}) = 2\pi i \cdot \mathrm{st}(\frac{\mu}{\nu}) = 2\pi i \mathrm{h} =: i\hbar.$$

Hence we have proved that

$$QP - PQ : |x\rangle \mapsto i\hbar|x\rangle. \tag{16}$$

Finally, to prove that P defines a connection on the bundle, note that (14) can be easily generalised to the action of an operator $f(Q) = c_0 I + c_1 Q + c_2 Q^2 + \dots$ where f is a smooth function

$$f(Q) : |x\rangle \mapsto f(x)|x\rangle.$$

It is well known that in this setting P satisfies the Leibnitz rule,

$$Pf(x)|x\rangle = f(x)P|x\rangle + f'|x\rangle,$$

that is P is a connection. $\square$

6.6 Remark. Without effecting the outcome (but somewhat complicating the calculation) Q an P can be defined as self-adjoint operators on the Hilbert space $\mathbf{V}$ as follows:

$$Q = \frac{U^{\epsilon} - U^{-\epsilon}}{2i\delta}, \quad P = \frac{V^{\delta} - V^{-\delta}}{2i\delta}.$$

6.7 Now we are going to work out how to associate a line bundle with a connection to a more general k-Hilbert space with actions of two unitary operators S and T.

We will assume that

$$S = q^s U^{\epsilon\kappa_{11}} V^{\delta\kappa_{12}}, \quad T = q^o U^{\epsilon\kappa_{21}} V^{\delta\kappa_{22}},$$

$$\kappa_{ij} \in {}^*\mathbb{Z}, \quad \kappa_{ij} << \mu$$

We denote

$$\mathbf{d} = \mathbf{d}(\kappa) := \det(\kappa_{ij}).$$

It is easy to calculate that

$$ST = q^{\mathbf{d}} TS. \tag{17}$$

Since S, T are definable in terms of U^{ϵ}, V^{δ}, these are operators on the k-Hilbert space $\mathbf{V}$.

6.8 In particular, in case $\mathbf{d} = 1$,

U^ϵ and V^δ are in the algebra $\langle S, T\rangle$ generated by S and T, moreover

$$\langle S, T\rangle = \langle U^\epsilon, V^\delta\rangle,$$

the k-Hilbert space with action of S, T is bi-definable with the k-Hilbert space defined in 6.1.

In this case we similarly define $E^{S,T}$ and canonical S-bases $\{\mathbf{s}(q^a) : a \in R\}$

$$\begin{array}{l} S : \mathbf{s}(q^a) \mapsto q^a\mathbf{s}(q^a) \\ T : \mathbf{s}(q^a) \mapsto \mathbf{s}(q^{a-1}) \end{array} \tag{18}$$

and consider such a base as a line bundle over R with actions of S and T.

We will also assume in this case, for reasons which should become obvious later, that all the κ_{ij} are in $\mathbb{Z}$, that is are finite.

We consider now the pair of operators on $\mathbf{V}$ which we call $\kappa_{11}Q + \kappa_{12}P$ and $\kappa_{21}Q + \kappa_{22}P$ which we define as

$$\kappa_{11}Q + \kappa_{12}P := \tfrac{S-1}{i\delta};$$

$$\kappa_{21}Q + \kappa_{22}P := \tfrac{T-1}{i\delta}$$

Applying respective definition and carrying out the same calculations as in 6.5 we will get in the limit the line bundle

$$\lim_\mu(a) = s \in \mathbb{R} \Leftrightarrow \lim_\mu \mathbf{s}(q^a) = |s\rangle \tag{19}$$

for the section $s \mapsto |s\rangle$,

$$(\kappa_{11}Q + \kappa_{12}P)|s\rangle = s|s\rangle. \tag{20}$$

While $\kappa_{21}Q + \kappa_{22}P$ acts on the bundle as a connection, by the same calculation as in 6.5. More precisely, we will of course have

$$(\kappa_{11}Q + \kappa_{12}P)(\kappa_{21}Q + \kappa_{22}P) - (\kappa_{21}Q + \kappa_{22}P)(\kappa_{11}Q + \kappa_{12}P) = i\hbar I.$$

We will call the line bundle of 6.5 the U^ϵ-bundle (or just U-bundle) and the one here the S-bundle.

6.9 Continuation of case d $= 1$. Now we wish to establish a relation between the U-bundle and the S-bundle. This will be given in approximation by the inner product

$$\langle \mathbf{u}(q^a)|\mathbf{s}(q^b)\rangle = r(a,b)$$

in the sense of the k-Hilbert space $\mathbf{V}$.

Then we will apply $\lim_\mu$ to give a meaning to this for the line bundles over $\mathbb{R}$. This would require a certain *renormalisation* to the naive meaning of $r(a,b)$.

7 Linear transformation on algebraic Hilbert spaces

7.1 Consider a linear transformation L given in terms of the U-basis, depending on parameters $a, b, c, g, e, \kappa, \kappa'$.

We assume $2a, b, 2c, e, \kappa, \kappa', g\kappa \in {}^*\mathbb{Q}$ are μ-small.

The domain H_L of L is the subspace spanned by $\{\mathbf{u}[n] : n \in \kappa' R\}$. Here and below

$$\mathbf{u}[n] := \mathbf{u}(q^n).$$

A linear transformation given as

$$L : \mathbf{u}[n] \mapsto c_L \sum_{m=0}^{\frac{N}{|e|}-1} q^{f(n,m)}\mathbf{u}[gn + em] \tag{21}$$

will be called **quadratic**, if $f(n,m) = an^2 + bnm + cm^2$.

We will require later that some more condition, in particular that L is unitary. The latter implies that the modulus of $L\mathbf{u}(q^n)$ is equal to 1 and so

$$|c_L| = \sqrt{\frac{|e|}{N}} \tag{22}$$

(here and above $e \in {}^*\mathbb{Z}$ is a parameter, not to be confused with e defined along with μ and ν).

We say L in (21) is **regular** if for some $\alpha, \beta, \alpha', \beta'$, μ-small such that $\alpha\alpha' - \beta\beta' = \kappa\kappa'$

(a)

$$q^{\frac{\gamma}{2}} U^\alpha V^\beta : q^{f(n,m)}\mathbf{u}[gn + em] \mapsto q^{f(n,m+1)+\kappa n}\mathbf{u}[gn + e(m+1)]$$

(b)

$$q^{\frac{\gamma'}{2}}U^{\beta'}V^{\alpha'}: q^{f(n,m)}\mathbf{u}[gn+em] \mapsto q^{f(n-\kappa',m)}\mathbf{u}[g(n-\kappa')+em]$$

7.2 Remark. (a) tells us that $L\mathbf{u}[n]$, $(n \in \kappa'{}^*\mathbb{Z})$ is an eigenvector $\mathbf{s}[\kappa n]$ of $q^{\frac{\gamma}{2}}U^{\alpha}V^{\beta}$ with eigenvalue $q^{\kappa n}$,

(b) tells that operator $q^{\frac{\gamma'}{2}}U^{\beta'}V^{\alpha'}$ shifts $\mathbf{s}[\kappa n]$ to $\mathbf{s}[\kappa n - \kappa\kappa']$ the eigenvector with eigenvalue $q^{\kappa n-\kappa\kappa'}$.

7.3 Example from [6]. Free particle.

$$\mathbf{u}[n] \mapsto c\sum_m q^{-\alpha\beta\frac{m^2}{2}}\mathbf{u}[-\beta m+n]$$

$-\frac{\beta}{\alpha} = t.$ $\gamma = \alpha\beta$, $k = \alpha$, $\kappa' = 1.$
$\alpha' = 1$, $\beta' = 0.$ $f(n,m) = m^2.$
$g = 1$, $e = -\beta.$

$$q^{\frac{\alpha\beta}{2}}U^{\alpha}V^{\beta}: q^{-\alpha\beta\frac{m^2}{2}}\mathbf{u}[-\beta m+n] \mapsto q^{-\alpha\beta\frac{m^2}{2}-\alpha\beta m-\frac{\alpha\beta}{2}+n\alpha}\mathbf{u}[-\beta(m+1)+n] =$$

$$= q^{\alpha n}q^{-\alpha\beta\frac{(m+1)^2}{2}}\mathbf{u}[-\beta(m+1)+n]$$

7.4 Proposition. *For any μ-small $\alpha, \alpha', \beta, \beta' \in {}^*\mathbb{Z}$, such that $\kappa\kappa' = \alpha\alpha' - \beta\beta'$ for some $\kappa, \kappa' \in {}^*\mathbb{Z}$, there are unique $a, b, c, \gamma, \gamma', g, e$ (a) and (b) of (21) hold and $2a, b, 2c, \gamma, \gamma', g\kappa, e \in {}^*\mathbb{Z}$ μ-small.*

In other words, given $\alpha, \alpha', \beta, \beta' \in {}^*\mathbb{Z}$, a choice of integer κ such that $L\mathbf{u}(n)$ of (7.1) is an eigenvector of $q^{\frac{\gamma}{2}}U^{\alpha}V^{\beta}$ with eigenvalue $q^{\kappa n}$ for all $n \in \kappa'\,{}^*\mathbb{Z}$ determines L uniquely up to the constant c_L.

Proof.

Calculate using (a): By definition

$$q^{\frac{\gamma}{2}}U^{\alpha}V^{\beta}: q^{f(n,m)}\mathbf{u}(q^{gn+em}) \mapsto q^{\frac{\gamma}{2}+f(n,m)+\alpha(gn+em)-\alpha\beta}\mathbf{u}(q^{gn+em-\beta})$$

hence we need for (a):

$$\frac{\gamma}{2} + f(n,m) + \alpha(gn+em) - \alpha\beta = f(n,m+1) + \kappa n \tag{23}$$

and

$$gn + em - \beta = gn + e(m+1) \tag{24}$$

Hence, from (24),

$$e = -\beta. \tag{25}$$

From (23):

$$\frac{\gamma}{2} = \alpha\beta + c = -\alpha e + c, \tag{26}$$

and for any n and m

$$n(b + \kappa - \alpha g) + m(2c - \alpha e) = 0,$$

hence

$$b + \kappa - \alpha g = 0 \tag{27}$$

and

$$2c - \alpha e = 0.$$

So, using (25)

$$c = -\frac{\alpha\beta}{2}, \quad \gamma = \alpha\beta \tag{28}$$

Now we calculate the requirement (b).

By definition

$$q^{\frac{\gamma'}{2}} U^{\beta'} V^{\alpha'} : q^{f(n,m)} \mathbf{u}(q^{gn+em}) \mapsto q^{f(n,m)+\frac{\gamma'}{2}+\beta'(gn+em-\alpha')} \mathbf{u}(q^{gn+em-\alpha'}).$$

Hence we need for (b):

$$gn + em - \alpha' = g(n - \kappa') + em \tag{29}$$

and

$$f(n,m) + \frac{\gamma'}{2} + \beta'(gn + em - \alpha') = f(n - \kappa', m). \tag{30}$$

Hence, from (29)

$$g = \frac{\alpha'}{\kappa'}, \tag{31}$$

From (30), comparing the terms with n, m and the terms free of n, m we get

$$\frac{\gamma'}{2} - \alpha'\beta' = a\kappa'^2, \tag{32}$$

$$\beta' g = -2a\kappa' \tag{33}$$

$$\beta' e = -b\kappa' \tag{34}$$

The last equation combined with (25) gives

$$b = \frac{\beta'\beta}{\kappa'}. \tag{35}$$

And, using all the previous, (33) gives us

$$a = -\frac{\alpha'\beta'}{2\kappa'^2} \tag{36}$$

Now we have found the values of a, b, c, g, e, γ and γ' in terms of $\alpha, \beta, \alpha', \beta'$ in (36), (35), (28), (31), (26) and (32).

Note that equation (27) is a consequence of the rest of the equations and the definition of k. Indeed, substituting in (27) the values of a, b and g from (36), (35) and (31) respectively, we get

$$\alpha\alpha' - \beta\beta' - \kappa\kappa' = 0$$

which is a valid equality by our assumptions. □

7.5 Summary

$$\begin{aligned} a &= -\tfrac{\alpha'\beta'}{2\kappa'^2} \\ b &= \tfrac{\beta'\beta}{\kappa'} \\ c &= -\tfrac{\alpha\beta}{2} \\ g &= \tfrac{\alpha'}{\kappa'}, \\ e &= -\beta \end{aligned}$$

7.6 Corollary. The range of $L_{\alpha,\beta,\alpha',\beta'}$ is a subspace of the k-space generated by

$$\{\mathbf{u}(q^l) : l \in \gcd(\alpha', \beta) \cdot R\}.$$

Proof. By the definition of $L_{\alpha,\beta,\alpha',\beta'}$ it is enough to have all l of the form

$$l = gn + em, \quad n \in \kappa' R, m \in R.$$

(Recall that $R = {}^*\mathbb{Z}/N$.)

By the above $gn = \alpha' \frac{n}{\kappa'} \in \alpha' R$ and em runs through all elements of βR. □

7.7 Let $n \in \kappa'^*\mathbb{Z}$. Then $gn \in {}^*\mathbb{Z}$.
Let $l \in gn + e^*\mathbb{Z}$, that is

$$m := \frac{l - gn}{e} \in {}^*\mathbb{Z}.$$

By definition we get then

$$\langle \mathbf{u}(q^l) | L\mathbf{u}(q^n) \rangle = c_L q^{f(n,m)} = c_L \exp 2\pi i \frac{an^2 + bnm + cm^2}{\mu^2 \mathrm{h}}. \tag{37}$$

Substituting the value $\frac{l-gn}{e}$ for m we get

$$an^2 + bnm + cm^2 = \frac{(ae^2 - beg + cg^2)n^2 + (be - 2cg)ln + cl^2}{e^2}$$

and so

$$\langle \mathbf{u}(q^l) | L\mathbf{u}(q^n) \rangle = \exp 2\pi i \mathrm{h} \frac{(ae^2 - beg + cg^2)n^2 + (be - 2cg)ln + cl^2}{e^2 \mu^2}.$$

Now note that by 7.5

$$\frac{ae^2 - beg + cg^2}{e^2} = -\frac{\alpha' \kappa}{2\beta\kappa'}, \quad \frac{be - 2cg}{e^2} = \frac{\kappa}{\beta}, \quad \frac{c}{e^2} = -\frac{\alpha}{2\beta}$$

Set

$$A := \frac{\alpha' \kappa}{\beta \kappa'}, \quad B := -\frac{\kappa}{\beta}, \quad C := \frac{\alpha}{\beta}. \tag{38}$$

7.8 Thus we have proved
For any $n \in \kappa'^*\mathbb{Z}$ and $l \in gn + e^*\mathbb{Z}$

$$\langle \mathbf{u}(q^l) | L\mathbf{u}(q^n) \rangle = c_L \exp -\frac{2\pi i}{\mathrm{h}} \{ A(\frac{n}{\mu})^2 + 2B\frac{n}{\mu}\frac{l}{\mu} + C(\frac{l}{\mu})^2 \}.$$

7.9 Matrix representation. We can equivalently represent L as a operator on the whole N-dimensional vector space with $N \times N$ matrix given by

$$L(l,n) = \begin{cases} c_L \exp -i\frac{2pi}{N}\{An^2 + 2Bnl + Cl^2\}, & \text{if } \kappa'|n \ \&\ e|(l - gn) \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$L(l,n) = c_L \cdot \delta(l,n) \cdot q^{-\frac{An^2+2Bnl+Cl^2}{2}} \tag{39}$$

where

$$\delta(l,n) = \begin{cases} 1 & \text{if } \kappa'|n \ \& \ e|(l-gn) \\ 0, & \text{otherwise.} \end{cases}$$

$$L : \mathbf{u}(q^n) \mapsto \begin{cases} c_L \sum_{l\in gn+eR} q^{-\frac{An^2+2Bnl+Cl^2}{2}} \mathbf{u}(q^l), & \text{if } \kappa'|n \\ 0, & \text{otherwise.} \end{cases}$$

Note that the support of L (as a map from the canonical U-basis E^U) is dense in E^U. This is analogous to the *linear transformation of* R of section 5.

7.10 Proposition. *Given $A, B, C \in {}^*\mathbb{Q}_{fin}$ with μ-small denominators and numerators, there are $\alpha, \beta, \kappa, \alpha', \beta', \kappa' \in {}^*\mathbb{Z}$ which satisfy 7.4 and (38).*

Every transformation representable in the form (39) is also representable in the form (21).

Proof. The rational numbers B and C can be represented in the form

$$B = -\frac{\kappa}{\beta}, \quad C = \frac{\alpha}{\beta}$$

where $\alpha, \beta, \kappa \in {}^*\mathbb{Z}$, $\beta > 0$. The rational number $\frac{A}{B}$ can be represented as

$$\frac{A}{B} = \frac{\alpha'}{\kappa'}, \quad \text{for some } \alpha', \kappa' \in {}^*\mathbb{Z}.$$

Now we have the relations (38) satisfied. Moreover, we may assume that $\alpha\alpha' - \kappa\kappa'$ is divisible by β, multiplying, if necessary, both α' and κ' by a divisor of β. Thus we get

$$\beta' = \frac{\alpha\alpha' - \kappa\kappa'}{\beta} \in {}^*\mathbb{Z}$$

and this satisfies the requirements of 7.4. □

7.11 Open Problem. Determine the eigenvalues and eigenvectors of L.

7.12 Continuous limit. Now denote

$$x := \mathrm{st}(\frac{n}{\mu}) = \lim_\mu(n), \quad y = \mathrm{st}(\frac{l}{\mu}) = \lim_\mu(l) \tag{40}$$

and assuming that $A, B, C \in {}^*\mathbb{Q}_{fin}$ (that is are finite non-standard rationals)

$$\mathrm{A} := \mathrm{st}(A), \quad \mathrm{B} := \mathrm{st}(B), \text{ and } \mathrm{C} := \mathrm{st}(C).$$

Then applying limit to the last equality we get for the expression in 7.8 :

$$\lim \exp -i\frac{2\pi}{\mathrm{h}}\{A(\frac{n}{\mu})^2 + 2B\frac{n}{\mu}\frac{l}{\mu} + C(\frac{l}{\mu})^2\} = \exp -i\frac{\mathrm{A}x^2 + 2\mathrm{B}xy + \mathrm{C}y^2}{\hbar}.$$

7.13 Equivalent transformations. Call $L_{\alpha_1,\beta_1,\alpha'_1,\beta'_1}$ **strongly equivalent to** $L_{\alpha_2,\beta_2,\alpha'_2,\beta'_2}$ if corresponding coefficients are equal:

$$A_1 = A_2, \ B_1 = B_2, \ C_1 = C_2.$$

In the case

$$\mathrm{A}_1 = \mathrm{A}_2, \ \mathrm{B}_1 = \mathrm{B}_2, \ \mathrm{C}_1 = \mathrm{C}_2$$

we say the transformations are **equivalent**.

Clearly, strong equivalence implies equivalence.

Lemma. *Let $\rho, \sigma \in {}^*\mathbb{Z}$ be μ-small. Then equivalence is preserved under the change*

$$\alpha, \beta, \kappa, \alpha', \beta', \kappa' \mapsto \rho\alpha, \rho\beta, \rho\kappa, \sigma\alpha', \sigma\beta', \sigma\kappa'.$$

Proof. Immediate by definitions. □

7.14 Rescaling to density and the relative value of c_L. By 7.1 we determine the absolute value of

$$c_L = \langle \mathbf{u}[l] | L\mathbf{u}[n] \rangle = \langle \mathbf{u}[l] | \mathbf{s}[n] \rangle$$

uniquely.

However, when considering the numerical limit, i.e. the values of the calculated parameters in $\bar{\mathbb{R}}_w$, we see that the natural limit value of c_L is 0. This is because $|c_L|$ is in fact a number for "the probability that the particle in position $\mathbf{u}(q^n)$ under the action of L gets into position $\mathbf{u}(q^l)$". We will replace this by a **Dirac-rescaled** value $c_{\mathrm{Dir}}(L)$, in essense, the "probability density" which we define in the current work axiomatically in terms of parameters of L given in 7.1.

7.15 According to 7.2 one should treat L as an isomorphism between two k-linear subspaces,

$$L : H_u \to H_s$$

where $H_u \subseteq H_L$ and $H_s = L(H_u)$.

H_s is a module under the operators $q^{\frac{\gamma}{2}} U^\alpha V^\beta$ and $q^{\frac{\gamma'}{2}} U^{\beta'} V^{\alpha'}$. A minimal such subspace has a canonical basis of the form

$$\{\mathbf{s}[\kappa\kappa' l] : \ l = 0, 1, \dots \frac{N}{\kappa\kappa'} - 1\}.$$

Respectively, H_u is a module under the operators U^κ, $V^{\kappa'}$ spanned over a canonical basis

$$\{\mathbf{u}^\kappa[\kappa\kappa' l] : \ l = 0, 1, \dots \frac{N}{\kappa\kappa'} - 1\}$$

of U^κ-eigenvectors. (Note that $\mathbf{u}[n]$, an eigenevector of U with eigenvalue q^n, is an eigenvector of U^κ with eigenvalue $q^{\kappa n}$ but this is just one form of U^κ-eigenvectors.)

We continue the analysis following [6], section 3.

In order for H_u to be of the required dimension $\frac{N}{\kappa\kappa'}$ (and to make L injective) it is necessary and sufficient, by 3.47 of [6], that the eigenvectors $\mathbf{u}^\kappa$ are of the form

$$\mathbf{u}^\kappa[\kappa n] := \frac{1}{\sqrt{|\kappa|}} \sum_{p=0}^{|\kappa|-1} q^{pd\frac{N}{\kappa}} \mathbf{u}[n + p\frac{N}{\kappa}], \tag{41}$$

for some μ-small integer $0 \le d < |\kappa|$ which determines the choice of H_u.

In accordance with above

$$L\mathbf{u}^\kappa[\kappa n] =: \mathbf{s}[\kappa n] = c_L \sum_{m=0}^{\frac{N}{|e|}-1} q^{f(n,m)} \mathbf{u}[gn + em], \tag{42}$$

the vector on the right hand side of (21).

Calculating the inner product in the ambient space we get, using the orthonormality of the basis $\mathbf{u}$,

$$\langle \mathbf{u}^\kappa[\kappa l] | \mathbf{s}[\kappa n] \rangle = c_L \frac{1}{\sqrt{|\kappa|}} \sum_{p=0}^{|\kappa|-1} \sum_{m=\frac{l-gn}{e}+p\frac{N}{\kappa e}} \langle q^{pd\frac{N}{\kappa}} \mathbf{u}[l + p\frac{N}{\kappa}] | q^{f(n,m)} \mathbf{u}[gn + em] \rangle =$$

$$= c_L \frac{1}{\sqrt{|\kappa|}} q^{an^2} \sum_{p=0}^{|\kappa|-1} q^{bn(m_l+p\frac{N}{\kappa e})+c(m_l+p\frac{N}{\kappa e})^2-pd\frac{N}{\kappa}}$$

where $m_l := \frac{l-gn}{e}$ is an integer.

Note that $(\frac{N}{\kappa e})^2 = N \cdot \frac{N}{\kappa^2 e^2}$ is an integer divisible by N. Hence, the above is equal to

$$= c_L \frac{1}{\sqrt{|\kappa|}} q^{an^2+bnm_l+m_l^2} \sum_{p=0}^{|\kappa|-1} q^{pD\frac{N}{\kappa}}$$

where $D = \frac{bn}{e} + 2\frac{cm_l}{e} - d$. Using 7.5 and the fact that $\frac{n}{\kappa'}$ is an integer, one deduces that D is an integer. Thus we calculate the sum of roots of unity,

$$\sum_{p=0}^{|\kappa|-1} q^{pD\frac{N}{\kappa}} = \begin{cases} |\kappa|, \text{ if } D \equiv 0 \mod \kappa \\ 0 \text{ otherwise} \end{cases}$$

Thus we conclude that in order for the H_u and H_s to be non-orthogonal we must choose d so that $D \equiv 0 \mod \kappa$ and as a consequence we get

$$|\langle \mathbf{u}^\kappa[\kappa l] | \mathbf{s}[\kappa n] \rangle| = |c_L| \sqrt{|\kappa|}. \tag{43}$$

Now we determine Δm according to the density of m in (21)

$$\Delta m = \frac{|\beta|}{\mu} = |\beta| \sqrt{\frac{1}{\mathrm{h}N}} \quad \text{and} \quad \Delta x := \hbar \Delta m = |\beta| \sqrt{\frac{2\pi\hbar}{N}}. \tag{44}$$

Thus we normalise the value of c_L (in fact switching to Dirac's bra-ket inner product instead of the product (37)) which we call $c_{\mathrm{Dir}}(L)$.

$$|c_{\mathrm{Dir}}(L)| = |\frac{\langle \mathbf{u}^\kappa[\kappa l] | \mathbf{s}[\kappa n] \rangle}{\Delta x}| = |\frac{c_L\sqrt{|\kappa|}}{\Delta x}| = \sqrt{|\frac{\kappa}{2\pi\hbar\beta}|} = \sqrt{\frac{|B|}{2\pi\hbar}}.$$

The angular value of $c_{\mathrm{Dir}}(L)$ can not be determined by our method at this point and we determine it here axiomatically as:

$$c_{\mathrm{Dir}}(L) := \frac{c_L\sqrt{|\kappa|}}{\Delta x} = e^{-\frac{\pi i}{4}} \sqrt{\frac{|B|}{2\pi\hbar}} \tag{45}$$

However, we will see later in 8.8 that *this value of angle is the only one compatible with the composition of linear transformations.*

7.16 Remark. *The definition of* $c_{\mathrm{Dir}}(L)$ *is invariant under equivalence.*

The limit value is accordingly

$$\lim c_{\mathrm{Dir}}(L) = \mathrm{st}(c_{\mathrm{Dir}}(L)) = e^{-\frac{\pi i}{4}} \sqrt{\frac{|\mathrm{B}|}{2\pi\hbar}}.$$

7.17 Example. Quantum Harmonic oscillator. The Hamiltonian is of the form

$$H_\omega = \frac{\mathrm{P}^2 + \omega^2 \mathrm{Q}^2}{2}$$

and

$$L = K^t = \exp\{-i\frac{H_\omega}{\hbar}t\}$$

Assume $\frac{\omega}{2\pi} =: w \in {}^*\mathbb{Z}$. We may always choose t so that $\sin \omega t$ and $\cos \omega t$ are rational.

Now we can present the rational numbers

$$A = \frac{\omega}{2\pi}\frac{\cos \omega t}{\sin \omega t}, \ B = -\frac{\omega}{2\pi \sin \omega t}, \quad C = \frac{\omega}{2\pi}\frac{\cos \omega t}{\sin \omega t}$$

and set

$$\langle x|L|y\rangle = c_{\mathrm{Dir}}(L) \cdot \exp i\omega \frac{(x^2+y^2)\cos \omega t - 2xy}{2\hbar \sin \omega t}$$

Accordingly,

$$c_{\mathrm{Dir}}(L) = e^{-i\frac{\pi}{4}} \sqrt{\frac{\omega}{2\pi\hbar |\sin \omega t|}}$$

7.18 The general form of quadratic transformations. We will also have to consider at some later point linear quadratic transformations with $L(l,n)$ of the form

$$L(l,n) = c_L \cdot \delta(l,n) \cdot \exp \frac{2\pi i}{\eta} \frac{An^2 + 2Bnl + Cl^2 + Dn + El}{2} \tag{46}$$

where $D, E \in {}^*\mathbb{Q}$ are bounded by a μ-small constant and may depend on l, n.

In this case, when passing to continuous limit as determined in 7.12, we will have

$$Dn \rightsquigarrow \frac{D\mathrm{h}}{\mu}\frac{n}{\mu} \text{ and } El \rightsquigarrow \frac{E\mathrm{h}}{\mu}\frac{l}{\mu}$$

and so these term vanish in the limit.

This proves the following.

Lemma. The generalised quadratic transformation (46) is equivalent to a quadratic of the form

$$L(l,n) = c_L \cdot \delta(l,n) \cdot \exp -2\pi i \frac{An^2 + 2Bnl + Cl^2}{2\eta}$$

7.19 Lemma. *Given $L_{\alpha_1,\beta_1,\alpha_1',\beta_1'}$ and $L_{\alpha_2,\beta_2,\alpha_2',\beta_2'}$ with commutation parameters κ_1, κ_1' and κ_2, κ_2' respectively, suppose the greatest common divisor (α_2', β_2) is divisible by κ_1'. Then*

$$\mathrm{Range}\, L_{\alpha_2,\beta_2,\alpha_2',\beta_2'} \subseteq \mathrm{Dom}\, L_{\alpha_1,\beta_1,\alpha_1',\beta_1'}.$$

In particular, for every $L_{\alpha_1,\beta_1,\alpha_1',\beta_1'}$ and $L_{\alpha_2,\beta_2,\alpha_2',\beta_2'}$ with commutation parameters κ_1, κ_1' and κ_2, κ_2' respectively we have that

$$\mathrm{Range}\, L_{\kappa_1'\alpha_2,\kappa_1'\beta_2,\kappa_1'\alpha_2',\kappa_1'\beta_2'} \subseteq \mathrm{Dom}\, L_{\alpha_1,\beta_1,\alpha_1',\beta_1'}.$$

Proof. Indeed, $\mathrm{Dom}\, L_{\alpha_1,\beta_1,\alpha_1',\beta_1'}$ is the subspace generated by $\{\mathbf{u}(q^n) : n \in \kappa_1' R\}$. And by 7.6 $\mathrm{Range}\, L_{\alpha_2,\beta_2,\alpha_2',\beta_2'}$ is a subspace of the subspace generated by $\{\mathbf{u}(q^l) : l \in (\alpha_2', \beta_2) \cdot R\}$. □

8 Composition of quadratic transformations

8.1 We will consider the composition $L = L_1 \circ L_2$ of two quadratic k-regular unitary quadratic transformations. The definition of composition is standard, for linear transformations in finite-dimensional k-spaces. In particular, we consider L_1, L_2 in matrix form and aim to calculate the product.

We have

$$L_1(l,n) = c_1 \cdot \delta(l,n) \cdot q^{\frac{A_1 n^2 + 2B_1 nl + C_1 l^2}{2}}$$

$$L_2(n,m) = c_2 \cdot \delta(n,m) \cdot q^{\frac{A_2 m^2 + 2B_2 mn + C_2 n^2}{2}}$$

where $\delta_1(l,n)$ and $\delta_2(n,m)$, are in terms of $\alpha_1, \beta_1, \ldots$ and $\alpha_2, \beta_2, \ldots$ respectively.

8.2 By above

$$L(l,m) = \sum_{n\in\kappa' R} L_1(l,n)\cdot L_2(n,m) =$$

$$= c_1c_2 \sum_{n\in\kappa' R} \delta_1(l,n)\delta_2(n,m)q[\frac{1}{2}(A_1n^2{+}2B_1nl{+}C_1l^2{+}A_2m^2{+}2B_2mn{+}C_2n^2)] =$$

$$= c_1c_2q[\frac{1}{2}(C_1l^2{+}A_2m^2)] \sum_{n\in\kappa' R} \delta_1(l,n)\delta_2(n,m)q[\frac{1}{2}\left((A_1+C_2)n^2+2(B_1l+B_2m)n\right)]$$

8.3 Now we need to evaluate the sum in the last expression, given fixed values of l and m.

Given an integer l,

$$\delta_1(l,n) = 1 \text{ iff } n \equiv 0 \bmod \kappa_1' \ \& \ \frac{\alpha_1' n}{\kappa_1'} \equiv l \bmod \beta_1 \tag{47}$$

This is immediate by definition once one recalls that $|e| = |\beta|$. The last congruence is equivalent to $e_1|(g_1n - l)$.

Given an integer m,

$$\delta_2(n,m) = 1 \text{ iff } m \equiv 0 \bmod \kappa_2' \ \& \ n \equiv \frac{\alpha_2' m}{\kappa_2'} \bmod \beta_2. \tag{48}$$

Claim. Given l and m

$$\delta_1(l,n)\delta_2(n,m) = 1 \text{ iff } m \equiv 0 \bmod \kappa_2' \ \& \ n \equiv 0 \bmod \kappa_1'$$

$$\exists n_0 \ n_0 \equiv 0 \bmod \kappa_1' \ \& \ \frac{\alpha_1' n_0}{\kappa_1'} \equiv l \bmod \beta_1 \ \& \ n_0 \equiv \frac{\alpha_2' m}{\kappa_2'} \bmod \beta_2 \ \&$$

$$n \equiv n_0 \bmod \rho \ \text{ where } \rho = \operatorname{lcm}(\kappa_1', \frac{\beta_1}{(\alpha_1', \beta_1)}, \beta_2)$$

Proof. The conditions on n_0 follows directly from (47) and (48). This is also the condition of solvability (with respect to n) of the system of the three congruences of claims 1 and 2. The general solution of the congruences is the last line.

Remark. By definition n_0 depends on l and m. Assuming that $l \equiv 0 \bmod \beta_1$ and $\frac{m}{\kappa_2'} \equiv 0 \bmod \beta_2$ one can choose $n_0 = 0$.

8.4 We will use the conclusion of the Claim in the form

$$n = n_0 + p\rho, \quad p \in {}^*\mathbb{Z}.$$

By substitution we get

$$(A_1 + C_2)n^2 + 2(B_1 l + B_2 m)n =$$
$$= \rho^2(A_1+C_2)p^2 + 2\rho(B_1 l + B_2 m + A_1 n_0 + C_2 n_0)p + \{(A_1+C_2)n_0^2 + 2(B_1 l + B_2 m)n_0\}.$$

So the expression of $L(l,m)$ from 8.2 can be rewritten as

$$L(l,m) = c_1 c_2 q[\frac{1}{2}(C_1 l^2 + A_2 m^2 + (A_1 + C_2)n_0^2 + 2(B_1 l + B_2 m)n_0)]\cdot$$
$$\cdot \sum_{p=0}^{\frac{N}{\rho}-1} q[\frac{1}{2}\rho\left((A_1 + C_2)\rho\, p^2 + 2(B_1 l + B_2 m + A_1 n_0 + C_2 n_0)p\right)] \tag{49}$$

Note that $C_1 l^2 + A_2 m^2 + (A_1 + C_2)n_0^2 + 2(B_1 l + B_2 m)n_0$ is integral since $p = 0$ is admissible for the given l, m.

8.5 Claim. $B_1 l + A_1 n_0$, $B_2 m + C_2 n_0$, $A_1\rho$ and $C_2\rho$ are integers.

Proof. $B_1 l = \frac{\kappa_1}{\beta_1} l$ and $A_1 n_0 = -\frac{\alpha_1' \kappa_1}{\beta_1 \kappa_1'} n_0$ and, according to Claim 1, $\frac{\alpha_1'}{\kappa_1'} n_0 = l + p_1\beta_1$ for some $p_1 \in {}^*\mathbb{Z}$. One sees that $B_1 l + A_1 n_0 = -p_1\kappa_1$.

$B_2 m = \frac{\kappa_2}{\beta_2} m$ and, according to Claim 2, $C_2 n_0 = -\frac{\alpha_2}{\beta_2} n_0 = -\frac{\alpha_2}{\beta_2}(\frac{\alpha_2' m}{\kappa_2'} + p_2\beta_2)$ for some integer p_2. Using the assumption $\alpha_2\alpha_2' = \beta_2\beta_2' + \kappa_2\kappa_2'$ we get

$$\frac{\alpha_2}{\beta_2}\frac{\alpha_2' m}{\kappa_2'} = \frac{\beta_2' m}{\kappa_2'} + \frac{\kappa_2 m}{\beta_2}, \quad \frac{\beta_2' m}{\kappa_2'} \in {}^*\mathbb{Z}$$

and

$$B_2 m + C_2 n_0 = -\frac{\beta_2' m}{\kappa_2'} - \alpha_2 p_2 \in {}^*\mathbb{Z}.$$

For

$$A_1\rho = -\frac{\alpha_1'\kappa_1}{\beta_1\kappa_1'}\frac{\kappa_1'\beta_1\beta_2}{(\alpha_1', \beta_1)\sigma} = -\frac{\alpha_1'\kappa_1\beta_2}{(\alpha_1', \beta_1)\sigma}$$

it is easy to see that (α_1', β_1) divides α_1' and σ divides β_2.

For

$$C_2\rho = -\frac{\alpha_2}{\beta_2}\frac{\kappa_1'\beta_1\beta_2}{(\alpha_1', \beta_1)\sigma}$$

one just needs to note that (α_1', β_1) divides β_1 and σ divides κ_1'.

Claim proved.

8.6 We need to evaluate the quadratic Gauss sum in (49), which we write in the form

$$\sum_{p=0}^{\frac{N}{\rho}-1} q^{\frac{\rho}{2}(ap^2+2bp)},$$

where we define integers

$$a = (A_1 + C_2)\rho, \quad b = B_1 l + B_2 m + N_0$$

where $N_0 = (A_1 + C_2)n_0$. In particular, $|N_0|$ is bounded by a $|(A_1 + C_2)\rho|$. So is μ-small.

Since $q^\rho = \exp 2\pi i \frac{\rho}{N}$, the Gauss formula gives us

$$\sum_{p=0}^{\frac{N}{\rho}-1} q^{\rho\frac{1}{2}(ap^2+2bp)} = \begin{cases} e^{\frac{\pi i}{4}}\sqrt{\frac{aN}{\rho}}q^{-\rho\frac{b^2}{2a}} \text{ if } \frac{b}{a} \in \mathbb{Z} \\ 0, \text{ otherwise} \end{cases} \tag{50}$$

$$\rho\frac{b^2}{2a} = \frac{B_1^2 l^2 + B_2^2 m^2 + 2B_1B_2 lm + 2B_1 l N_0 + 2B_2 m N_0 + N_0^2}{A_1 + C_2}$$

By the above, substituting into (49) we get, for l, m such that $a|b$,

$$L(l,m) = c_1 c_2 e^{\frac{\pi i}{4}}\sqrt{|\frac{aN}{\rho}|}\; q[\frac{1}{2}(C_1 l^2 + A_2 m^2 - \rho b^2/a)] =$$

$$c_1 c_2 e^{\frac{\pi i}{4}}\sqrt{N|A_1 + C_2|}\; q[\frac{1}{2}(Am^2 + Blm + Cl^2 + Dm + El + F)] \tag{51}$$

where

$$A = A_2 - \frac{B_2^2}{A_1 + C_2} = \kappa_1'\kappa_2\frac{\alpha_1'\alpha_2'\kappa_1 + \beta_1\beta_2'\kappa_1'}{\kappa_2'(\alpha_1'\kappa_1\beta_2 + \alpha_2\kappa_1'\beta_1)} \tag{52}$$

$$B = \frac{B_1 B_2}{A_1 + C_2} = \frac{\kappa_1\kappa_1'\kappa_2}{\alpha_1'\kappa_1\beta_2 + \alpha_2\kappa_1'\beta_1} \tag{53}$$

$$C = C_1 - \frac{B_1^2}{A_1 + C_2} = -\frac{\alpha_1\alpha_2\kappa_1' + \beta_2\beta_1'\kappa_1}{\alpha_1'\kappa_1\beta_2 + \alpha_2\kappa_1'\beta_1} \tag{54}$$

and

$$D = -\frac{2B_2N_0}{A_1+C_2}, \quad E = -\frac{2B_1N_0}{A_1+C_2}, \quad F = -\frac{N_0^2}{A_1+C_2}$$

and so D, E, F are non-standard rationals with numerators and denominators bounded in terms of $\alpha_1, \beta_1, \kappa_1, \alpha_1', \beta_1', \kappa_1'$ $\alpha_2, \beta_2, \kappa_2, \alpha_2', \beta_2', \kappa_2'$ and hence are μ-small. In other words, L is a generalised quadratic transformation (see 7.18).

8.7 By restricting the domain of L_1L_2 so that $\frac{m}{\kappa_2'}, l \in \beta_1\beta_2\ {}^*\mathbb{Z}$ we get a choice $n_0 = 0$ in 8.3 and so $N_0 = 0$. It follows that D, E, F are all 0 for L_1L_2 with the smaller domain.

Hence L *is strongly equivalent to the quadratic transformation of the form 7.9 with* A, B, C *determined above.*

8.8 Also note that if we identify the multipliers

$$c_1 = e^{-\frac{\pi i}{4}}\sqrt{\frac{|\beta_1|}{N}}, \quad c_2 = e^{-\frac{\pi i}{4}}\sqrt{\frac{|\beta_2|}{N}}$$

then the multiplier in the product is

$$c_{L_1L_2} = c_1c_2e^{\frac{\pi i}{4}}\sqrt{|A_1+C_2|N} = e^{-\frac{\pi i}{4}}\sqrt{\frac{|\beta_1\beta_2(A_1+C_2)|}{N}} = e^{-\frac{\pi i}{4}}\sqrt{|\frac{\alpha_1'\kappa_1\beta_2+\alpha_2\kappa_1'\beta_1}{\kappa_1'N}|}$$

8.9 Now we assume, without loss of generality (see 7.19), that (α_2', β_2) is divisible by κ_1' and introduce, in accordance with (52)-(54), a regular linear transformation with parameters

$$\alpha := \frac{\alpha_1\alpha_2\kappa_1' + \beta_2\beta_1'\kappa_1}{\kappa_1'},$$

$$\beta := \frac{\alpha_1'\kappa_1\beta_2 + \alpha_2\kappa_1'\beta_1}{\kappa_1'}$$

$$\alpha' := \frac{\alpha_1'\alpha_2'\kappa_1 + \beta_1\beta_2'\kappa_1'}{\kappa_1'}$$

$$\beta' := \frac{\beta_2'\alpha_1\kappa_1' + \alpha_2'\beta_1'\kappa_1}{\kappa_1'}$$

$$\kappa := \frac{\kappa_1\kappa_2}{\kappa_1'}$$

$$\kappa' := \frac{\kappa_1 \kappa_2'}{\kappa_1'}$$

which corresponds to the A, B, C above in the form (39).

By our assumption all these are integers.

Note that these can be obtained as follows:

$$\begin{pmatrix} \alpha/\kappa & \beta/\kappa \\ \beta'/\kappa' & \alpha'/\kappa' \end{pmatrix} = \begin{pmatrix} \alpha_2/\kappa_2 & \beta_2/\kappa_2 \\ \beta_2'/\kappa_2' & \alpha_2'/\kappa_2' \end{pmatrix} \begin{pmatrix} \alpha_1/\kappa_1 & \beta_1/\kappa_1 \\ \beta_1'/\kappa_1' & \alpha_1'/\kappa_1' \end{pmatrix} \tag{55}$$

Note that under the notations, 8.8 gives us

$$c_{L_1 L_2} = e^{-\frac{\pi i}{4}} \sqrt{\frac{|\beta|}{N}} \tag{56}$$

as is the case for regular linear quadratic transformation with respective parameters.

8.10 Definition. Set the product of two quadratic linear transformations L_1, L_2 to be $L := L_1 L_2$, where L in the form (21) is determined by $\alpha, \beta, \kappa, \alpha', \beta', \kappa'$ in 8.9. Equivalently, determined by A, B, C in the form (39) as in (52)-(54).

8.11 Proposition. The product defined 8.10 is associative. Moreover,

8.12 Calculation in bra-ket notation. In this case, we have to adjust 8.2 to

$$L(l,m)_{Dir} = \sum_{\kappa_1' | n} L_1(l,n)_{Dir} \cdot L_2(n,m)_{Dir} \cdot \Delta n$$

Now we can continue using our calculation in (49) and (51) with adjusted multipliers

$$= e^{\frac{\pi i}{4}} c_{Dir}(L_1) c_{Dir}(L_2) \sqrt{N|A_1 + C_2|}\, \Delta n\, q[\frac{1}{2}(Am^2 + Blm + Cl^2 + Dm + El + F)]$$

Now we need to evaluate Δn, which by definition is the step in the summation over n. By looking at (50)[5] we identify the step for n

$$\Delta n = \frac{\rho}{a} \frac{1}{\mu} = \frac{1}{|A_1 + C_2| \sqrt{N} \mathrm{h}} \text{ and } \Delta x = |A_1 + C_2|^{-1} \sqrt{\frac{2\pi\hbar}{N}}$$

[5]Define Δn for canonical Gaussian integral properly.

and so in coordinates compatible with x we have the multiplier

$$c_{Dir}(L_1)c_{Dir}(L_2)\sqrt{N|A_1+C_2|}\,\Delta x = e^{-\frac{\pi i}{4}}\sqrt{\frac{|B_1|}{2\pi\hbar}}e^{-\frac{\pi i}{4}}\sqrt{\frac{|B_2|}{2\pi\hbar}}e^{\frac{\pi i}{4}}\sqrt{\frac{2\pi\hbar}{|A_1+C_2|}} =$$

$$= e^{-\frac{\pi i}{4}}\sqrt{\frac{|B|}{2\pi\hbar}}$$

the last equality by (53).

Thus we have proved

8.13 Theorem. *The product $L_1 \circ L_2$ of two regular linear quadratic transformations in the form (39) is equivalent to a regular linear quadratic transformations with parameters given in (52), (53), (54) and (56).*

The rescaled multiplier

$$c_{Dir}(L_1L_2) = e^{-\frac{\pi i}{4}}\sqrt{\frac{|B|}{\hbar}}$$

.

8.14 Example (Continuation of Example 7.17). Consider two real values t_1 and t_2 with the same ω. We take L_1 to correspond to the data with $\sin\omega t_1 = \frac{\beta_1}{\kappa_1'}$ and $\cos\omega t_1 = \frac{\alpha_1'}{\kappa_1'}$ and L_2 to correspond to $\sin\omega t_2 = \frac{\beta_2}{\kappa_2'}$ and $\cos\omega t_2 = \frac{\alpha_2'}{\kappa_2'}$.

Also by 7.17 $\kappa_1 = \frac{\omega}{2\pi}\kappa_1'$, $\alpha_1 = \frac{\omega}{2\pi}\alpha_1'$, and $\kappa_2 = \frac{\omega}{2\pi}\kappa_2'$, $\alpha_2 = \frac{\omega}{2\pi}\alpha_2'$.

Substituting this into the trigonometric data from 7.17 we get

$$\cos\omega t_1 \cdot \sin\omega t_2 + \cos\omega t_2 \cdot \sin\omega t_1 = \frac{\alpha_1'\beta_2\kappa_1 + \alpha_2\kappa_1'\beta_1}{\kappa_1'\kappa_1\kappa_2'} = \frac{\beta}{\kappa'}$$

or

$$\frac{\beta}{\kappa'} = \sin\omega(t_1+t_2).$$

Similarly, we can calculate other parameters of $L = L_1 \circ L_2$ and eventually see that L corresponds to quantum harmonic oscillator with frequency ω and time $t_1 + t_2$.

9 Compositions and path integration

We prove the path integral formula for the quantum harmonic oscillator.

9.1 Following conventional methods we model the operator

$$K^t = \mathrm{e}^{-i\frac{\mathrm{P}^2+V(\mathrm{Q})}{2\hbar}t}$$

for "very small" value of t using its approximate evaluation

$$\mathrm{e}^{-i\frac{\mathrm{P}^2+V(\mathrm{Q})}{2\hbar}\Delta t} \approx \mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar}\Delta t} \cdot e^{-i\frac{V(\mathrm{Q})}{2\hbar}\Delta t}$$

In our model we set $\Delta t = \frac{1}{\eta}$, for a $\eta \in {}^*\mathbb{Z}$, $1 << \eta << \mu$. We are particularly interested in case $V(Q) = Q^2$.

9.2 We constructed the operator corresponding to $\mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar\eta}}$ in section 7 ($\alpha = \eta, \beta = -1, \alpha' = 1, \beta' = 0, \kappa = \eta, \kappa' = 1$)

$$\mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar\eta}} : \mathbf{u}(q^n) \mapsto c(\eta) \sum_{m=0}^{N-1} q^{\eta\frac{(n-m)^2}{2}} \mathbf{u}(q^m)$$

if $n \in \eta \cdot {}^*\mathbb{Z}$ and

$$\mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar\eta}} : \mathbf{u}(q^{n+\rho}) \mapsto 0, \quad \text{for } \rho = 1, \ldots, \eta - 1$$

For $\mathrm{e}^{-i\frac{\omega^2\mathrm{Q}^2}{2\hbar\eta}}$ we have a straightforward

$$\mathrm{e}^{-i\frac{\omega^2\mathrm{Q}^2}{2\hbar\eta}} : \mathbf{u}(q^n) \mapsto q^{-\frac{n^2\omega^2}{2\eta}} \mathbf{u}(q^n)$$

for $n \in \eta^*\mathbb{Z}$ which corresponds to $\alpha = 1, \alpha' = \eta, \beta = 0, \beta' = \omega^2$.

9.3 Combining the two into a product we have, according to (55), for

$$K_\eta := e^{-i\frac{\mathrm{P}^2}{2\hbar\eta}} \cdot e^{-i\frac{\omega^2\mathrm{Q}^2}{2\hbar\eta}}$$

the (α, β)-matrix (55)

$$\begin{pmatrix} \eta & -1 \\ \eta\omega^2 & \eta^2 - \omega^2 \end{pmatrix}, \quad \kappa = \eta, \ \kappa' = \eta^2$$

respectively,

$$A = \frac{\eta^2 - \omega^2}{\eta},\ B = -\eta,\ C = \eta$$

and in the form of subsection 7.9

$$K_\eta : \mathbf{u}(n) \mapsto c_{K_\eta} \sum_{l=0}^{N-1} q[\frac{\eta^2(l-n)^2 - \omega^2 n^2}{2\eta}]\mathbf{u}(l) \tag{57}$$

for $n \in \eta^2\, {}^*\mathbb{Z}$ and

$$c_{K_\eta} = e^{-\frac{\pi i}{4}} \sqrt{\frac{1}{N}}$$

is defined as in (7.14). The Dirac-delta renormalised (density) coefficient is respectively determined by (45) as

$$c_{Dir}(K_\eta) = e^{-\frac{\pi i}{4}} \sqrt{\frac{\eta}{2\pi\hbar}}.$$

9.4 Our plan is to calculate the limit value of the k-th power K_η^k of K_η for k a μ-small non-standard integer such that

$$t = \frac{k}{\eta} = k\Delta t \in {}^*\mathbb{Q}_{fin},\quad 0 < \omega t \leq T < \frac{\pi}{2}$$

for some positive $T \in \mathbb{R}$.

It is not difficult to see that K_η^k is a nonstandard approximation to the integral operator which conventionally is being presented as Feynman's path integral

$$\lim_{k\to\infty} \lambda(k) \int_{\mathbb{R}} \int_{\mathbb{R}} \dots \int_{R} \exp i f_{V,\hbar}(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $f_{V,\hbar}(x_1, \dots, x_k)$ is a quadratic form whose construction depends on V, k and the parameter $\hbar$, and $\lambda(k)$ is a normalising coefficient.

We choose T small enough (but standard) so that it satisfies the extra assumption:

for all $0 < x < y < T$

$$\frac{\cot y}{\cot x + \cot y} < \frac{x}{y} \tag{58}$$

$$\frac{\csc y}{\csc x + \csc y} < \frac{x}{y} \tag{59}$$

Such $T \in \mathbb{R}$ exists by (60) and (61) below.

We use induction on k which is legitimate since ${}^*\mathbb{Z}$ is a model of arithmetic.

9.5 By the composition theorem applied k times K_η^k can be represented in the form 7.9 with $A = A_k, B = B_k, C = C_k$ for some non-standard rationals A_k, B_k, C_k with μ-small numerators and denominators (since k is μ-small) and related $\alpha_k, \alpha'_k, \beta_k, \beta'_k, \kappa_k, \kappa'_k$. We will also have the multipliers $c_k := c_{K_\eta^k}$ and $c_{Dir}(K_\eta^k)$ which can be expressed in terms of the other parameters by (7.14) and (45).

Note that $\alpha_1, \alpha'_1, \beta_1, \beta'_1, \kappa_1, \kappa'_1$ and A_1, B_1, C_1 are given in 9.3.

9.6 Theorem. *The transformation K_η^k is equivalent to the time evolution operator for the quantum harmonic oscillator described in subsection 7.17, that is*

$$K_\eta^k \approx K^t = \exp\{-i\frac{H_\omega}{\hbar}t\} \text{ for } H_\omega = \frac{\mathrm{P}^2 + \omega^2\mathrm{Q}^2}{2}, \quad t = \mathrm{st}(\frac{k}{\eta})$$

The rest of the section is devoted to the proof of the theorem.

9.7 Inductive hypothesis. We assume that

$$A_k = \omega(\cot\frac{\omega k}{\eta} + a(k)); \quad B_k = -\omega(\csc\frac{\omega k}{\eta} + b(k)), \quad C_k = \omega(\cot\frac{\omega k}{\eta} + c(k)),$$

where $a(k), b(k)$ and $c(k)$ are non-standard real numbers such that

$$|a(k)|, |b(k)|, |c(k)| < \omega\frac{\ln k + L_0}{\eta} \leq \omega\frac{\ln\eta}{\eta}$$

9.8 Note that by the Laurent series decomposition, for z near 0,

$$\cot z = \frac{1}{z} - \frac{1}{3}z + o(z^2) \tag{60}$$

$$\csc z = \frac{1}{z} + \frac{1}{6}z + o(z^2) \tag{61}$$

Also we recall the trigonometric identities

$$\cot(z_1 + z_2) = \frac{\cot z_1 \cot z_2 - 1}{\cot z_1 + \cot z_2} \tag{62}$$

$$\cot^2 z - \csc^2 z = 1 \tag{63}$$

$$\csc(z_1 + z_2) = \frac{\csc z_1 \csc z_2}{\cot z_1 + \cot z_2} \tag{64}$$

9.9 Case $k = 1$. We have by 9.3

$$A_1 = \eta - \frac{\omega^2}{\eta} = \omega(\cot \frac{\omega}{\eta} + a(1))$$

where

$$a(1) = -\frac{2\omega}{3\eta} + o(\frac{1}{\eta^2}),$$

$$B_1 = -\eta = -\omega(\csc \frac{\omega}{\eta} + b_1)$$

where

$$b(1) = -\frac{\omega}{6\eta} + o(\frac{1}{\eta^2})$$

$$C_1 = \eta = \omega(\cot \frac{\omega}{\eta} + c(1))$$

where

$$c(1) = \frac{\omega}{3\eta} + O(\frac{1}{\eta^3})$$

Here and below $O(\frac{1}{\eta^3})$ means a nonstandard number such that $O(\frac{1}{\eta^3}) : \frac{1}{\eta^3}$ is a finite nonstandard real.

Clearly, the inductive hypothesis for $k = 1$ holds.

Also note that

$$|a(1) + c(1) - 2b(1)| = O(\frac{\omega}{\eta^3}). \tag{65}$$

9.10 Inductive step. Calculate by product formulas (52), (53) and (54)

$$A_{k+1} = \frac{A_1C_1 + C_1C_k - B_1^2}{C_k + A_1} =$$

$$= \omega \frac{(\cot\frac{\omega}{\eta} + a(1))(\cot\frac{\omega}{\eta} + c(1)) + (\cot\frac{\omega}{\eta} + c(1))(\cot\frac{\omega k}{\eta} + c(k)) - (\csc\frac{\omega}{\eta} + b(1))^2}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} + a(1) + c(k)} =$$

$$= \omega \frac{\cot^2\frac{\omega}{\eta} + \cot\frac{\omega}{\eta}\cot\frac{\omega k}{\eta} - \csc^2\frac{\omega}{\eta}}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} + a(1) + c(k)} +$$

$$+\omega \frac{(a(1) + c(1))\cot\frac{\omega}{\eta} + c(k)\cot\frac{\omega}{\eta} + c(1)\cot\frac{\omega k}{\eta} - 2b(1)\csc\frac{\omega}{\eta} + O(\frac{1}{\eta^2})}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} + a(1) + c(k)}$$

Now we use (63) and (62) to get

$$\frac{\cot^2\frac{\omega}{\eta} + \cot\frac{\omega}{\eta}\cot\frac{\omega k}{\eta} - \csc^2\frac{\omega}{\eta}}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta}} = \cot\frac{\omega(k+1)}{\eta}$$

and

$$\frac{\cot^2\frac{\omega}{\eta} + \cot\frac{\omega}{\eta}\cot\frac{\omega k}{\eta} - \csc^2\frac{\omega}{\eta}}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} + a(1) + c(k)} = \cot\frac{\omega(k+1)}{\eta} + \cot\frac{\omega(k+1)}{\eta} \cdot O(\frac{1}{\eta^2}))$$

since $\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} \geq \frac{\eta}{\omega}$. The estimate of the second summand is

$$|a'(k+1)| = |\frac{(a(1) + c(1))\cot\frac{\omega}{\eta} + c(k)\cot\frac{\omega}{\eta} + c(1)\cot\frac{\omega k}{\eta} - 2b(1)\csc\frac{\omega}{\eta} + o(\frac{1}{\eta^2})}{\cot\frac{\omega k}{\eta} + \cot\frac{\omega}{\eta} + a(1) + c(k)}| \leq$$

$$\leq |a(1) + c(1) - 2b(1) + c(k) + \frac{c(1)}{k}| + o(\frac{1}{\eta^2}) \leq \frac{\omega}{3\eta k} + c(k) + o(\frac{1}{\eta^2})$$

Hence $|a(k+1)| = |a'(k+1) + \cot\frac{\omega(k+1)}{\eta} \cdot O(\frac{1}{\eta^2}))| < \omega\frac{ln(k+1) + L_0}{\eta}$

(use (65), the above estimate of $c(1)$ and (58) for the last inequality).

This proves the inductive hypothesis for A_{k+1} and a_{k+1}.

Case C

$$C_{k+1} = \frac{A_kC_k + A_1A_k - B_k^2}{C_k + A_1} =$$

$$= \omega \frac{(\cot \frac{\omega k}{\eta} + a(k))(\cot \frac{\omega k}{\eta} + c(k)) + (\cot \frac{\omega}{\eta} + a(1))(\cot \frac{\omega k}{\eta} + a(k)) - (\csc \frac{\omega k}{\eta} + b(k))^2}{\cot \frac{\omega k}{\eta} + \cot \frac{\omega}{\eta} + a(1) + c(k)} =$$

$$= \omega \frac{\cot^2 \frac{\omega k}{\eta} + \cot \frac{\omega}{\eta} \cot \frac{\omega k}{\eta} - \csc^2 \frac{\omega k}{\eta}}{\cot \frac{\omega k}{\eta} + \cot \frac{\omega}{\eta} + a(1) + c(k)} +$$

$$+ \omega \frac{(a(k) + a(1) + c(k)) \cot \frac{\omega k}{\eta} + a(k) \cot \frac{\omega}{\eta} - 2b(k) \csc \frac{\omega k}{\eta} + o(\frac{1}{\eta})}{\cot \frac{\omega k}{\eta} + \cot \frac{\omega}{\eta} + a(1) + c(k)} =$$

$$= \omega \cot \frac{\omega(k+1)}{\eta} \cdot (1 + O(\frac{1}{\eta^2})) + c'(k+1)$$

where

$$|c'(k+1)| = |\omega \frac{(a(k) + a(1) + c(k)) \cot \frac{\omega k}{\eta} + a(k) \cot \frac{\omega}{\eta} - 2b(k) \csc \frac{\omega k}{\eta} + o(\frac{1}{\eta})}{\cot \frac{\omega k}{\eta} + \cot \frac{\omega}{\eta} + a(1) + c(k)}| =$$

hence

$$c(k+1) \approx a(k) + \frac{a(1) + a(k) + c(k) - 2b(k)}{k}$$

as required.

Case B can be proven similarly using (53), (59) and (61).

This completes the proof of the theorem. □

9.11 Remark. Compare this with existing proof of the same formula, [9], 7.7.4, which leads to a non-convergent limit (page 552) that requires a "special summation method" to get a correct answer.

The reason why the existing proof requires the "special method", from our point of view, is that the conventional calculation uses two limits: once in calculating the integrals for a finite (N-multiple) product formula, and then in calculating the limit for $N \to \infty$. The composition of the two limits (which do not commute) is replaced in our calculation by the single limit ($\frac{k}{\eta} \to t$ by applying the standard part map). This balances the two limits in a correct way.

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