

Dirac - von Neumann axioms in the setting of Continuous Model Theory

B.Zilber

January 10, 2026

Abstract

We recast the well-known axiom system of quantum mechanics used by physicists (the Dirac calculus) in the language of Continuous Logic. For the basic version of the axiomatic system we prove that along with the canonical continuous model the axioms have approximate finite models of large sizes, in fact the continuous model is isomorphic to an ultraproduct of finite models. We analyse the continuous logic quantifier corresponding to Dirac integration and show that in finite context it has two versions, local and global, which coincide on Gaussian wave-functions.

1 Introduction

1.1 The axiomatic formulation of quantum mechanics was introduced by Paul Dirac in 1930 [1] through a description of Hilbert space, and later developed with greater mathematical rigor in a monograph of 1932 by John von Neumann. Since 1930, Dirac went through several rewritings and new editions to refine his calculus to a level he considered satisfactory. In fact, von Neumann himself expressed his dissatisfaction not long after publication of his book, and spent considerable effort looking for alternatives [20].

This discussion continues presently, see e.g. [12] questioning physicality of the complete Hilbert space \mathcal{H} of quantum mechanics, effectively arguing for a dense subspace \mathcal{H}^{Def} of physically meaningful states instead.

1.2 In section 2 we survey the axioms of quantum mechanics following [2]. Readers with a background in logic will notice that what physicists refer to

as axioms is very far from what is a conventional set of axioms in a formal language even in its early form as presented e.g. by Hilbert's axiomatisation of geometry [3]. We noted earlier, in [5] and in [7], that the language that Dirac introduced is that of *continuous logic*, CL, and the (rigged) Hilbert space which the axioms describe is a close analogue of cylindric algebra of Tarski, see [6], in a CL-version. In section 4 we go further and prove our main Theorem 4.8 stating that the continuous theory of Dirac - von Neumann given in Hilbert-space form (with integral operators) has two kinds of models for the rigged Hilbert space $L^2(\mathbb{R})$ which we call \mathcal{H} -structures:

- the canonical model $\mathbb{U}(\infty)$ based on wave-functions/continuous predicates on \mathbb{R}^m , and
- an asymptotic class of finite approximate models $\mathbb{U}(n)$.

More precisely we prove that the canonical continuous model $\mathbb{U}(\infty)$ is isomorphic to a CL-ultraproduct of the finite $\mathbb{U}(n)$, $n \in \mathbb{N}$.

1.3 It is crucial that to represent elements of the rigged Hilbert space which are outside $L^2(\mathbb{R})$, including all-important position and momentum eigenvectors, one has to use the model-theoretic techniques of imaginary elements and imaginary sorts. Moreover, this technique allows to immediately generalise the setting of $L^2(\mathbb{R}^m)$, of quantum mechanics over the space \mathbb{R}^m , to the general $L^2(\mathbb{M})$, of quantum mechanics over a manifold \mathbb{M} which is interpretable in \mathbb{R} .

Note the key difference between the Hilbert space - representation of quantum mechanics and the CL-models representation: the latter has an explicit geometric universe \mathbb{M} similar to the universe of Newtonian physics.

1.4 As a matter of fact our $\mathbb{U}(n)$ are specific lattice models. Of course, a variety of lattice models have been in use in physics. They play an important role as toy models, as vehicles for specific calculations as well as modelling specific phenomena. However, the fact that the specific cyclic lattice models $\mathbb{U}(n)$ hosting the full set of integral operators represent in a mathematically exact way Dirac's axioms of quantum mechanics is novel. In particular, the Dirac integral is shown to correspond to a summation formula which is "local" in a certain well-defined sense, namely the summation domain has to be much shorter than the length of the full cycle. All unitary operators of the form e^{iL} where L is self-adjoint in a form of a polynomial of P and Q with rational coefficients, are represented in the $\mathbb{U}(n)$.

The proof of the theorem is based on an analysis of Dirac's integral (equivalently of the structure of the underlying rigged Hilbert space) in the context

of a CL-quantifier. Modelling Dirac’s integral in finite CL-structures $\mathbb{U}(n)$ we expose a specific *local* nature of this quantifier and compare it with another possible *global* quantifier, based on discrete number-theoretic Gauss summation. Our second main result Theorem 5.6 states that for theories restricted to Gaussian states (whose Hamiltonian includes quadratic terms only) the local and the global quantifiers act equivalently.

Gaussian part of quantum mechanics is the backbone of the theory. From it the theory extends by considering perturbed Gaussian states. We demonstrate in subsection 5.7 that the global quantifier is applicable to perturbed Gaussian states with result close to ones of perturbation theory.

2 Dirac’s calculus and axiomatisation of quantum mechanics

Below we reproduce a slightly edited version of axioms from [2], 6.3.

2.1 Axiom 1. The state of a quantum system is described by a vector $|\psi\rangle$ belonging to a complex Hilbert space \mathcal{H} . This state is usually called “ket ψ ”. A complex Hilbert space \mathcal{H} is a vector space, which can be finite dimensional or infinite dimensional, equipped with the complex scalar product (also called inner product) $\langle\psi|\psi'\rangle$ between any pair of states $|\psi\rangle, |\psi'\rangle$ in \mathcal{H} . The norm, or modulus, of a generic vector $|\psi\rangle \in \mathcal{H}$ is defined as

$$\|\psi\| = |\langle\psi|\psi\rangle|$$

and usually $|\psi\rangle$ is normalized to one, i.e. $\|\psi\| = 1$. The symbol $\langle\psi|$ which appears in the definition of the norm is called “bra ψ ” and it can be interpreted as the function

$$\langle\psi| : \mathcal{H} \rightarrow \mathbb{C}.$$

For any $|\psi'\rangle \in \mathcal{H}$ this function gives a complex number $\langle\psi|\psi'\rangle$ obtained as scalar product of $|\psi\rangle$ and $|\psi'\rangle$. In a complex Hilbert space \mathcal{H} it exists a set of basis vectors $|\phi_\alpha\rangle$ which are orthonormal, i.e. $\langle\phi_\alpha|\phi_\beta\rangle = \delta(\alpha - \beta)$, and such that

$$|\psi\rangle = \sum_{\alpha} c_{\alpha} |\phi_{\alpha}\rangle \tag{1}$$

for any $|\psi\rangle$, where the coefficients c_{α} belong to \mathbb{C} .

Axiom 2. Any observable (measurable quantity) of a quantum system is described by a self-adjoint linear operator $F : \mathcal{H} \rightarrow \mathcal{H}$ acting on the Hilbert space of state vectors.

For any classical observable F it exists a corresponding quantum observable F .

Axioms 3. The possible measurable values of an observable F are its eigenvalues f , such that

$$F|f\rangle = f|f\rangle$$

with $|f\rangle$ the corresponding eigenstate. The observable F admits the spectral resolution

$$F = \sum_f f|f\rangle\langle f| \quad (2)$$

where $\{|f\rangle\}$ is the set of orthonormal eigenstates of F , and the mathematical object $\langle f|$, called “bra of f ”, is a linear map that maps into the complex number. This also satisfy the identity

$$\sum_f |f\rangle\langle f| = \mathbf{I}.$$

Axiom 4. The probability P of finding the state $|\psi\rangle$ in the state $|f\rangle$ (both of norm 1) is given by

$$P = |\langle f|\psi\rangle|^2$$

This probability P is also the probability of measuring the value f of the observable F when the system is in the quantum state $|\psi\rangle$.

Axiom 5. The time evolution of states and observables of a quantum system with Hamiltonian H is determined by the unitary operator

$$K^t := \exp(-iHt/\hbar)$$

, such that $|\psi(t)\rangle = K^t|\psi\rangle$ is the time-evolved state $|\psi\rangle$.

2.2 The dynamical reformulation of quantum mechanics. This is based on the Stone Theorem:

For each self-adjoint operator A on \mathcal{H} there is a well-defined one parameter group of unitary operators on \mathcal{H}

$$\{e^{iAt} : t \in \mathbb{R}\}$$

and A can be recovered uniquely from the group.

Thus, we may reduce the theory to the equivalent theory of Hilbert spaces with unitary operators of the form above. One advantage of such a theory is that the unitary operators, unlike unbounded self-adjoint operators, are defined on the whole of \mathcal{H} and their treatment is mathematically more straightforward. The framework is also called the Heisenberg picture of quantum mechanics.

2.3 Now we make several comments on the axioms.

The term “Hilbert space” here should actually be read as *the rigged Hilbert space* (see [11]) because it differs from the standard definition by accommodating both a Hilbert space Φ and the dual space Φ^* with

$$\Phi \subseteq \mathcal{H} \subseteq \Phi^*.$$

The summation formulas like (1) and (2) are presented in a form of an integral if the family $|\psi_\alpha\rangle$ is continuous but seems natural in the summation form when α runs in the discrete spectrum of an operator.

2.4 Remark. Rigged Hilbert spaces provide a powerful mathematical framework to extend quantum mechanics, allowing distributions and generalized eigenfunctions to be rigorously handled. However, as is almost generally accepted, not every element corresponds to a physically realisable state – some are purely mathematical artifacts, see e.g. [12].

In the more general context of quantum field theories Wightman axioms explicitly postulate that physically meaningful part of the rigged Hilbert space \mathcal{H} is a dense subset $\mathcal{D} \subset \mathcal{H}$.

3 The axioms in the setting of Continuous Logic

3.1 We discuss in this section the possible interpretation of the above axioms in terms of (the most general versions of) Continuous Logic (CL) and Continuous Model Theory.

Recall that in a most general terms the language of CL consists of *predicate* symbols (we will ignore function symbols for now), a collection of *connectives*, that is continuous functions $\mathbb{C}^n \rightarrow \mathbb{C}$, and quantifiers, that is continuous transformations of predicates.

A basic CL-formula is made of predicate symbols using connectives and quantifiers.

An interpretation of symbols and formulas begins with a choice of a *universe* M , which may be a metric space or, in more recent application, a measure space.

Symbols of n -ary predicates P are interpreted as maps $P : M^n \rightarrow \mathbb{C}$. If $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a connective and ψ_1, \dots, ψ_n are formulas, equivalently, definable predicates, then the formula $f(\psi_1, \dots, \psi_n)$ is interpreted as the composition of the maps defined by ψ_1, \dots, ψ_n with $f(x_1, \dots, x_n)$. Quantifiers are interpreted in a special way as transformations of formulas in $n+1$ variables into formulas in n variables.

The uniformity of interpretation of language symbols across different M is ensured by certain uniform continuity moduli for the symbols P .

A universe M together with interpretation of predicates P of the language constitutes a *structure* in continuous model theory. Importantly, the definable sets in a structure are obtained not just by CL-formulas but also as limits in the families of formulae-definable sets.

See [4], [14] and [?] for further details.

3.2 Recall that the historical prototype of a vector $|\psi\rangle$ of the Hilbert space has been a wave-function, that is a function

$$\psi : \mathcal{M} \rightarrow \mathbb{C}$$

from a configuration space \mathcal{M} into a bounded domain of the complex numbers \mathbb{C} .

These can be seen as predicates on a domain which, as the matter of fact is identified in Dirac calculus of quantum mechanics with \mathbb{R}^n , where \mathbb{R} is the real line seen as a measure space. Definable predicates of norm 1 will be referred to as states.

Of special significance are the **momentum and position states**. Momentum states were defined by Dirac as the definable family of predicates of the form

$$|p\rangle := \frac{1}{\sqrt{2\pi}} e^{-ipx}, \quad p \in \mathbb{R}. \quad (3)$$

One can consider a \mathbb{C} -linear space generated by the momentum states and define Hermitian inner product, first between the momentum states

$$\langle p_1 | p_2 \rangle := \delta(p_1 - p_2) \quad (4)$$

where δ is the Dirac delta. However, in the context of rigged Hilbert spaces one can identify the inner product above with the Kronecker delta.

The **position** states $|x\rangle$, $x \in \mathbb{R}$, by their physical meaning are characteristic functions of one-point subsets $\{x\}$

$$|x\rangle := \delta(x - z) \quad (5)$$

(as a function of z) which for convenience of continuous mathematical manipulations have been replaced here and in (4) by Dirac's delta-functions, that is by distributions. In this sense

$$|x\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixp} |p\rangle dp \quad (6)$$

Equivalently, position states can be represented by linear functionals (bra-vectors)

$$\langle x| : |\psi\rangle \mapsto \psi(x)$$

or equivalently, for ψ running in $\{|p\rangle : p \in \mathbb{R}\}$,

$$\langle x| : |p\rangle \mapsto \frac{1}{\sqrt{2\pi}} e^{-ipx} \quad (7)$$

In model theory terms, the linear functionals $\langle x|$ are imaginary elements in the structure, the interpretation of which is given by (7).

The basic unitary operators (Weyl operators) can be defined by their action on the basis:

$$\begin{aligned} e^{iP} : |p\rangle &\mapsto e^{ip} |p\rangle \\ e^{iQ} : |x\rangle &\mapsto e^{ix} |x\rangle. \end{aligned}$$

In particular, the former can be equivalently, using (6), written as

$$e^{iP} : |x\rangle \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ip(x+1)} |p\rangle dp$$

The time-evolution operator K_{Free}^t for a free particle is $e^{-it\frac{P^2}{2}}$, $t \in \mathbb{R}$, that is

$$K^t : |p\rangle \mapsto e^{-it\frac{p^2}{2}} |p\rangle \quad (8)$$

which yields by (6)

$$K^t : |x_0\rangle \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(px_0 - t\frac{p^2}{2})} |p\rangle dp$$

and

$$\langle x|K^t|x_0\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(px_0 - t\frac{p^2}{2})} \langle x|p\rangle dp$$

Substituting (3) one gets

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(px_0 - t\frac{p^2}{2})} \langle x|p\rangle dp &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(p(x_0 - x) - t\frac{p^2}{2})} dp = \\ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\frac{(p - \frac{(x-x_0)}{t})^2}{2} + i\frac{(x-x_0)^2}{2t}} dp &= \frac{1}{2\pi} e^{i\frac{(x-x_0)^2}{2t}} \cdot \left(\int_{\mathbb{R}} e^{-it\frac{p^2}{2}} dp \right) \\ &= \frac{1}{2\pi} e^{i\frac{(x-x_0)^2}{2t}} \cdot \sqrt{\frac{2\pi}{it}} = \frac{1}{\sqrt{2\pi it}} e^{i\frac{(x-x_0)^2}{2t}} \end{aligned}$$

and one obtains the well-known formula

$$\langle x|K^t|x_0\rangle = \frac{1}{\sqrt{2\pi it}} e^{i\frac{(x-x_0)^2}{2t}} \quad (9)$$

3.3 In a more abstract axiomatic setting the theory of Dirac integration in the context of Gaussian states can be expressed in formulae:

for $a \neq 0$,

$$\int_{\mathbb{R}} e^{\pi i(ax^2 + 2bx)} dx = \sqrt{\frac{1}{ia}} e^{-\pi i\frac{b^2}{a}} = e^{-\frac{\pi i}{4}} \sqrt{\frac{1}{a}} e^{-\pi i\frac{b^2}{a}} \quad (10)$$

and,

$$\int_{\mathbb{R}} e^{-2\pi ibx} dx = \delta(b) = b^{-1} \delta(0) \quad (11)$$

More advanced calculus, going beyond Gaussian states, requires methods of perturbation theory. This can be illustrated by the following typical calculation related to *anharmonic oscillator* in quantum mechanics and also setting a pattern for crucial calculations in QFT

$$\int_{\mathbb{R}} e^{i\frac{x^2 + \lambda x^4}{h}} dx = e^{\frac{\pi i}{4}} \sqrt{2\pi h} (1 + i\lambda h + o(\lambda h)) \quad (12)$$

for $h > 0$ small, see [17], section 2.

3.4 In terms of structures, let \mathcal{H}_m be the set of all m -ary predicates on \mathbb{R} . This by definition has structure of \mathbb{C} -vector spaces

$$\mathbb{C} = \mathcal{H}_0 \subset \dots \subset \mathcal{H}_m \subset \dots \subset \mathcal{H}_{m+1} \dots \mathcal{H}.$$

Also, one uses quantifiers, linear maps written as integrals

$$\phi(z_1, \dots, z_n) \mapsto \int_{\mathbb{R}} \phi(z_1, \dots, z_n) dz_n.$$

In fact, this is a collection of linear maps

$$\int : \mathcal{H}_{m+1} \rightarrow \mathcal{H}_m,$$

the rules of calculation of which defined by Dirac's improper integration.

For a discrete basis the sum in (1) can be represented in CL-setting as a definable countable sum, see e.g. [14], Example 5.2.

A special binary operation in the spaces, inner product,

$$\mathcal{H}_m \times \mathcal{H}_m \rightarrow \mathbb{C}; \quad \langle \phi(z_1, \dots, z_n), \psi(z_1, \dots, z_n) \rangle = \int_{\mathbb{R}^m} \phi^* \cdot \psi dz_1 \dots dz_m$$

where ϕ^* is the complex conjugate of ϕ and $\int_{\mathbb{R}^m}$ is m -multiple integral. $\langle \phi | \psi \rangle$ can be seen as a continuous predicate of equality $\phi = \psi$.

One restricts the notion of **state** to predicates ϕ such that $\langle \phi | \phi \rangle = 1$.

An important role in the theory is played by a collection of linear maps (operators)

$$L : \mathcal{H}_m \rightarrow \mathcal{H}_m$$

with physical meanings. These can be of the integral form

$$\phi(\bar{z}_1, \bar{z}_2) \mapsto \int_{\mathbb{R}^k} \alpha(\bar{y}, \bar{z}_1) \cdot \phi(\bar{y}, \bar{z}_2) d\bar{y}$$

where $|\bar{y}| = |\bar{z}_1| = k$, $\alpha \in \mathcal{H}_{2k}$, or as classical linear operators

$$P : \phi(x, \bar{z}) \mapsto i\hbar \frac{\partial \phi(x, \bar{z})}{\partial x} \quad \text{or} \quad Q : \phi(x, \bar{z}) \mapsto x \cdot \phi(x, \bar{z})$$

The **time evolution operator** $\exp(-iHt/\hbar)$ acts on states as a unitary operator determining the evolution of a state in time t with a given

Hamiltonian H . A state ϕ_{t_0} determining a system at time t_0 evolves into a state $\phi_t := \exp(-iH(t - t_0)/\hbar)$ with the *probability amplitude* equal to $\langle \phi_{t_0} | \phi_t \rangle$, which is a complex number of modulus 1. The calculation of the CL-formulae ϕ_t and $\langle \phi_{t_0} | \phi_t \rangle$ (which involve mainly calculations of the application of quantifier \int) is the central problem of quantum theory, equivalent to solving the associated Schrödinger equation.

All of the above together makes the \mathcal{H}_m a collection of Hilbert spaces with linear operators and \mathcal{H} an ambient Hilbert space.

It should be mentioned that the Hilbert state formalism of quantum mechanics can be fully reduced to the unitary setting, that is the setting with a Hilbert space equipped with unitary operators only. This is our preferred formalism and by the remark above, with enough functions α in the formalism, one can reduce all the operators to the integration operator.

Below we explain how the Hilbert space axiomatisation of QM can be represented as a formal theory in the language of Continuous Logic.

3.5 Remarks on Dirac integration and measure. Let $\int_R f(x)\delta_x$ stand for the Dirac integral and $\int_R f(x)dx$ for the proper Riemann integral.

$$\int_R f(x)\delta_x = \int_R f(x)dx \tag{13}$$

if the latter is well-defined.

In particular, for $f(x)$ continuous on \mathbb{R} ,

$$\int_R f(x)\delta_x = \lim_{m \rightarrow \infty} \int_{-m}^m f(x)dx \tag{14}$$

if the right-hand-side is well-defined.

If a finite limit in (14) does not exist the integral is understood in the sense of distributions, in particular one writes

$$\int_{\mathbb{R}} e^{-2\pi bx} \delta_x = \delta(b)$$

However, in the setting of rigged Hilbert spaces it is consistent to renormalise to the Kronecker delta-symbol:

$$\int_{\mathbb{R}} e^{-2\pi bx} \delta_x := \delta_{0,b}^{\text{Kr}}. \tag{15}$$

3.6 Remarks on rigged Hilbert spaces. .

Recall (see e.g. [9]) that a Gelfand triple is:

$$\Phi \subset \mathcal{H} \subset \Phi^*$$

where Φ is a space of test functions (e.g. the space of continuous functions on \mathbb{R} with compact support), \mathcal{H} is the Hilbert space, Φ^* is the space of continuous linear functionals on Φ , i.e., distributions.

In the context of continuous model theory with universe \mathbb{R} it is natural to take for Φ continuous functions which are zero outside the interval $[-m, m]$, which agrees with (14) and further remarks above.

An element $\phi \in \Phi^*$ acts on a test function f via the application of inner product in Φ^* : $f \mapsto \langle f | \phi \rangle$

The ket $|x\rangle \in \Phi^*$ is not a vector in \mathcal{H} , but a generalized eigenvector.

The pairing $\langle x | \phi \rangle$ can be interpreted as the evaluation $\phi(x)$ of ϕ at the point x , if such evaluation makes sense. If $\phi \in \mathcal{H}$, this is a well-defined function $\phi : \mathbb{R} \rightarrow \mathbb{C}$

4 Hilbert space formalism and H-structures

4.1 The axiomatic description of quantum mechanical theory in the form of rigged Hilbert space may be quite confusing from the logician point of view – there are no logical sentences which can be called axioms. What Axioms 1 – 5 render instead is the topological-algebraic structure of a Hilbert space with operators. This brings us to the *algebraisation of logic* approach introduced by A.Lindenbaum, A.Tarski, P.Halmos for the first order setting. It is quite natural to see the Hilbert space formalism as the form of algebraic logic in the context of the continuous logic of physics.

As explained in our less formal note [5] the Hilbert space \mathcal{H} of quantum mechanics, or rather the tower of Hilbert spaces $\mathcal{H}^{\otimes n}$, plays the role of the Tarski *cylindric algebra*, see [6].

4.2 \mathcal{H} -structure.

Let \mathcal{H} be a rigged Hilbert space, $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ and \mathcal{O} a collection of linear operators on the \mathcal{H}_n . We will often write \mathcal{H} for the union of the tower $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots$

Let $\mathcal{H}_n^{\text{Def}}$ be subspaces of the \mathcal{H}_n closed under operators from \mathcal{O} .

Given $(\mathcal{H}^{\text{Def}}, \mathcal{O})$ we will associate with it a structure $(\mathbb{U}; \mathcal{H}^{\text{Def}}, \mathcal{O})$ in continuous model theory (sometimes called an \mathcal{H} -structure). \mathbb{U} is the metric universe represented as a union of finite diameter domains, $(\mathcal{H}^{\text{Def}}, \mathcal{O})$ is the signature.

There is a quantifier E_x on $(\mathbb{U}; \mathcal{H}^{\text{Def}}, \mathcal{O})$, realised as the integral over a certain measure. Using the integral we define a Hermitian inner product on the linear spaces $\mathcal{H}_n^{\text{Def}}$. This gives us a structure of continuous model theory.

Then we can use the construction of imaginary elements and imaginary sorts to recover in \mathbb{U} elements of spaces dual to $\mathcal{H}_n^{\text{Def}}$.

We aim to represent in $(\mathbb{U}; \mathcal{H}^{\text{Def}}, \mathcal{O})$ enough of physically meaningful states, at least all-important position $|x\rangle$ and momentum $|p\rangle$ eigenvectors, and \mathcal{O} to contain at least the unitary operators like the Weyl operators e^{iQ} and e^{iP} and general integral kernel operators.

4.3 Constructing an \mathcal{H} structure \mathbb{U} . The **universe** of \mathbb{U} , which we also call \mathbb{U} , is a complete metric space with a distinguished point 0 , a measure μ and metric $\text{dist}(u_1, u_2)$.

There are two cases of how the universe can be presented. In case $\mathbb{U} = \mathbb{U}(\infty)$ is infinite, define $\mathbb{U}(\infty) := \mathbb{R}$, $\text{dist}(u_1, u_2) = |u_1 - u_2|$ and the measure is the Dirac's delta-measure as determined in 3.5.

In case $\mathbb{U} = \mathbb{U}(\mathfrak{n})$ is finite of size \mathfrak{n} (may assume \mathfrak{n} even and $\sqrt{\mathfrak{n}}$ is an integer) we identify

$$\mathbb{U}(\mathfrak{n}) := \left[-\frac{\mathfrak{n}}{2}, \frac{\mathfrak{n}}{2}\right) \cap \mathbb{Z}$$

with metric given by $\text{dist}(u_1, u_2) = \sqrt{\frac{2\pi}{\mathfrak{n}}} \cdot |u_1 - u_2|$, the measure of a point $dx = \sqrt{\frac{2\pi}{\mathfrak{n}}}$, where $|u_1 - u_2|$ is the natural integer valued distance in the cyclic group $\mathbb{Z}/\mathfrak{n}\mathbb{Z}$.

The universe \mathbb{U} contains family of closed subdomains $[k, m]$ of finite diameters, $k, m \in \mathbb{R}$, $k < m$

$$[k, m] = \{u \in \mathbb{U} : \text{dist}(k, u) + \text{dist}(u, m) = m - k\}.$$

and we assume that the end-points of the intervals are named.

In $\mathbb{U}(\infty) = \mathbb{R}$ we consider Lipschitz-continuous n -ary predicates

$$\psi : [k_1, m_1] \times \dots \times [k_n, m_n] \rightarrow \mathbb{C}.$$

The names of all such predicates is \mathcal{H}^{Def} . There is a natural uniform continuity modulus for each name ψ .

Equivalently, we can associate with each such name the map $\psi' : \mathbb{U}^n \rightarrow \mathbb{C}$ which coincides with ψ on $[k_1, m_1] \times \dots \times [k_n, m_n]$ and is 0 outside. Call such ψ' **test functions** with name ψ .

For each n , let Φ_n be the \mathbb{C} -linear space of functions $\phi : \mathbb{U}^n \rightarrow \mathbb{C}$ which are representable as finite linear combinations of test functions.

In case of the discrete $\mathbb{U}(\mathbf{n})$ there is a natural embedding

$$[k, m]^{\mathbb{U}(\mathbf{n})} \subset [k, m]^{\mathbb{U}(\infty)}; u \mapsto \sqrt{\frac{2\pi}{\mathbf{n}}} u \quad (16)$$

and thus we define the predicate with the name ψ on $\mathbb{U}(\mathbf{n})$ as the restriction of ψ to $[k, m]^{\mathbb{U}(\mathbf{n})}$. This is obviously extendable to n -ary predicates.

We may use notation $\Phi_n(\infty)$, $\Phi_n(\mathbf{n})$, for spaces of predicates on $\mathbb{U}(\infty)$ and $\mathbb{U}(\mathbf{n})$ respectively.

It is well known, for $\mathbb{U}(\infty) = \mathbb{R}$, that Φ_n is a dense inner product subspace of $L^2(\mathbb{R}^n)$ which in many cases is taken to be \mathcal{H} of the axioms. Moreover,

$$\Phi_n \subset \mathcal{H} \subset \Phi_n^*$$

where Φ_n^* is the dual to Φ_n , that is the inner product space of anti-linear functionals on Φ_n .

We introduce **quantifier** $E : \Phi_{n+1} \rightarrow \Phi_n$ uniformly on all domains $[-m, m]^{n+1}$

$$E_x : \psi(x, y_1, \dots, y_n) \mapsto \int_{\mathbb{R}} \psi(x, y_1, \dots, y_n) dx = \int_{-m}^m \psi(x, y_1, \dots, y_n) dx$$

In case of discrete $\mathbb{U}(\mathbf{n})$ the quantifier is expressible by the summation formula

$$E_x : \psi(x, y_1, \dots, y_n) \mapsto \sum_{-m \leq k \sqrt{\frac{2\pi}{\mathbf{n}}} < m} \psi(k \sqrt{\frac{2\pi}{\mathbf{n}}}, y_1, \dots, y_n) \cdot \sqrt{\frac{2\pi}{\mathbf{n}}}$$

where k runs in $\mathbb{Z} \cap [-\frac{\mathbf{n}}{2}, \frac{\mathbf{n}}{2})$.

The quantifier allows the definition of the inner product on Φ (with the natural extension to Φ_n)

$$\langle \psi | \phi \rangle := E_x \psi(x) \cdot \phi(x)^* \quad (17)$$

where $\phi(x)^*$ stands for complex conjugation of $\phi(x)$.

This implies that ψ can be identified with the anti-linear functional

$$\phi \mapsto \langle \psi | \phi \rangle,$$

an element of Φ^* .

In a slight **extension** of the standard continuous model theory setting we consider the continuous n -ary maps

$$\psi : \mathbb{U}^n \rightarrow \mathbb{C}$$

such that

$$\psi = \bigcup_{m \in \mathbb{N}} \psi|_{[-m, m]^n} \tag{18}$$

the union of basic continuous predicates.

We call these **generalised predicates** and consider them definable. This passes the main test of definability. It follows from the representation (18):

Given a generalised predicate ψ on \mathbb{U} and an ultrafilter \mathcal{D} , the ultraproduct $\psi^{\mathcal{D}}$ of ψ in the ultrapower $\mathbb{U}^{\mathcal{D}}$, is a generalised predicate.

On the other hand, for a generalised ψ the inner product $\langle \phi | \psi \rangle$ with $\phi \in \Phi$ is well-defined since the effective domain of integration is bounded. So ψ can be identified with the linear functional on Φ

$$\phi \mapsto \langle \phi | \psi \rangle,$$

that is an imaginary element $\langle \psi | \in \Phi^*$.

We define **momentum eigenfunctions** $\mathbf{v}[p]$, $p \in \mathbb{R}$ as generalised continuous predicates on $\mathbb{U}(\infty) = \mathbb{R}$

$$\mathbf{v}[p] : x \mapsto \frac{1}{\sqrt{2\pi}} e^{\frac{ixp}{\hbar}}.$$

Another way of representing $\mathbf{v}[p]$ is by considering the predicate \mathbf{vu} on $\mathbb{U}^2(\infty)$:

$$\mathbf{vu} : \mathbb{R}^2 \rightarrow \mathbb{C}; \quad (x, p) \mapsto \langle x | p \rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{ixp}{\hbar}}$$

This defines the family

$$\mathbb{V} = \{\mathbf{v}[p] : p \in \mathbb{U}\}$$

as an imaginary sort (in the sense slightly broader than that in [14]) with a metric given by

$$\text{dist}(\mathbf{v}[p_1], \mathbf{v}[p_2]) = \text{dist}(p_1, p_2).$$

It is clear that $\mathbb{V} \cong \mathbb{U}$ as metric spaces. \mathbb{V} is an imaginary sort in $(\mathbb{U}; \mathcal{H}^{\text{Def}})$.

Define, for $x \in \mathbb{U}$, the maps

$$\mathbf{u}[x] : \mathbb{V} \rightarrow \mathbb{C}; \mathbf{v}[p] \mapsto \mathbf{v}[p](x) = \frac{1}{\sqrt{2\pi}} e^{\frac{ixp}{\hbar}}$$

These are generalised continuous predicates on sort \mathbb{V} . Moreover, the above can be extended to a unique linear functional on Φ and so $\mathbf{u}[x] \in \Phi^*$.

$$\mathbf{u}[x] : \Phi \rightarrow \Phi; \psi \mapsto \psi(x) = \langle \psi | x \rangle \quad (19)$$

This determines $\mathbf{u}[x]$ (or $|x\rangle$ in bra-ket notations) as imaginary elements and

$$\mathbb{U}' := \{\mathbf{u}[x] : x \in \mathbb{U}\}$$

a sort of imaginaries in a natural isomorphism with \mathbb{U} . The $\mathbf{u}[x]$ are not in \mathcal{H} but in Dirac calculus one defines their norm to be the symbolic Dirac $\delta(0)$. In the linear space Φ^* one can equivalently introduce the inner product

$$\langle \mathbf{u}[x] | \mathbf{u}[y] \rangle := \delta_{x,y} \text{ (Kronecker } \delta \text{)}.$$

In this sense \mathbb{U}' is an orthogonal basis for Φ in the sense that, for each $\psi \in \Phi$,

$$\psi = E_x \psi(x) \mathbf{u}[x] \quad (20)$$

since, by linearity of E ,

$$\langle E_x \psi(x) \mathbf{u}[x] | \mathbf{u}[y] \rangle = \psi(y).$$

Operators. There are two definable ways of introducing operators on Φ .

1. Integral kernel operators. Let $\kappa : \mathbb{U}^2 \rightarrow \mathbb{C}$ be a generalised predicate in variable z, x . Then for all $\psi \in \Phi_n$ in variables x, \bar{y}

$$L_\kappa : \psi \mapsto E_x \kappa(z, x) \cdot \psi(x, \bar{y})$$

is well-defined and determines a linear operator $\Phi_n \rightarrow \Phi_n$.

2. Operators defined by their actions on bases.

We assume that \mathcal{O} contains a pair of (Weyl) operators U and V on Φ defined on the $\mathbf{u}[r]$, $r \in \mathbb{U}$:

$$U : \mathbf{u}[r] \mapsto e^{ir\nu} \cdot \mathbf{u}[r] \text{ and } V : \mathbf{u}[r] \mapsto \mathbf{u}[r + \mathbf{h}], \text{ if } \mathbb{U} = \mathbb{U}(\infty) \quad (21)$$

where

$$\nu = 1 \text{ if } \mathbb{U} = \mathbb{U}(\infty), \text{ and } \nu = \frac{2\pi}{\mathbf{n}} \text{ if } \mathbb{U} = \mathbb{U}(\mathbf{n}).$$

$\mathbf{h} = \hbar$ a positive real, if $\mathbb{U} = \mathbb{U}(\infty)$, and $\mathbf{h} = h_{\mathbf{n}}$, a positive integer, if $\mathbb{U} = \mathbb{U}(\mathbf{n})$.

(Note the correspondence (16) between units of $\mathbb{U}(\infty)$ and $\mathbb{U}(\mathbf{n})$. We will later assume that $\sqrt{\mathbf{n}}$ is an integer and $\lim_{\mathbf{n} \rightarrow \infty} \frac{h_{\mathbf{n}}}{\sqrt{\mathbf{n}}} = \hbar$).

The following commutation relation then holds:

$$UV = e^{i\nu\mathbf{h}}VU.$$

Using (20) we then get

$$\begin{aligned} U\psi &= E_x \psi(x) \cdot e^{i\nu x} \mathbf{u}[x] \\ V\psi &= E_x \psi(x - \mathbf{h}) \mathbf{u}[x] \end{aligned}$$

Thus U transforms function $\psi(x)$ on $[k, m]$ to $\psi(x) \cdot e^{i\nu x}$ and V shifts $\psi(x)$ on $[k, m]$ to $\psi(x - \mathbf{h})$ defined as a continuous test function on $[k + \mathbf{h}, m]$. (Note that test function $\psi(x)$ vanishes outside $[k, m]$).

On the dual set of vectors $\mathbf{v}[p]$, $p \in \mathbb{U}$:

$$V : \mathbf{v}[p] \mapsto e^{-i\nu p \mathbf{h}} \mathbf{v}[p] \text{ and } U : \mathbf{v}[p] \mapsto \mathbf{v}[p - 1].$$

4.4 Remarks.

1. There is a Fourier duality operator between the bases $\mathbf{u}[r]$ and $\mathbf{v}[p]$ of Φ^* .

In the finite case:

$$\mathbf{v}[p] = \frac{1}{\sqrt{\mathbf{n}}} \sum_{r \in \mathbb{U}(\mathbf{n})} e^{\frac{2\pi i h_{\mathbf{n}} r p}{\mathbf{n}}} \mathbf{u}[r]; \quad \mathbf{u}[r] = \frac{1}{\sqrt{\mathbf{n}}} \sum_{p \in \mathbb{U}(\mathbf{n})} e^{-\frac{2\pi i h_{\mathbf{n}} p r}{\mathbf{n}}} \mathbf{v}[p] \quad (22)$$

and in the continuous case, in agreement with (5),

$$\mathbf{v}[p] = \frac{1}{\sqrt{2\pi}} e^{irp}; \quad \mathbf{u}[r] = \delta(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{v}[p] e^{-ipr} dp$$

2. Note that unitary operators of the form e^{iL} , where L is self-adjoint can be represented in the integral form with respective kernels κ when L is a polynomial of the basic operators P and Q , see [19].

3. The time evolution operator $e^{-iHt/\hbar}$ for the quantum harmonic oscillator of frequency ω can be represented by an integral operator with kernel

$$\sqrt{\frac{\omega}{2\pi i\hbar \sin \omega t}} \exp i\omega \frac{(x_0^2 + x^2) \cos \omega t - 2x_0x}{2\hbar \sin \omega t}$$

4.5 Proposition. *Given \mathcal{H} -structure $(\mathbb{U}(\infty); \mathcal{H}^{\text{Def}}, \mathbb{O})$ constructed in 4.3, there is a unique rigged Hilbert space \mathcal{H} which contains a dense inner product subspace of continuous predicates on bounded domains isomorphic to \mathcal{H}^{Def} with operators \mathbb{O} .*

Given the \mathcal{H} -structure $(\mathbb{U}(\mathbf{n}); \mathcal{H}^{\text{Def}}, \mathbb{O})$ constructed in 4.3, the space of predicates on bounded domains with respect to the inner product is an \mathbf{n} -dimensional Hilbert space with operators \mathbb{O} .

Proof. This is the content of 4.3.

4.6 Other metric universes. Suppose $\mathbb{M} \subseteq \mathbb{R}^m = \mathbb{U}(\infty)^m$ is a real manifold definable in $(\mathbb{U}(\infty); \mathcal{H}^{\text{Def}}, \mathbb{O})$ along with a metric $\text{dist}_{\mathbb{M}}$ and a measure $d^{\mathbb{M}}x$ on it, compatible with the metric and measure on $\mathbb{U}(\infty)$. Then bounded domains in $\mathbb{U}(\infty)^m$ induce bounded domains on \mathbb{M}^n and we consider all the definable (in $(\mathbb{U}(\infty); \mathcal{H}^{\text{Def}}, \mathbb{O})$) continuous maps on bounded domains of \mathbb{M}^n to be definable continuous predicates on domains in \mathbb{M} .

Define $\Phi_n(\mathbb{M})$ to be the space generated by all $\psi' : \mathbb{M}^n \rightarrow \mathbb{C}$ where $\psi'(x) = \psi(x)$ for a definable continuous predicate ψ on a bounded domain $D_\psi \subseteq \mathbb{M}^n$, when $x \in D_\psi$, and $\psi'(x) = 0$, when $x \notin D_\psi$. The measure $d^{\mathbb{M}}x$ on $\mathbb{U}(\infty)$ induces the respective quantifier $E^{\mathbb{M}} : \Phi_{n+1}(\mathbb{M}) \rightarrow \Phi_n(\mathbb{M})$. As a result we have a well-defined inner product on $\Phi_n(\mathbb{M})$ and the closure of $\Phi_n(\mathbb{M})$ in the inner product topology gives us a Hilbert space $\mathcal{H}_n(\mathbb{M})$ which is contained in the dual space $\Phi_n(\mathbb{M})^*$. This provides us with the rigged Hilbert space of quantum mechanics on the manifold \mathbb{M} .

4.7 CL-ultraproduct. We will work with the asymptotic class of finite-dimensional \mathcal{H} -structures $\mathbb{U}(n)$ of signature $(\mathcal{H}^{\text{Def}}, \mathbb{O})$.

Note that the predicates of \mathcal{H}^{Def} on $\mathbb{U}(\mathbf{n})$ can be written in the form

$$\psi(k, \dots, k_m) = f\left(k_1 \sqrt{\frac{2\pi}{\mathbf{n}}}, \dots, k_m \sqrt{\frac{2\pi}{\mathbf{n}}}\right),$$

where $f(x_1, \dots, x_m)$ is a differentiable function into \mathbb{C} on finite diameter domains in \mathbb{R}^m . (The predicate with the same name ψ on the continuous \mathbb{U} is just $f(x_1, \dots, x_m)$.)

The quantifier \mathbb{E} acting on generalised predicates ψ can be represented as a family of quantifiers

$$\mathbb{E} := \{\mathbb{E}^{(m_1, m_2)} : m_1 < m_2 \in \mathbb{Z}\},$$

on finite intervals of diameter $m_2 - m_1 < \sqrt{\frac{\mathbf{n}}{2\pi}}$ is defined as follows:

$$\mathbb{E}_k^{(m_1, m_2)} \psi(k, \bar{p}) = \text{st} \left(\sqrt{\frac{2\pi}{\mathbf{n}}} k \sum_{\substack{k\sqrt{\frac{2\pi}{\mathbf{n}}} \leq m_2 \\ k\sqrt{\frac{2\pi}{\mathbf{n}}} \geq m_1}} \psi(k, \bar{p}) \right), \text{ for } \mathbb{U}(\mathbf{n}) \quad (23)$$

which corresponds to

$$\mathbb{E}_x^{(m_1, m_2)} \psi(x, \bar{y}) = \int_{m_1}^{m_2} \psi(x, \bar{y}) dx, \text{ for } \mathbb{U}(\infty). \quad (24)$$

We also consider:

$$\mathbb{E}_k^{\text{loc}} \psi(k, \bar{p}) := \lim_{m \rightarrow \infty} \mathbb{E}_k^{(-m, m)} \psi(k, \bar{p}) \quad (25)$$

when the limit exists.

Here, in case \mathbf{n} is finite, the limit should be understood as the value for the maximal m satisfying $2m \leq \sqrt{\frac{\mathbf{n}}{2\pi}}$.

For a pseudo-finite \mathbf{n} it is assumed that m runs in \mathbb{N} , the standard positive integers. In the context of continuous model theory $\mathbb{E}_k^{\text{loc}}$ is definable in terms of $\mathbb{E}^{(-m, m)}$ which we will write as just $\mathbb{E}^{(m)}$.

While $\mathbb{E}_k^{(m)} \psi$ is well-defined for all ψ , $\mathbb{E}_k^{\text{loc}} \psi$ might be not, for some ψ for infinite \mathbf{n} .

However, we use notation \mathbb{E}^{loc} for the family $\{\mathbb{E}^{(m)} : m \in \mathbb{N}\}$ when it does not lead to confusion.

Recall that the definition of an \mathcal{H} -structure furnishes a definable constant \mathbf{h} which in the finite-dimensional case can also be expressed in the form $\mathbf{h}(n) = \frac{h_n}{n}$, see (21).

4.8 Theorem. *Let \mathcal{D} be a non-principal ultrafilter on \mathbb{N} such that*

$$\lim_{n/\mathcal{D}} \frac{h_n}{n} = \mathbf{h}(\infty).$$

Then the continuous model theory ultraproduct $(\mathbb{U}^, \mathcal{H}^{\text{Def}}, \mathbb{O})$ of finite \mathcal{H} -structures $(\mathbb{U}(n), \mathcal{H}^{\text{Def}}, \mathbb{O})$ is an \mathcal{H} -structure isomorphic to $(\mathbb{U}(\infty), \mathcal{H}^{\text{Def}}, \mathbb{O})$.*

Equivalently, for every sentence σ and every positive ϵ there is a subset $D_{\sigma, \epsilon} \in \mathcal{D}$ such that for all $n \in D_{\sigma, \epsilon}$ the value of σ on $(\mathbb{U}, \mathcal{H}^{\text{Def}}, \mathbb{O})$ differs from the value of σ on $(\mathbb{U}(n), \mathcal{H}^{\text{Def}}, \mathbb{O})$ by no more than ϵ .

Proof. The metric universe \mathbb{U}^* of the ultraproduct is defined as the union of sorts of finite diameter $2m$, which are limits along the ultrafilter \mathcal{D} of sorts of the same diameter of $\mathbb{U}(n)$. This means that for a limit non-standard number \mathbf{n} and numbers $k \in \mathbb{U}(\mathbf{n})$ we set the limit point $x = k/\mathcal{D}$ so that

$$\text{dist}(0, x) = \text{dist}(0, k)/\mathcal{D}$$

This brings us to

$$x := \text{st}\left(k\sqrt{\frac{2\pi}{\mathbf{n}}}\right),$$

(the standard part map). In particular, the interval $[-m\sqrt{\frac{\mathbf{n}}{2\pi}}, m\sqrt{\frac{\mathbf{n}}{2\pi}}]$ in $\mathbb{U}(\mathbf{n})$ corresponds to the interval $[-m, m]$ in \mathbb{R} .

This also agrees with the definition of predicates ψ on the ultraproduct and

Next we prove the correspondence for quantifiers. It is enough to consider unary $\psi : \mathbb{R} \rightarrow \mathbb{C}$. By definition $\psi(k) = f(k\sqrt{\frac{2\pi}{\mathbf{n}}})$, $f(x)$ differentiable on \mathbb{R} .

Claim. Given any positive $\epsilon \in \mathbb{R}$,

$$\left| \sqrt{\frac{2\pi}{\mathbf{n}}} \sum_{-m \leq k\sqrt{\frac{2\pi}{\mathbf{n}}} < m} f\left(k\sqrt{\frac{2\pi}{\mathbf{n}}}\right) - \int_{-m}^m f(x)dx \right| < \epsilon$$

Indeed, the discrete formula is a Riemann sum with spacing $\Delta x = \sqrt{\frac{2\pi}{\mathbf{n}}}$. By the left Riemann sums estimate for an interval (a, b)

$$\text{Err} \leq M_f \frac{(a-b)^2}{2N} = M_f \frac{(2m)^2}{4m\sqrt{\mathbf{n}}}$$

where N is the number of points between $a = m$ and $b = -m$ and $M_f = \max\{f'(x) : b \leq x < a\}$. Clearly, $M_f \frac{(2m)^2}{4m} \in \mathbb{R}$ and thus $M_f \frac{(2m)^2}{4m\sqrt{n}} < \epsilon$ because $\sqrt{\frac{2\pi}{n}}$ is a non-standard infinitesimal.

Thus the application of the quantifier in the asymptotic class agrees with the quantifier in the ultraproduct. It follows that the inner product operation in the asymptotic class agrees with the inner product operation in the ultraproduct, once it is determined by integration. This is enough to obtain the correspondence for the construction of interpretable linear functionals and the rigged Hilbert space in the asymptotic class and in the ultraproduct. It follows that the inner product operation $\langle \psi_1 | \psi_2 \rangle$ is preserved by the ultraproduct for all $\psi_1, \psi_2 \in \mathcal{H}^{\text{Def}}$.

Finally, the operators in \mathcal{O} are preserved by the ultraproduct because they are expressible in terms of E^{loc} . This includes Weyl operators.

□

5 Gaussian and perturbation-Gaussian states

5.1 Gaussian predicates.

Call an m -predicate $\psi(k_1, \dots, k_m)$ on $\mathbb{U}(\mathfrak{n})$ basic Gaussian if there is a $\eta \in \mathbb{C}$ and a positive-definite quadratic form $Q(x_1, \dots, x_m)$ over \mathbb{Q} such that

$$\psi(k_1, \dots, k_m) = \eta \cdot e^{-\pi i \frac{Q(k_1, \dots, k_m)}{n}} = \eta \cdot e^{-i \frac{Q(k_1 \sqrt{\frac{2\pi}{n}}, \dots, k_m \sqrt{\frac{2\pi}{n}})}{2}}$$

For the continuous \mathbb{U} :

$$\psi(x_1, \dots, x_m) = \eta \cdot e^{-i \frac{Q(x_1, \dots, x_m)}{2}},$$

where Q is a over \mathbb{R} .

Since by definition e^{-irp} is a Gaussian predicate, we consider the Fourier-dual one point characteristic function $\mathbf{u}[r]$ to be a Gaussian state.

Note that $Q(x_1, \dots, x_m)$ can be written in the form that singles out a particular variable, say x_1 ,

$$Q(x_1, \dots, x_m) = ax^2 + 2xb(\bar{y}) + c(\bar{y})$$

where $x = x_1$, \bar{y} is the rest of the variables, $b(\bar{y})$ a linear form and $c(\bar{y})$ a quadratic form.

Now a Gaussian predicate can be written as

$$\psi(k, \bar{p}) = \eta \cdot e^{-\pi i \frac{c(\bar{p})}{n}} \cdot e^{-\pi i \frac{ak^2 + 2kb(\bar{p})}{n}}$$

in the discrete setting, and

$$\psi(x, \bar{y}) = \eta \cdot e^{-i \frac{c(\bar{y})}{2}} \cdot e^{-i \frac{ax^2 + 2xb(\bar{y})}{2}}$$

in continuous setting.

For the discrete version, if $a \neq 0$, we call the rational number a the **period of ψ** with respect to variable k .

If $a = 0$ and $b(\bar{p}) = b \cdot (L_1 p_1 + \dots + L_m p_m)$ with L_1, \dots, L_m coprime tuple of integers, then the **period of ψ** with respect to variable k is equal to b .

Definitions. Call non-standard integer \mathbf{n} **highly divisible** if it is divisible by all standard integers.

Let \mathbf{n} be highly divisible and $|\mathbb{U}| = \mathbf{n}$. We say that a subset $X \subset \mathbb{U}^m$ is **\mathbf{d} -dense** if X contains a submodule of \mathbb{U}^m of finite index.

5.2 Lemma (Gauss summation). *On $\mathbb{U}(\mathbf{n})$, for \mathbf{n} highly divisible:*

$$\frac{1}{\sqrt{\mathbf{n}}} \sum_{-\frac{\mathbf{n}}{2a} \leq k < \frac{\mathbf{n}}{2a}} e^{-\pi i \frac{ak^2 + 2kb(\bar{p})}{n}} = \sqrt{\frac{1}{a}} \cdot e^{\frac{\pi i}{4}} \cdot e^{-\pi i \frac{b(\bar{p})^2}{an}} \quad (26)$$

for all \bar{p} in a \mathbf{d} -dense subset of \mathbb{U}^{m-1} , and equals 0 outside the \mathbf{d} -dense subset.

In case $a = 0$ and $b(\bar{p}) = b \cdot p$, for $b \in \mathbb{Q}$,

$$\frac{1}{\sqrt{\mathbf{n}}} \sum_{-\frac{\mathbf{n}}{2b} \leq k < \frac{\mathbf{n}}{2b}} e^{\pi i \frac{2bkp}{n}} = b^{-1} \delta^{(\mathbf{n})}(p) \quad (27)$$

where

$$\delta^{(\mathbf{n})}(p) = \begin{cases} 0 & \text{if } p \neq 0 \\ \sqrt{\mathbf{n}} & \text{otherwise} \end{cases}$$

Proof. Let $a = \frac{A}{D} > 0$ where $A, D \in \mathbb{Z}$. We choose D so that $D \cdot b(\bar{p})$ is over \mathbb{Z} . Note that $\frac{\mathbf{n}}{2a}$ is an integer because \mathbf{n} is divisible by A by assumptions.

Now let

$$X_a = \{\bar{p} \in \mathbb{U}^{m-1} : A | Db(\bar{p})\}.$$

This is a dense subset of \mathbb{U}^{m-1} . For a $\bar{p} \in X_a$ the function

$$e^{-\pi i \frac{ak^2 + 2kb(\bar{p})}{n}}$$

of variable k has period $\frac{n}{a}$ and $\frac{b(\bar{p})}{a}$ is an integer. Thus the summands in

$$ak^2 + 2kb(\bar{p}) = a\left(k + \frac{b(\bar{p})}{a}\right)^2 - \frac{b(\bar{p})^2}{a} = an^2 - \frac{b(\bar{p})^2}{a}$$

are integer and we can write

$$\sum_{-\frac{n}{2a} \leq k < \frac{n}{2a}} e^{-\pi i \frac{ak^2 + 2kb(\bar{p})}{n}} = e^{i\pi \frac{b(\bar{p})^2}{n}} \sum_{0 \leq n < \frac{n}{a}} e^{-\pi i \frac{an^2}{n}} = e^{i\pi \frac{b(\bar{p})^2}{an}} \cdot \sqrt{\frac{n}{a}} e^{\frac{\pi i}{4}}$$

where at the last step we used the classical Gauss' quadratic sums equality.

In case $\bar{p} \notin X_a$ the Gauss sum is equal 0.

Now consider the case $a = 0$, $b = \frac{B}{D}$, for $B, D \in \mathbb{N}$ coprime. If $p = 0 \pmod{B}$ then all the summands in (27) are equal to 1 and we get $\frac{\sqrt{n}}{b}$ for the value of the formula. Alternatively, if $p \neq 0 \pmod{B}$ then we get all the roots of 1 of order $\frac{n}{b}$ as summands, and the sum is equal 0.

□

5.3 Lemma. *For any d -dense subset $X \subset \mathbb{U}(\mathfrak{n})^m$ for any $\bar{y} \in \mathbb{R}^m$ there is $\bar{p} \in X$ such that $\text{st}(\sqrt{\frac{2\pi}{n}}\bar{p}) = \bar{y}$.*

Proof. When p runs in $\mathbb{U}(\mathfrak{n}) = [-\frac{n}{2}, \frac{n}{2}]$ the numbers $\text{st}(\sqrt{\frac{2\pi}{n}}p)$ run continuously between $-\infty$ and $+\infty$. Thus there is $\bar{p} \in \mathbb{U}(\mathfrak{n})^m$ such that $\text{st}(\sqrt{\frac{2\pi}{n}}\bar{p})$.

Density of X implies that there is a tuple of non-negative standard integers \bar{d} such that $\bar{p} + \bar{d} \in X$. But $\text{st}(\sqrt{\frac{2\pi}{n}}\bar{d}) = \bar{0}$ and thus we can assume $\bar{p} \in X$. □

5.4 Corollary

Let $(\mathbb{U}^, \mathcal{H}^{\text{Def}}, \mathcal{O})$ be the ultraproduct constructed in 4.8, \mathfrak{n} highly divisible, and $e^{-\pi i ax^2 + 2xb(\bar{y})}$ a Gaussian predicate on \mathbb{U}^* , where a, b are finite rational (possibly non-standard). Then,*

$$\sqrt{\frac{2\pi}{\mathfrak{n}}} \sum_{-\frac{n}{2a} < k \leq \frac{n}{2a}} e^{-\pi i \frac{ak^2 + 2kb}{n}} = \int_{\mathbb{R}} e^{-i \frac{ax^2 + 2xb}{2}} dx \quad (28)$$

if $a > 0$, and

$$\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2b} < k \leq \frac{\mathfrak{n}}{2b}} e^{-\frac{2\pi i k b}{\mathfrak{n}}} - \int_{\mathbb{R}} e^{-2\pi i x b} dx \quad (29)$$

where we dropped $\delta^{(\mathfrak{n})}$ on the left and Dirac delta on the right of equality (see (15)).

Remark. (29) makes sense when $b = 0$, in which case one has a constant function of value 1 on the right and the constant sequence on the left. These are Gaussian predicates as well (the case $a = 0 = b$). The formula can also serve for calculating norms.

5.5 Global versions of quantifiers.

The **global** quantifier is defined for finite and pseudo-finite \mathbb{U} :

$$E_k^{\text{glob}} \psi(k) := \text{st} \left(\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2P} < k \leq \frac{\mathfrak{n}}{2P}} \psi(k) \right)$$

where P is the period of ψ .

5.6 Theorem. *Given a pseudo-finite H -structure with $\mathbb{U} = \mathbb{U}(\mathfrak{n})$ with \mathfrak{n} highly divisible, and a Gaussian predicate ψ with variables k, \bar{p} :*

$$E_k^{\text{glob}} \psi(k, \bar{p}) = E_k^{\text{loc}} \psi(k, \bar{p})$$

for each \bar{p} in a d -dense subset.

Proof. This is a direct consequence of 5.4 and 4.8. \square

5.7 Beyond Gaussian. Anharmonic oscillator. The Gaussian fragment of quantum mechanics modelled above (with a little more work includes also quantum harmonic oscillator) is the only part of QM which allows exact solutions. The more general version of QM would include states of the form $e^{-i\frac{x^2+f(x)}{2\hbar}}$ where $f(x)$ is a polynomial of degree > 2 . In fact, the theory, due to physical and mathematical issues only deal with quite specific forms of such states. The key example is that of an anharmonic oscillator $e^{-i\frac{x^2+\lambda x^4}{2\hbar}}$ as analysed in [18]. This is also a much simplified analogue of so called ϕ^4 -quantum field theory.

The important difference with the Gaussian case is that Dirac calculus over such states, namely the key calculation

$$\int_{\mathbb{R}} e^{-i\frac{x^2+\lambda x^4}{2\mathbf{h}}} dx, \quad \lambda > 0$$

can only be carried out using *perturbation* methods, which impose specific restriction on coefficients, in particular \mathbf{h} have to be infinitesimally small in the example.

This leads us to restrict our analysis to discrete states of the form

$$\psi(k) := e^{-\pi i \frac{H(k^2 + \frac{1}{L}k^4)}{2\mathbf{n}}}$$

perturbed Gaussian states where H, L positive integers and

$$\mathbf{h} = \frac{1}{2\pi H}.$$

As in the Gaussian case we assume that H divides \mathbf{n} . Note that H also plays here a role of **asymptotic period** for ψ . Perturbed Gaussian states in general are not of period H .

Set $x = \sqrt{\frac{1}{\mathbf{n}}}k$. Then¹

$$e^{-\pi i \frac{H(k^2 + \frac{1}{L}k^4)}{\mathbf{n}}} = e^{-i\frac{x^2+\lambda x^4}{2\mathbf{h}}}, \quad \lambda = \frac{\mathbf{n}}{L}$$

5.8 As for the above Gaussian states the application of the global quantifier to a perturbed state ψ is defined as

$$E_k^{\text{glob}} \psi(k) := \sqrt{\frac{1}{\mathbf{n}}} \sum_{-\frac{\mathbf{n}}{2H} \leq k < \frac{\mathbf{n}}{2H}} \psi(k)$$

That is for ψ as above

$$E_k^{\text{glob}} \psi(k) := \sqrt{\frac{1}{\mathbf{n}}} \sum_{-\frac{\mathbf{n}}{2H} \leq k < \frac{\mathbf{n}}{2H}} e^{-\pi i \frac{Hk^2 + \frac{H}{L}k^4}{\mathbf{n}}} = \sqrt{\frac{2}{\mathbf{n}}} \sum_{0 \leq k < \frac{\mathbf{n}}{2H}} e^{-\pi i \frac{Hk^2 + \frac{H}{L}k^4}{\mathbf{n}}}$$

¹Physicists also consider perturbative states with term λx^d for $d \geq 3$. In this case $\lambda := \frac{\mathbf{n}^{d/2-1}}{L}$.

assumin $\frac{\mathfrak{n}}{2H} > 1$ is an integer.

We will assume that

$$\lambda = O(1).$$

Then under the restriction $0 \leq k < \frac{\mathfrak{n}}{2H}$ we have

$$\frac{Hk^4}{L\mathfrak{n}} < \frac{\mathfrak{n}^3}{LH^3} = O\left(\frac{\mathfrak{n}^2}{H^3}\right) \quad (30)$$

5.9 Let $\phi(k) = 1 - e\left(\frac{Hk^4}{2\mathfrak{n}}\right)$. Consider the partition of the sum

$$\begin{aligned} E_k^{\text{glob}} \psi &= 2\mathfrak{h}^{\frac{1}{2}} \sqrt{\frac{H}{\mathfrak{n}}} \sum_{0 \leq k < \frac{\mathfrak{n}}{2H}} e^{-\pi i \frac{Hk^2 + \frac{H}{L}k^4}{\mathfrak{n}}} = \\ &= \mathfrak{h}^{\frac{1}{2}} \left(2\sqrt{\frac{H}{\mathfrak{n}}} \sum_{0 \leq k < \frac{\mathfrak{n}}{2H}} e^{-\pi i \frac{Hk^2}{\mathfrak{n}}} + 2\sqrt{\frac{H}{\mathfrak{n}}} \sum_{0 \leq k < \frac{\mathfrak{n}}{2H}} \phi(k) e^{-\pi i \frac{Hk^2}{2\mathfrak{n}}} \right) = \\ &= \mathfrak{h}^{\frac{1}{2}} (T_0(\mathfrak{h}) + T_\phi(\mathfrak{h})). \end{aligned}$$

We know that

$$T_0(\mathfrak{h}) = e^{-\frac{\pi i}{4}}.$$

So our aim is to evaluate $T_\phi(\mathfrak{h})$.

5.10 Note that

$$\phi(k) = 1 - e^{\pi i \frac{Hk^4}{L\mathfrak{n}}} = \pi i \frac{Hk^4}{L\mathfrak{n}} + O\left(\left(\frac{Hk^4}{L\mathfrak{n}}\right)^2\right) = \pi i \frac{\lambda Hk^4}{\mathfrak{n}^2} + \epsilon$$

Note that $k^4 \leq \left(\frac{\mathfrak{n}}{H}\right)^4$ and thus $\frac{Hk^4}{\mathfrak{n}^2} \leq 2\pi\mathfrak{h}\left(\frac{\mathfrak{n}}{H}\right)^2$ and $\epsilon = o(\mathfrak{h})$, so

$$|\phi(k)| \leq \mathfrak{h} \cdot O\left(\lambda \left(\frac{\mathfrak{n}}{H}\right)^2\right)$$

$$\begin{aligned} T_\phi(\mathfrak{h}) &= \left| 2\sqrt{\frac{H}{\mathfrak{n}}} \sum_{0 \leq k < \frac{\mathfrak{n}}{2H}} \phi(k) e^{-\pi i \frac{Hk^2}{\mathfrak{n}}} \right| \leq \left| 2\sqrt{\frac{H}{\mathfrak{n}}} \sum_{0 \leq k < \frac{\mathfrak{n}}{2H}} \phi(k) \right| \leq \mathfrak{h} \sqrt{\frac{H}{\mathfrak{n}}} \cdot \frac{\mathfrak{n}}{H} \cdot O\left(\lambda \frac{\mathfrak{n}^2}{H^2}\right) = \\ &= \lambda \mathfrak{h} \cdot O\left(\frac{\mathfrak{n}}{H}\right)^{2+1/2} \end{aligned}$$

We will say

$$\mathfrak{h} \rightarrow 0 \text{ iff } \frac{\mathfrak{n}}{H} = O(1).$$

5.11 Corollary

$$T_\phi(H) \leq O(1) \cdot \lambda \mathbf{h} \text{ when } \mathbf{h} \rightarrow 0.$$

5.12 Corollary

$$E_k^{\text{glob}} \psi(k, \mathbf{h}) = \mathbf{h}^{\frac{1}{2}} \sqrt{2\pi} e^{\frac{\pi i}{4}} (1 + I(\mathbf{h})), \text{ where } I(\mathbf{h}) = O(1) \cdot \lambda \mathbf{h} \text{ when } \mathbf{h} \rightarrow 0.$$

This is in a good agreement with $E_k^{\text{loc}} \psi(k, \mathbf{h})$ calculated in [17] and [18] as an *asymptotic (non-convergent) series* of \mathbf{h} .

References

- [1] P.A.M. Dirac, **The Principles of Quantum Mechanics**. Third Edition. Oxford University Press, 1948
- [2] L.Salasnich, **Modern Physics**, Springer, 2022
- [3] D.Hilbert, **Grundlagen der Geometrie**, Leipzig, Teubner, 1899
- [4] C.Chang and H.Kiesler, **Continuous model theory**, Princeton U.Press, 1966
- [5] B.Zilber, *Axioms of quantum mechanics in light of continuous model theory*, arxiv
- [6] L.Henkin, J.D.Monk and A.Tarski, **Cylindric Algebras**, Part I, North-Holland, 1971.
- [7] B.Zilber, *On the logical structure of physics and continuous model theory*, Monatshefte für Mathematik, May 2025
- [8] E.Zeidler, **Quantum Field Theory II: Quantum Electrodynamics. A Bridge between Mathematicians and Physicists**, Springer, 2009
- [9] A. Bohm, **The Rigged Hilbert Space and Quantum Mechanics**, Lecture Notes in Physics, Springer, 1978
- [10] E.Hrushovski, *On the Descriptive Power of Probability Logic*, In **Quantum, Probability, Logic**, 2020

- [11] R. de la Madrid, *The role of the rigged Hilbert space in quantum mechanics*, Eur. J. Phys. 26 (2005), 287
- [12] Carcassi, G., Caldern, F. Aidala, C.A. *The unphysicality of Hilbert spaces*. Quantum Stud.: Math. Found. 12, 13 (2025)
- [13] B.Zilber, *Perfect infinities and finite approximation*. In: **Infinity and Truth**. IMS Lecture Notes Series, V.25, 2014
- [14] B.Hart, *An introduction into continuous model theory*, In **Model Theory of Operator Algebras** de Gruyter, 2023
- [15] E.Hrushovski, *Ax's theorem with an additive character*, EMS Surveys in Mathematical Sciences 8(1):179 – 216
- [16] T. Estermann, *On the sign of the Gaussian sum*, J. London Math. Soc. 20 (1945), 66–67
- [17] P. Etingof, “Mathematical ideas and notions of quantum field theory”, arXiv:2409.03117
- [18] C. M. Bender and T. T. Wu, *Anharmonic oscillator*, The Physical Review, Second Series, Vol. 184, No. 5, 1969
- [19] L. Hörmander **The Analysis of Linear Partial Differential Operators**, Vol. III
- [20] M. Redei, *Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead)*, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 27, 493 (1996).