Structural approximation and a Minkowski space-time lattice with Lorentzian invariance

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Abstract

We introduce a framework of structural approximation to represent Lorentz-invariant Minkowski space-time as the limit of finite cyclic lattices, each equipped with the action of a finite quasi-Lorentz group. This construction provides a discrete model preserving Lorentz symmetry and offers new insights into the algebraic and geometric structure of space-time.

1 Introduction

1.1 A physical theory is an approximation to reality. But what is an approximation? In [Z14] we discussed this problem from the perspective of model theory. One of the necessary properties of a physical theory is that the model preserves the same structural properties that is being assumed for reality. Such properties can often be formulated in formal languages studied by logicians and we used well-studied model-theoretic tools to introduce in [Z14] the definition of **structural approximation** and respective notion of limit, which the current paper heavily relies upon. An earlier application of this notion in context of quantum and statistical mechanics is in the recent paper [Z25].

Among the fundamental properties that physics may ascribe to the universe is the property of being symmetric with regard to the action

*The data that supports the findings of this study are available within the article of a certain group of transformations. On the other hand we might allow the possibility of the universe of being discrete or even finite (consisting of enormous number of points) while the present theory treats it as continuous. In such a setting structural approximation considers a sequence of $\mathbf{M}_i$, $i \in \mathbb{N}$, of finite structures each equipped with a finite group G_i acting on $\mathbf{M}_i$ in a certain prescribed way. The definition in section 2, following [Z14], makes precise the statement that $(\mathbf{M}_i, G_i)$, $i \in \mathbb{N}$, **approximates** a continuous model $\mathbf{M}$ with an action of continuous group G along an ultrailter $\mathcal{D}$. This can be written as

$$\mathsf{lm}_{\mathcal{D}}(\mathbf{M}_i, G_i) = (\mathbf{M}, G).$$

However, such a straightforward form of structural approximation gives intuitively expected outcomes only in simple cases, basically when G is abelian and compact. In more interesting cases, it turns out that both $\mathbf{M}$ and G have to be both representable as *compact complex manifolds*, see 2.3. As a result G is a *compactified group* of transformations, that is a group extended by *infinitary* transformations which send elements of $\mathbf{M}$ to a connected components of its boundary (obtained as a result of approximation) rather than a single point. For this reason we call the approximation as defined in [Z14] **global**.

This is, in effect, due to the fact that in the discrete setting the algebra of the meaningful $\mathbf{M}_i$ and G_i is based on rings of the form $\mathbb{Z}/n_i\mathbb{Z}$ (residue ring modulo n_i) with $n_i \to \infty$. It turns out that in the limit of structural approximation such rings have to be identified as the compactified field of complex numbers $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that is

$$\mathsf{lm}_{\mathcal{D}}\, \mathbb{Z}/n_i\mathbb{Z} = \bar{\mathbb{C}}$$

along any non-principal ultrafilter.

1.2 There is another, more standard and intuitive form of approximation, which we refer to as **local approximation**. This corresponds to "an observer located inside $\mathbf{M}$ with a limited observation span". Namely, suppose a part of $\mathbf{M}_i$, has a form of a 1-dim lattice of a large size l_i which still is much smaller than the size of $\mathbf{M}_i$. It is convenient in this context to use the language and the setting of non-standard numbers (see e.g. [Alb86] for definitions and examples) and assume that we work in some pseudo-finite (equivalently, hyper-finite) structure ${}^*\mathbf{M}$ with a lattice of pseudo-finite size $\mathfrak{l}$.

Let us call $\mathfrak{u}$ the spacing of the lattice. Set a non-standard distance on the lattice so that the shift by $k \cdot \mathfrak{u}$ corresponds to the distance $\frac{k}{\mathfrak{l}}$.

The latter by definition is an element of the set ${}^*\mathbb{Q}$ of non-standard rational numbers. Now use the standard part map (of non-standard analysis)

$$\mathrm{st} : {}^*\mathbb{Q} \to \mathbb{R} \cup \{+\infty, -\infty\}$$

into compactified reals to convert the interval $\{k\mathfrak{u} : \ 0 \leq k \leq \mathfrak{l}\}$ in the lattice into the interval $[0, 1] \subset \mathbb{R}$ and more generally, the union of increasing intervals

$$\mathrm{st} : \bigcup_{m\in\mathbb{N}} \{k\mathfrak{u} : \ -m\mathfrak{l} \leq k \leq m\mathfrak{l}\} \ \twoheadrightarrow \ \mathbb{R}$$

where $\mathbb{R}$ is the real line as a metric space. In case that the $\mathbf{M}_i$ have an algebraic structure of rings or fields, as e.g. in the case of $\mathbb{Z}/p_i\mathbb{Z}\,(= \mathrm{F}_{p_i})$ p_i prime, one can arrange that map st preserves also relations of the structure, see [Z25], Theorem 4.11. The proof of this theorem in [Z25] furnishes a construction of $\mathsf{lm}_{\mathcal{D}}$ so that the limit map coincides with the standard part map on the subset of non-standard rational numbers embedded in the pseudo-finite field. Thus this construction *locally approximates* $\mathbb{R}$ *while globally approximates* $\bar{\mathbb{C}}$ *by a sequence of finite fields.*

1.3 In the paper we apply these techniques and construct a sequence of finite groups acting on finite 4-dim lattices which **locally approximates** the Lorentz group $\mathrm{SO}^+(1, 3)$ acting on the Minkowski spacetime $\mathcal{M}(\mathbb{R})$ endowed with the Minkowski metric.

The same sequence of finite structures **globally approximates** a compact complexification $\overline{\mathcal{M}}(\mathbb{C})$ of Minkowski space acted upon by a compactification of the group $\overline{\mathrm{SO}}(4, \mathbb{C})$. As it happens the structure is almost identical with Penrose's twistor space, see [PenRin84]. The same approximation map considered **locally** induces a unique compactification $\overline{\mathcal{M}}(\mathbb{R})$ of $\mathcal{M}(\mathbb{R})$ which turns out to be a conformal compactification. More precisely,

$$\overline{\mathcal{M}}(\mathbb{R}) \subset \overline{\mathcal{M}}(\mathbb{C}) \hookrightarrow \mathbb{C}\mathbf{P}^5,$$

with embedding into $\mathbb{C}\mathbf{P}^5$ as the Grassmanian $\mathrm{Gr}(2, 4)$, and

$$\overline{\mathcal{M}}(\mathbb{R}) \cong \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^3$$

(note that its double cover is $\mathbb{S}^1 \times \mathbb{S}^3$).

Unlike Penrose's this compactification is truly cyclic: null rays are closed, in fact isomorphic to the projective line $\mathbb{R}\mathbf{P}^1$. This agrees well with the cyclic structure of the discrete lattices which structurally approximate Minkowski spacetime in our construction.

1.4 Finally we would like to mention without going here into details that the 4-dimensional lattice modelling Minkowski space and the structure around it fits well with constructions in [Z25]. Essentially, the 4-dimensional universe would be representable as $\mathbb{U}^4$ where $\mathbb{U}$ is the 1-dimensional universe of the earlier paper.

The last section of the paper illustrates how $\mathbb{U}^4$ can be treated as a lattice model of Minkowski space-time endowed with solutions of Klein-Gordon equation invariant under the action of quasi-Lorentz group.

2 Structural approximation by rings

In this section we give precise definitions and quote main theorems related to approximation, in particular, in the class of rings.

2.1 Ultraproducts and pseudo-finite structures. A structure $\mathbf{M}$ is a set M with a collection Σ of n-ary relations $S \subset M^n$, for some n, called the language (or the vocabulary) of $\mathbf{M} = (M; \Sigma)$.

Suppose we are given a sequence $\{\mathbf{M}_i : i \in \mathbb{N}\}$, $\mathbf{M}_i = (M_i, \Sigma)$, of structures in language Σ. One can choose a Fréchet ultrafilter D on $\mathbb{N}$ and construct the ultraproduct

$$^*\mathbf{M} := \prod_{i \in \mathbb{N}} \mathbf{M}_i / D$$

which is a structure in language Σ with the key property (the Łoś theorem): given a first-order sentence σ in the language Σ,

$$\sigma \text{ is true in } {^*\mathbf{M}} \text{ if and only if } \sigma \text{ is true in } \mathbf{M}_i \text{ along } \mathcal{D} \qquad (1)$$

$^*\mathbf{M}$ is often referred to as *the model-theoretic limit of* $\mathbf{M}_i$ along D (not a structural approximation yet).

In case the $\mathbf{M}_i$'s are finite, $^*\mathbf{M}$ is said to be a **pseudo-finite** structure (or hyper-finite in the terminology of [Alb86]).

In particular, if $\mathbf{M}_i = \mathrm{F}_{p_i}$, p_i prime numbers, structure $^*\mathbf{M}$ is a pseudo-finite field of characteristic zero.

2.2 Structural approximation. It is convenient to consider the system $\mathcal{T}_n$ of topologies on $\mathbf{M}^n$, all n, the basic closed sets of which are realisations $S(\mathbf{M}) \subset \mathbf{M}^n$ of the n-ary $S \in \Sigma$. In geometric/physics setting the closed sets could be introduced as the zero sets of appropriate systems of equations.

Such a system is said to be quasi-compact (or complete) if the projection maps $\mathrm{pr}_{n+1,n} : \mathbf{M}^{n+1} \to \mathbf{M}^n$ preserve closed subsets, that is $\mathrm{pr}_{n+1,n}(S)$ closed, for S closed.

This definition makes sense for the $\mathbf{M}_i$, $i \in I$, and indeed for any structure in language Σ.

A **structural approximation** of $\mathbf{M}$ by $\{\mathbf{M}_i : i \in I\}$ along D is determined by a surjective map (**limit map**)

$$\mathsf{lm} : {}^*\mathbf{M} \twoheadrightarrow \mathbf{M} \tag{2}$$

which has the property, for any n-ary closed $S \in \Sigma$ and $s \in {}^*\mathbf{M}$:

$$s \in S({}^*\mathbf{M}) \text{ closed} \Rightarrow \mathsf{lm}(s) \in S(\mathbf{M})$$

As it happens, below, most of the time $\mathbf{M}_i$, $\mathbf{M}$ and ${}^*\mathbf{M}$ are rings in the language $\{x + y = z,\ x \cdot y = z\}$ or groups in the language $x * y = z$, and *closed* for $S \subset \mathbf{M}^n$ means S is the set of solutions of a system of algebraic equations in n-variables with parameters in $\mathbf{M}$, or equivalently, closed in **Zariski topology**.

Note that despite the coarse topology we still are able to use the intuition of infinitesimals: two elements $a, a' \in {}^*\mathbf{M}^n$ are seen to be "infinitesimally close" if $\mathsf{lm}\, a = \mathsf{lm}\, a'$.

Thus, in view of (1) and (2) structural approximation is a formalisation of the statement *a very large structure* $\mathbf{M}_i$ *looks like* $\mathbf{M}$ *from afar.*

2.3 Approximation and compactness. It was established in [Z14] that in main cases $\mathbf{M}$ has to be quasi-compact in the formal topology (complete in Zariski topology) in order for it to be approximated by a non-trivial sequence $\mathbf{M}_i$ as in (2).

For fields it takes the following form (Theorem 5.2 of [Z14]):

The compactification $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{C}\mathbf{P}^1$ *of the field of complex numbers can be approximated by a sequence of finite fields* F_{p_i} *along an ultrafilter* D *on prime numbers* p_i.

Moreover, for any zero-characteristic pseudo-finite field E *there exists a structural approximation*

$$\mathsf{lm}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}. \tag{3}$$

The field of complex numbers is the only locally compact field that can be structurally approximated by a sequence of finite fields.

The limit map in (3) is far of being unique but we can pick ones with some specific and useful properties as in [Z25], which allows to mimic complex analysis in the pseudo-finite field E.

2.4 Scales and scale-dependence of approximation.

The interplay between the domain and the range of the approximation map lm_{E} as in (3) brings in some features not encountered in the limit construction with inherent metrics. By its nature field E is of a pseudo-finite (non-standard) characteristic $\mathfrak{p}$ ("the limit" of p_i) while $\mathbb{C}$ is characteristic zero field with a natural metric.

More precisely, as in [Z25] let ${}^*\mathbb{Z}$ be an $\aleph_0$-saturated model of arithmetic, $\mathfrak{p} \in {}^*\mathbb{Z}$ an infinite prime number and $\mathrm{E} = {}^*\mathbb{Z}/\mathfrak{p}{}^*\mathbb{Z}$.

Note that in E usual integers $1, 2, 3\ldots$ are represented, and the sequence continuous so far as it does not reach the infinite pseudo-finite number $\mathfrak{p}$, which is bigger than any standard number.

It is clear by algebraic considerations that lm_{E} sends standard integers $1, 2, 3\ldots$ of E to respective integers $1, 2, 3\ldots$ of $\mathbb{C}$. Moreover, on the domain of standard integers lm_{E} acts as an isomorphism of subrings. The same is true for rational numbers, where we represent rational numbers $\frac{k}{m}$ as pairs (k, m) of integer numbers satisfying the identity

$$(k_1, m_1) = (k_2, m_2) \Leftrightarrow k_1 m_2 = k_2 m_1.$$

In [Z25], 3.8-3.9 we defined a convex subring O (denoted $\mathrm{O}(\mathcal{F})$ therein) of the ring ${}^*\mathbb{Z}$ of **small scale** non-standard integers. Namely,

$$-\mathfrak{p} << \mathrm{O} << \mathfrak{p}$$

and O is closed under applications of all arithmetic functions. In fact

$$\mathrm{O} \prec {}^*\mathbb{Z}, \tag{4}$$

is an elementary submodel of ${}^*\mathbb{Z}$ containing an infinite integers $\mathfrak{l}$.

Note that the field of fractions Fr(O) of O can exactly be represented as

$$\mathrm{Fr(O)} = \{(k, m) : \ k, m \in \mathrm{O}, \ m \neq 0\}$$

and by construction

$$\mathrm{Fr(O)} \subset {}^*\mathbb{Q}, \tag{5}$$

the non-standard rational numbers, i.e. the fraction field of ${}^*\mathbb{Z}$.

We also use the field of *finite* rational numbers, that is

$$\mathrm{Fr}_{\mathrm{fin}}(\mathrm{O}) = \{(k, m) \in \mathrm{Fr(O)}, \ \exists N \in \mathbb{N} : \ |k| < mN\}.$$

Note that by assumptions the natural map

$$k \mapsto k \bmod p;\ \mathrm{O} \to \mathrm{E}$$

is an embedding of rings, which extends to the embedding of fields, and we will assume for convenience of notation the inclusions $\mathrm{Fr}_{\mathrm{fin}}(\mathrm{O}) \subseteq \mathrm{Fr}(\mathrm{O}) \subseteq \mathrm{E}$.

Define

$$\mathrm{E}_{\mathrm{Loc}} := \mathrm{Fr}_{\mathrm{fin}}(\mathrm{O})$$

The construction in [Z25], section 4, defines lm_{E} on a subfield $\mathrm{Fr}(\mathcal{F})$ of a pseudofinite field to be **the standard part map** st, that is

$$\mathsf{lm}_{\mathrm{E}}(x) := \mathrm{st}(x),\ \text{ for } x \in \mathrm{E}_{\mathrm{Loc}} \tag{6}$$

while the values $\mathsf{lm}_{\mathrm{E}}(x)$ for generic points of E have no explicit definition.

Note that (6) is well-defined because the standard part map is well-defined on $^*\mathbb{Q}$.

We will call an approximation (limit) map lm_{E} satisfying (6) as **locally canonical**. In the paper all our limit maps are locally canonical by default.

It is clear that (6) along with the fact that $\mathrm{E}_{\mathrm{Loc}}$ contains all finite rational numbers of a non-standard model of arithmetic implies

$$\mathsf{lm}_{\mathrm{E}}:\ \mathrm{E}_{\mathrm{Loc}} \twoheadrightarrow \mathbb{R}. \tag{7}$$

In other words, an observer which has only access to small scale elements of E can think of E as being $\mathbb{R}$. *Locally field* E *approximates the field of reals.*

However, as we continue along the natural order $1, 2, 3, \ldots$ of E, inevitably, (by 5.2(ii) of [Z14]) we will encounter elements of $\bar{\mathbb{C}}$ which are not in $\mathbb{R}$, in particular an element $\mathbf{i} \in \mathrm{E}$ such that

$$\mathsf{lm}_{\mathrm{E}} : \mathbf{i} \mapsto i = \sqrt{-1}; \quad \mathbf{i} \cdot \mathrm{E}_{\mathrm{Loc}} \twoheadrightarrow i \cdot \mathbb{R}.$$

Again by (3), we will also have a non-empty domain

$$\mathrm{E}_{\infty} = \{x \in \mathrm{E} : \mathsf{lm}_{\mathrm{E}}(x) = \infty\}.$$

So, an observer which has tools to explore the global characteristics of E has to think of E as a Riemann sphere $\bar{\mathbb{C}}$.

E *locally looks like* $\mathbb{R}$ *while globally looks like* $\bar{\mathbb{C}}$.

This, somewhat informal introduction of local and global approximations is based on a mathematically rigorous definition in [Z26].

2.5 Approximation by rings. Consider a more general case of a ring of nonstandard integers ${}^*\mathbb{Z}$ modulo $\mathcal{N}$

$$\mathrm{K} = {}^*\mathbb{Z}/\mathcal{N}{}^*\mathbb{Z}$$

where $\mathcal{N} \in {}^*\mathbb{Z}$ divisible by all standard integers, the condition used in [Z25]. (In the current setting it suffices to assume that $\mathcal{N}$ is divisible by unboundedly large standard prime numbers).

Lemma. *There is a surjective ring-homomorphism*

$$\mathsf{Im}_{\mathrm{K}}: \ \mathrm{K} \to \bar{\mathbb{C}}.$$

Im_{K} *is the composition of two Zariski homomorphisms*

$$\mathrm{res}_{\mathfrak{p}} : \mathrm{K} \twoheadrightarrow \mathrm{E} \ \textit{and} \ \mathsf{Im}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}$$

where $\mathrm{E} = \mathrm{E}_{\mathfrak{p}} = {}^*\mathbb{Z}/\mathfrak{p}{}^*\mathbb{Z}$ *for some infinite prime, a pseudo-finite field.*

There is an embedding $\mathrm{Fr}(\mathrm{O}) \subset \mathrm{K}$ *such that*

$$\mathsf{Im}_{\mathrm{K}}(x) := \mathrm{st}(x) \ \textit{for} \ x \in \mathrm{Fr}_{\mathrm{fin}}(\mathrm{O})$$

and, setting $\mathrm{K}_{\mathrm{Loc}} := \mathrm{Fr}_{\mathrm{fin}}(\mathcal{F})$,

$$\mathsf{Im}_{\mathrm{K}}: \ \mathrm{K}_{\mathrm{Loc}} \twoheadrightarrow \mathbb{R} \tag{8}$$

that is Im_{K} *is locally canonical.*

Proof. Since $\mathcal{N}$, the order of K, is divisible by every standard prime p, there is a residue map, ring-homomorphism $\mathrm{res}_{\mathfrak{p}}: \ \mathrm{K} \twoheadrightarrow \mathrm{E}_{\mathfrak{p}}$, for an infinite non-standard $\mathfrak{p}$. As in 2.4 $\mathrm{E}_{\mathfrak{p}} = \mathrm{E}$ contains the non-standard rational numbers Fr(O) and the subring $\mathrm{E}_{\mathrm{Loc}}$ which can be naturally lifted to $\mathrm{K}_{\mathrm{Loc}}$.

Now, consider a large enough non-standard model ${}^*\mathbb{C}$ of complex numbers so that it contains ${}^*\mathbb{Z}$ and thus embeds O, that is we may assume $\mathrm{O} \subset {}^*\mathbb{C}$. Since ${}^*\mathbb{C}$ is algebraically closed of large enough transcendence degree there is an embedding

$$\mathrm{I}_{\mathrm{E}} : \mathrm{E} \hookrightarrow {}^*\mathbb{C}$$

which is an identity on Fr(O).

Define $\mathsf{Im}_{\mathrm{E}} = \mathrm{st} \circ \mathrm{I}$, the composite of two maps. This is a Zariski homomorphism. Note that $\mathrm{st}(\mathrm{Fr}(\mathrm{O})) = \mathbb{R} \cup \{\infty\}$, that is $\mathsf{Im}_{\mathrm{E}}(\mathrm{E}) \supseteq \mathbb{R} \cup \{\infty\}$. However $\mathsf{Im}_{\mathrm{E}}(\mathrm{E}) \neq \mathbb{R} \cup \{\infty\}$ by Theorem 5.2(ii) of [Z14]. It follows $\mathsf{Im}_{\mathrm{E}}(\mathrm{E}) = \bar{\mathbb{C}}$.

□

2.6

$$\mathrm{K}_{\mathrm{Loc}} := \{z \in \mathrm{K} : \mathrm{res}_{\mathfrak{p}}(z) \in \mathrm{E}_{\mathrm{Loc}}\}.$$

This is a subring which contains O and all inverses z^{-1} modulo $\mathfrak{p}$ of elements $z \in \mathrm{O}$.

2.7 Approximation of algebraic varieties. Let $V \subset \mathrm{A}^n$ be an affine variety over $\mathbb{Z}$ and $V(\mathrm{K})$ its K points. By definition V is given by a system of polynomial equations $P_V(x_1, \ldots, x_n) = 0$ over $\mathbb{Z}$ and

$$V(\mathrm{K}) = \{(a_1, \ldots, a_n) \in \mathrm{K}^n : \ P_V(a_1, \ldots, a_n) = 0\}.$$

We consider V as a Zariski structure, that is a structure with universe $V(\mathrm{K})$ and basic k-ary relations given as Zariski closed subsets $S \subseteq V(\mathrm{K})^m$ defined over $\mathbb{Q}$.

Along with $V(\mathrm{K})$ consider the complex variety $V(\mathbb{C})$, also a Zariski structure by the same definition. There exists a projective variety $\mathbf{P}$ over $\mathbb{Q}$ (not unique) such that V can be embedded over $\mathbb{Q}$ into $\mathbf{P}$ as a quasi-projective subvariety. Let

$$e : V \hookrightarrow \mathbf{P}$$

be the embedding.

Set $V^{\mathbf{P}} \subseteq \mathbf{P}$ be the Zariski closure of $e(V)$ in $\mathbf{P}$. It follows that $V^{\mathbf{P}}(\mathbb{C})$ is a complex projective variety, and so compact (complete). So $e(V(\mathbb{R})) \subset e(V(\mathbb{C})) \subset V^{\mathbf{P}}(\mathbb{C})$ and $e(V(\mathbb{R}))$ is a real algebraic variety isomorphic to $V(\mathbb{R})$ over $\mathbb{Q}$.

Note that $V(\mathbb{C})$ is a Zariski substructure of $V^{\mathbf{P}}(\mathbb{C})$ since every Zariski predicate $S \subset (V(\mathbb{C}))^m$ is a restriction of $S^{\mathbf{P}} \subset (V^{\mathbf{P}}(\mathbb{C}))^m$, the closure of S in $V^{\mathbf{P}}(\mathbb{C})$, to $V(\mathbb{C})$.

2.8 Theorem. *Suppose $V(\mathbb{Q})$ is dense in $V(\mathbb{C})$ in the metric topology. There exists a surjective Zariski homomorphism*

$$\mathsf{lm}_{V(\mathrm{K})} : \ V(\mathrm{K}) \twoheadrightarrow V^{\mathbf{P}}(\mathbb{C})$$

such that the restriction of $\mathsf{lm}_{V(\mathrm{K})}$ to $V(\mathrm{K}_{\mathrm{Loc}})$ is coordinate-wise the standard part map[1] *and*

$$\mathsf{lm}_{V(\mathrm{K})} : \ V(\mathrm{K}_{\mathrm{Loc}}) \twoheadrightarrow e(V(\mathbb{R})) \subset V^{\mathbf{P}}(\mathbb{C}).$$

[1]Define $\mathsf{lm}_{V(\mathrm{K})}$ on $v \in V(\mathrm{K}_{\mathrm{Loc}})$ (defined in 2.6) as

$$\mathsf{lm}_{V(\mathrm{K})}(v) := \mathrm{st}(\mathrm{res}_{\mathfrak{p}}(v)).$$

Proof. First consider the Zariski homomorphism

$$\mathrm{res}_{\mathfrak{p}} : \mathrm{K} \twoheadrightarrow \mathrm{E}; \quad k \mapsto k \bmod \mathfrak{p}$$

for the prime $\mathfrak{p}$ and field $\mathrm{E} = \mathrm{E}_{\mathfrak{p}}$ described in 2.5.

Extend $\mathrm{res}_{\mathfrak{p}}$ to $V(\mathrm{K})$ coordinate-wise to get the Zariski homomorphism

$$\mathrm{res}_{\mathfrak{p}} : V(\mathrm{K}) \twoheadrightarrow V(\mathrm{E}).$$

This reduces the problem of constructing $\mathsf{Im}_{V(\mathrm{K})}$ to the problem of constructing

$$\mathsf{Im}_{V(\mathrm{E})} : \; V(\mathrm{E}) \twoheadrightarrow V^{\mathbf{P}}(\mathbb{C}).$$

We carry out the construction below.

Define

$$\mathsf{Im}_0 : V(\mathrm{E}_{\mathrm{Loc}}) \to V(\mathbb{R}); \;\; \mathsf{Im}_0(\bar{a}) = \mathrm{st}(\bar{a}),$$

where $\bar{a} = (a_1, \ldots, a_n) \in \mathrm{E}^n$ and $\mathrm{st}(\bar{a}) = (\mathrm{st}(a_1), \ldots, \mathrm{st}(a_n))$.

By definition st is the map that preserves the order relation on rationals of $\mathrm{E}_{\mathrm{Loc}}$ and preserves polynomial equations. It follows that Im_0 is a Zariski homomorphism and tt is surjective due to (7) and the assumption on density of rational points in $V(\mathbb{R})$.

At the next stage we require a slightly stronger version of (4).

Claim. *There exist a model* $^*\mathbb{Z}$ *of the ring of integers, prime* $\mathfrak{p} \in {}^*\mathbb{Z}$ *and a convex elementary submodel* $\mathrm{O} \prec {}^*\mathbb{Z}$ *such that* $\mathbb{Z} \neq \mathrm{O}$ *and*

$$\mathrm{tr.d.}(\mathrm{E}/\mathrm{O}) \geq 2^{\aleph_0}. \tag{9}$$

Proof. First consider an $\aleph_0$-saturated model $\mathcal{Z}$ of the ring of integers and note that $\mathbb{Z} \prec \mathcal{Z}$, by definition. It follows that there exists a prime $\mathfrak{p} \in \mathcal{Z}$ such that the field $\mathrm{E}_0 = \mathcal{Z}/\mathfrak{p}\mathcal{Z}$ contains an infinite subset A of algebraically independent elements, that is $\mathrm{tr.d.}(\mathrm{E}/\mathbb{Z}) \geq |A| \geq \aleph_0$.

Now let D be a non-rpincipal ultrafilter over $\mathbb{N}$ and set the ultrapowers

$$\mathrm{O} := \mathbb{Z}^{\mathbb{N}}/D, \;\; {}^*\mathbb{Z} := \mathcal{Z}^{\mathbb{N}}/D, \; \mathrm{E} := \mathrm{E}_0^{\mathbb{N}}/D, \; {}^*A := A^{\mathbb{N}}/D.$$

We then get

$$\mathrm{tr.d.}(\mathrm{E}/\mathrm{O}) \geq |{}^*A| \geq 2^{\aleph_0}$$

which proves the claim.

As a corollary we get: *There exists a subset of n-tuples $B \subset V(\mathrm{E}) \subset \mathrm{E}^n$ such that $|B| \geq 2^{\aleph_0}$ and any n-tuples*

$$(b_{11} \dots b_{1n}), \dots, (b_{m1} \dots b_{mn}) \in B$$

with distinct b_{ij} are algebraically independent over $\mathrm{E}_{\mathrm{Loc}}$, that is for any non-zero polynomial $P(x_{11} \dots x_{1n}, \dots, x_{m1} \dots x_{mn})$ over $\mathrm{E}_{\mathrm{Loc}}$,

$$P(b_{11} \dots b_{1n}, \dots, b_{m1} \dots b_{mn}) \neq 0.$$

This implies that any map $\mathsf{Im}_B : \mathrm{E}_{\mathrm{Loc}} \cup B \to \mathbf{V}^{\mathbf{P}}(\mathbb{C})$ extending Im_0 is a Zariski homomorphism. We define the extension Im_B so that

$$\mathsf{Im}_B(B) = V^{\mathbf{P}}(\mathbb{C}) \tag{10}$$

It remains to extend Im_B to $A = V(\mathrm{E}) \setminus B$.

We start by enumerating A by ordinal numbers $\alpha < \kappa$ where $\kappa = |A|$, the cardinality of A,

$$A = \{a_\alpha : \ a < \kappa\}.$$

We also extend the Zariski topologies on cartesian powers of $V(\mathrm{E})$ and $V^{\mathbf{P}}(\mathbb{C})$ by adding, for every Zariski closed relation $S(x_1, \dots, x_m)$ on mth cartesian power of the variety the new $k-1$-ary relation $\exists x_{m-k} \dots x_m S(x_1, \dots, x_m)$ (this procedure was called in [Z14] *the formal completion of the topology*). Note that each relation $\exists x_m S(x_1, \dots, x_m)$ on $V^{\mathbf{P}}(\mathbb{C})$ defines a Zariski closed subset since $V^{\mathbf{P}}(\mathbb{C})$ is projective and thus complete.

Let

$$A_\beta := \{a_\alpha \in A : \ \alpha < \beta\}.$$

Assume that for $\beta < \kappa$, there is $\mathsf{Im}_{B,\beta} : \ B \cup A_\beta \to V^{\mathbf{P}}(\mathbb{C})$ such that,

$$\begin{array}{l} \text{for any } c_1, \dots, c_{k-1} \in B \cup A_\beta, \\ V(\mathrm{E}) \vDash \exists x_k \dots x_m S(c_1, \dots, c_{k-1}, x_k \dots x_m) \Rightarrow \\ \quad V^{\mathbf{P}}(\mathbb{C}) \vDash \exists x_k \dots x_m S(c_1^{\mathsf{Im}}, \dots, c_{k-1}^{\mathsf{Im}}, x_k \dots x_m) \end{array} \tag{11}$$

where c^{Im} stands for $\mathsf{Im}_{B,\beta}(c)$.

This is the case for $\beta = 0$ since $\mathsf{Im}_{B,0} := \mathsf{Im}_B$.

Now we construct $\mathsf{Im}_{B,\beta+1}$ extending $\mathsf{Im}_{B,\beta}$. We need to find an element $a_\beta^{\mathsf{Im}} \in V^{\mathbf{P}}(\mathbb{C})$ such that

$$\begin{array}{l} V(\mathrm{E}) \vDash \exists x_k \dots x_{m-1} S(c_1, \dots, c_{k-1}, x_k \dots x_{m-1}, a_\beta) \Rightarrow \\ \quad V^{\mathbf{P}}(\mathbb{C}) \vDash \exists x_k \dots x_{m-1} S(c_1^{\mathsf{Im}}, \dots, c_{k-1}^{\mathsf{Im}}, x_k \dots x_{m-1}, a_\beta^{\mathsf{Im}}) \end{array}$$

Consider the type over A_β, collection of formulas with parameters A_β,

$$p(y) = \begin{array}{l}\{\exists x_k \dots x_{m-1} S(c_1, \dots, c_{k-1}, x_k \dots x_{m-1}, y) : \\ \qquad V(\mathrm{E}) \vDash \exists x_k \dots x_{m-1} S(c_1, \dots, c_{k-1}, x_k \dots x_{m-1}, a_\beta)\}\end{array}$$

We claim that $p(y)$ can be realised in $V^{\mathbf{P}}(\mathbb{C})$. Indeed, each formula of $p(y)$ defines a Zariski closed subset of $V^{\mathbf{P}}(\mathbb{C})$ and the type corresponds to the intersection of all the Zariski closed subsets. By the descending chain condition of Zariski topologies $p(y)$ is equaivalent to the intersection of finitely many sets, equivalently to a single set defined by a formula $\exists x_k \dots x_{m-1} S(c_1, \dots, c_{k-1}, x_k \dots x_{m-1}, y)$. Thus we need to see that

$$V^{\mathbf{P}}(\mathbb{C}) \vDash \exists y, x_k, \dots, x_{m-1}\, S(c_1, \dots, c_{k-1}, x_k \dots x_{m-1}, y)$$

which just follows from the indection assumption (11).

We have proved the induction step from $\mathsf{Im}_{B,\beta}$ to $\mathsf{Im}_{B,\beta+1}$. For a limit ordinal $\gamma \le \kappa$ define $\mathsf{Im}_{B,\gamma} = \bigcup_{\beta<\gamma} \mathsf{Im}_{B,\beta}$. This determines all the steps of the construction by induction and we set

$$\mathsf{Im}_{V(\mathrm{E})} := \mathsf{Im}_{B,\kappa}.$$

This finishes the proof of the theorem. □

2.9 Rational subvarieties. Assume $Q \subset V$ a subvariety of dimension d over $\mathbb{Q}$, rational over $\mathbb{Q}$. That is there are Zariski open subsets $D \subseteq A^d$, $Q^0 \subseteq Q$ and rational map

$$f : D \twoheadrightarrow Q^0$$

all defined over $\mathbb{Q}$. Let $Q^{\mathbf{P}}$ be the Zariski closure of Q in $\mathbf{P}$ and assume that $Q^{\mathbf{P}}(\mathbb{C})$ is non-singular. It is well-known that under the assumption $Q(\mathbb{Q})$ is dense in $Q(\mathbb{R})$ and in $Q^{\mathbf{P}}(\mathbb{R})$ in metric topology.

Claim 1. For $\mathsf{Im}_{V(\mathrm{E})}$ of 2.8

$$\mathsf{Im}_{V(\mathrm{E})}(\mathrm{E}_{\mathrm{Loc}}) = Q(\mathbb{R}) \text{ and } \mathsf{Im}_{V(\mathrm{E})}(Q({}^*\mathbb{Q})) = Q^{\mathbf{P}}(\mathbb{R})$$

where ${}^*\mathbb{Q}$ is an $\aleph_0$-saturated model of the field of rationals embedded in E.

Proof. The first statement is a property of Im^0 established in 2.8.

For the second statement consider $Q(\mathbb{Q})$ which is dense in $Q(\mathbb{R})$ and so dense in $Q^{\mathbf{P}}(\mathbb{R})$ since Zariski density implies metric density.

Since the standard part map st sends a Cauchy sequence in $Q(\mathbb{Q})$ to a Cauchy sequence in $Q^{\mathbf{P}}(\mathbb{R})$ and $Q({}^*\mathbb{Q})$ is saturated, for each $a \in Q^{\mathbf{P}}(\mathbb{R})$ there is $\alpha \in Q({}^*\mathbb{Q})$ such that $\mathsf{lm}_{V(\mathrm{E})}(\alpha) = a$. This finishes the proof.

Claim 2.

$$\mathsf{lm}_{V(\mathrm{E})}(Q(\mathrm{E})) = Q^{\mathbf{P}}(\mathbb{C}).$$

Proof. Recall that by [Z14] $\mathsf{lm}_{V(\mathrm{E})}(\mathrm{E}) \neq \mathbb{R} \cup \{\infty\}$ and so there is $\gamma \in \mathrm{E} \setminus {}^*\mathbb{Q}$ such that $\mathsf{lm}_{V(\mathrm{E})}(\gamma) \in \mathbb{C} \setminus \mathbb{R}$. Set $j := \mathsf{lm}_{V(\mathrm{E})}(\gamma)$. Thus $\mathbb{Q} + j \cdot \mathbb{Q}$ is dense in $\bar{\mathbb{C}}$ in metric topology. Note that since $\mathsf{lm}_{V(\mathrm{E})}$ is a homomorphism

$$\mathsf{lm}_{V(\mathrm{E})}(Q({}^*\mathbb{Q} + \gamma \cdot {}^*\mathbb{Q})) = Q(\mathrm{st}({}^*\mathbb{Q}) + j \cdot \mathrm{st}({}^*\mathbb{Q})).$$

Define an absolute value for $a+gb \in \mathbb{Q}+\gamma\mathbb{Q} \subset \mathrm{E}$ as $|a+gb| := |a|+|b|$, and similarly for $a + jb \in \mathbb{R} + j\mathbb{R} \subset \mathbb{C}$ as $|a + jb| := |a| + |b|$. It follows that $\mathsf{lm}_{V(\mathrm{E})}$ preserves the absolute value and so sends Cauchy sequences of $\mathbb{Q} + \gamma\mathbb{Q}$ to Cauchy sequences of $\mathbb{Q} + j\mathbb{Q}$. By the same argument as in the proof of Claim 1 we conclude that

$$Q(\mathrm{st}({}^*\mathbb{Q}) + j \cdot \mathrm{st}({}^*\mathbb{Q})) = Q^{\mathbf{P}}(\mathbb{C}),$$

which completes the proof Claim 2.

2.10

3 Pseudo-finite Minkowski space structure and its limit

3.1 It is well-known that $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to the double cover of the Lorentz group $\mathrm{SO}^+(1, 3)$ and it acts in agreement with this on the Minkowski space .

More precisely (see e.g. [Kl15]), one represents a vector with components $(t, x, y, z) \in \mathbb{R}^4$ (Minkowski space) as a 2×2 matrix

$$X := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

with $X^\dagger = X$ and $\det(X) = t^2 - x^2 - y^2 - z^2$, and considers

$$X \mapsto MXM^\dagger \text{ with } M \in \mathrm{SL}(2, \mathbb{C}). \tag{12}$$

This preserves $\det X$ and thus the Minkowski metric, which leads to the proof that (12) is a Lorentz transformation and all Lorentz transformations can be expressed in this way. The fact that $\pm M$ both give the same transformation of X corresponds to the fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of the Lorentz group, that is

$$\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \tag{13}$$

We denote

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2)$$

the structure which consists of $\mathbb{R}$-linear Minkowski space $\mathcal{M}(\mathbb{R})$ with metric given by $X \mapsto \det X$ along with the group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ acting on the space as describe in (12).

We note that the isomorphism of groups induces the isomorphism of structures

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2) \cong \big(\mathcal{M}(\mathbb{R}), \mathrm{SO}^+(1,3)\big) \tag{14}$$

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we will treat as a Zariski structure, that is the set with Zariski closed relations $R \subset \bar{\mathbb{C}}^n$ on it.

3.2 Complexification of a ring. Let A be a commutative unitary ring. Define

$A^{(2)}$ to be the unitary ring obtained from the ring A as follows:

$$A^{(2)} := \{(a,b) \in A \times A\};$$

$$(a_1,b_1)+(a_2,b_2) := (a_1+a_2, b_1+b_2),\ \ (a_1,b_1)\cdot(a_2,b_2) := (a_1a_2-b_1b_2, a_1b_2+a_2b_1).$$

Clearly, $a \mapsto (a,0)$ is an embedding of A into $A^{(2)}$ as a subring $(A,0)$ and

$$(a,b) \mapsto (a,-b) \text{ an automorphism of } A^{(2)}.$$

3.3 Let $\mathrm{M}(2,A^{(2)})$ be the set of 2×2 matrices over $A^{(2)}$ which we treat as an 8-dim A-module and let $\mathrm{SL}(2,A^{(2)})$ be the group of matrices of determinant 1.

A **Minkowski A-lattice** is the A-submodule $\mathcal{M}(A)$ of $\mathrm{M}(2,A^{(2)})$ consisting of matrices $X_{t,x,y,z}$ over $A^{(2)}$ of the form

$$X_{t,x,y,z} = X := \begin{pmatrix} (t+z,0) & (x,-y) \\ (x,y) & (t-z,0) \end{pmatrix}, \quad t,x,y,z \in A.$$

We have

$$\det(X) = (t^2 - x^2 - y^2 - z^2, 0) \in A \times \{0\}$$

and this defines Minkowski A-metric length of (t, x, y, z).

For the general $A^{(2)}$-matrix

$$Y = \begin{pmatrix} (a_1, a_2) & (b_1, b_2) \\ (c_1, c_2) & (d_1, d_2) \end{pmatrix}$$

define the adjoint matrix

$$Y^\dagger := \begin{pmatrix} (a_1, -a_2) & (c_1, -c_2) \\ (b_1, -b_2) & (d_1, -d_2) \end{pmatrix}$$

Clearly, $X^\dagger = X$ for $X \in \mathcal{M}(A)$. In general

$$(YZ)^\dagger = Z^\dagger Y^\dagger.$$

In particular, Y is self-adjoint ($Y = Y^\dagger$) iff $a_2 = 0 = d_2$ and $b_1 = c_1$, $b_2 = -c_2$.

It follows that for any $M \in \mathrm{SL}(2, A^{(2)})$, $X \in \mathcal{M}(A)$

$$MXM^\dagger \in \mathcal{M}(A) \text{ and } \det X = \det MXM^\dagger \tag{15}$$

Let

$$C = \{M \in \mathcal{M}(A) : MXM^\dagger = X \text{ for all } X \in \mathcal{M}(A).\}$$

Let $M_0 \in C$. In particular, $M_0 M_0^\dagger = \mathrm{I}$. It is equivalent to $M_0^\dagger = M_0^{-1}$ and thus $M_0 X M_0^{-1} = X$ for all $X \in \mathcal{M}(A)$. This implies that M_0 is diagonal and belongs to the centre of $\mathrm{SL}(2, A^{(2)})$, thus

$$C = \{M = \begin{pmatrix} (a_1, a_2) & 0 \\ 0 & (a_1, a_2) \end{pmatrix}; \quad a_1^2 - a_2^2 = 1 \ \& \ (a_1 = 0 \vee a_2 = 0)\} \tag{16}$$

Thus we have established:

3.4 Proposition. *The 2-sorted structure*

$$\left(\mathcal{M}(A), \mathrm{SL}(2, A^{(2)})/C\right)$$

is interpretable in the ring A along with the group action $X \mapsto MXM^\dagger$ and A-Minkowski metric.

The action and Minkowski metric are defined by systems of polynomial equations over $\mathbb{Z}$.

In particular, $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$ *is the group of K-linear transformations of $\mathcal{M}(\mathrm{K})$ preserving Minkowski* K*-valued metric.*

3.5 Lemma.

$$\mathrm{SL}(2,\mathbb{C}^{(2)})/C \cong \mathrm{SO}(4,\mathbb{C})$$

where C is the centre of $\mathrm{SL}(2,\mathbb{C}^{(2)})$ and

$$C \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. By 3.4 $\mathrm{SL}(2,\mathbb{C}^{(2)})/C$ is the group of transformations of $\mathcal{M}(\mathbb{C})$ preserving Minkowski $\mathbb{C}$-valued metric, that is the form $x_0^2 + x_1^2 + x_2^2 + x_3^2$. But this is also the definition of group $\mathrm{SO}(4,\mathbb{C})$.

The form of C is determined by (16). □

3.6 Compactification of Minkowski space. Consider $\mathcal{M}(\mathbb{C})$, the complexification of Minkowski space $\mathcal{M}(\mathbb{R})$. Clearly, setwise $\mathcal{M}(\mathbb{C}) = \mathbb{C}^4$. Note that the metric in $\mathcal{M}(\mathbb{R})$ in presence of the additive structure is determined by the distance s from 0, $s^2 = -t^2 + x_1^2 + x_2^2 + x_3^2$, that is a Zariski closed subset, quadric $Q(\mathbb{R}) \subset \mathcal{M}(\mathbb{R}) \times \mathbb{R}$. Similarly, $\mathrm{SL}(2,\mathbb{R}^{(2)})$ can be setwise identified with a Zariski closed subset of $\mathbb{R}^8$ and the action of the group by a Zariski closed subset $\Gamma(\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R}^{(2)}) \times \mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R})$.

These all can be represented as a Zariski closed subsets in cartesian powers $(\mathcal{M}(\mathbb{R}) \times \mathbb{R})^n$, equivalently in $\mathbb{R}^{5n}$, for some n.

Let

$$V(\mathbb{R}) = \mathcal{M}(\mathbb{R}) \times \mathbb{R} \times \mathrm{SL}(2,\mathbb{R}^{(2)}) \times \Gamma(\mathbb{R}) \tag{17}$$

the affine variety which represents in the form of Zariski closed sets the universes and the relation of the structure $\left(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{R}^{(2)})\right)$, which is the structure of the Minkowski space-time with its metric given by $Q(\mathbb{R})$ and the action of the Lorentz group $\mathrm{SO}^+(1,3)$ represented by $\mathrm{SL}(2,\mathbb{R}^{(2)})$ and $\Gamma(\mathbb{R})$.

The same is applicable to $\mathcal{M}(\mathbb{C})$, $\mathrm{SL}(2,\mathbb{C}^{(2)})$ and $\Gamma(\mathbb{C})$, so we get $V(\mathbb{C})$, the $\mathbb{C}$ points of the same variety, to represent the structure $\left(\mathcal{M}(\mathbb{C}), \mathrm{SL}(2,\mathbb{C}^{(2)})\right)$, the complexification of Minkowski structure.

Following 2.8, for variety V choose an embedding into a projective variety

$$e : V \to \mathbf{P}$$

(to be determined in the next section) and set $V^{\mathbf{P}}$ to be Zariski closure of $e(V)$ in $\mathbf{P}$. We assume that $\mathbf{P}$ is chosen so that the quadric Q defining the Minkowski metric has a non-singular completion in $\mathbf{P}$.

This gives us the $\mathbf{P}$-compactification of the complex structure $\left(\mathcal{M}(\mathbb{C}), \mathrm{SL}(2,\mathbb{C}^{(2)})\right)^{\mathbf{P}}$, with compact ingredients $\overline{\mathcal{M}}(\mathbb{C})$, $\overline{\mathrm{SL}}(2,\mathbb{C}^{(2)})$

and $\bar{\Gamma}(\mathbb{C})$ as well as **P**-compactification of the real structure $\big(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2, \mathbb{R}^{(2)})\big)^{\mathbf{P}}$ which will contain the compactification $\overline{\mathcal{M}}(\mathbb{R})$ of Minkowski space.

3.7 Theorem. *There is a limit map, Zariski homomorphism of structures,*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}), \mathrm{SL}(2, \mathrm{K}^{(2)})\Big) \twoheadrightarrow \Big(\mathcal{M}(\mathbb{C}), \mathrm{SL}(2, \mathbb{C}^{(2)})\Big)^{\mathbf{P}} \tag{18}$$

Its restriction to the structure over $\mathrm{K}_{\mathrm{Loc}}$ *is a Zariski homomorphism*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}_{\mathrm{Loc}}), \mathrm{SL}(2, \mathrm{K}^{(2)}_{\mathrm{Loc}})\Big) \twoheadrightarrow (\mathcal{M}(\mathbb{R}), \mathrm{SL}(2, \mathbb{C})) \tag{19}$$

Proof. Consider V as defined in (17). Note that by the form of the factors of the direct product $\mathbb{Q}$-points are dense in each of the factors of $V(\mathbb{R})$, or equivalently, $V(\mathbb{Q})$ is dense in $V(\mathbb{R})$.

By 2.8 there is a Zariski homomorphism

$$\mathsf{lm}_{V(\mathrm{K})} : V(\mathrm{K}) \twoheadrightarrow V^{\mathbf{P}}(\mathbb{C})$$

with the property

$$\mathsf{lm}_{V(\mathrm{K})} : V(\mathrm{K}_{\mathrm{Loc}}) \twoheadrightarrow V(\mathbb{R}).$$

It remains to check that

$$\mathsf{lm}_{V(\mathrm{K})} : Q(\mathrm{K}) \twoheadrightarrow Q^{\mathbf{P}}(\mathbb{C})$$

and

$$\mathsf{lm}_{V(\mathrm{K})} : Q(\mathrm{K}_{\mathrm{Loc}}) \twoheadrightarrow Q(\mathbb{R}).$$

As in 2.8 we can reduce the problem to proving

$$\mathsf{lm}_{\mathbf{V}(\mathrm{E})} : Q(\mathrm{E}) \twoheadrightarrow Q^{\mathbf{P}}(\mathbb{C})$$

and

$$\mathsf{lm}_{V(\mathrm{E})} : Q(\mathrm{E}_{\mathrm{Loc}}) \twoheadrightarrow Q(\mathbb{R}).$$

For this note that quadric Q in the 5-dim affine space contains a $\mathbb{Q}$-point and so is rational over $\mathbb{Q}$. Now 2.9 gives us both required properties.

□

4 Algebraic compactification of Minkowski space

In this section we construct the complex projective variety **P** in which $\mathcal{M}(\mathbb{C})$ can be suitably embedded according to 3.6. Since we at the same time consider $\mathcal{M}(\mathbb{R})$, we carry out the construction over arbitrary field of characteristic 0 to substitute $\mathbb{C}$ and $\mathbb{R}$ later.

The construction, which is usually applied to $\mathcal{M}(\mathbb{R})$, leading to a conformal compactification, is well known, although some small variations are possible. We found the algebraic presentation in [Yadczyk84] useful for our purposes. The author is grateful to M.Kabenyuk for the help with the algebra involved.

4.1 Let E be a field of characteristic 0, $\mathcal{M}(\mathrm{E}) = \mathrm{E} \times \mathrm{E}^3$ be the Minkowski space with the quadratic form encoding Minkowski metric

$$Q(t, x_1, \dots, x_3) = -t^2 + x_1^2 + \dots + x_3^2.$$

We define an embedding Φ of $\mathcal{M} = \mathcal{M}(\mathrm{E})$ into a higher-dimensional space $V = \mathrm{E}^2 \times \mathrm{E}^4$ as

$$\Phi: \ (t, x_1, \dots, x_3) \mapsto (\frac{1}{2} - \frac{q}{2}, t, x_1, ..., x_3, \frac{1}{2} + \frac{q}{2})$$

where $q = Q(t, x_1, ..., x_3)$. The equation

$$\bar{Q}(s, t, x_1, ..., x_3, x_4) = -s^2 - t^2 + x_1^2 +, ..., x_3^2 + x_4^2 = 0$$

defines a hypersurface L in V. It is easy to verify that $\Phi(\mathcal{M}) \subseteq L$.

Let $\pi(V)$ be the projective space obtained by projectivisation

$$\pi : V \to \pi(V); \ (y_0, \dots, y_5) \mapsto [y_0 : \dots : y_5].$$

We have dim $\dim \pi(V) = 5$, $\dim \pi(L) = 4$ and

$$\pi(\mathcal{M}) \subset \pi(L) \subset \pi(V).$$

Moreover, π bijectively maps $\Phi(\mathcal{M})$ onto $\pi(\Phi(\mathcal{M}))$, (because $\Phi(\mathcal{M})$ lies in $s + x_4 = 1$) so $\dim \pi(\Phi(\mathcal{M})) = 4 = \dim \pi(L)$.

4.2 Thus the map

$$\pi \circ \Phi : (t, x_1, \dots, x_3) \mapsto [\frac{1}{2} - \frac{q}{2} : t : x_1 : ..., x_3 : \frac{1}{2} + \frac{q}{2}]$$

$$\mathcal{M} \twoheadrightarrow \pi(\Phi(\mathcal{M})) \subset \pi(L) \subset \mathrm{E}\mathbf{P}^5$$

is bijective onto $\pi(\Phi(\mathcal{M}))$ and $\pi(\Phi(\mathcal{M}))$ is in the open subset of the projective variety $\pi(L)$ defined by equation $s + x_4 = 1$.

Moreover, by the dimensional equalities $\pi(\Phi(\mathcal{M}))$ is dense in $\pi(L)$. Thus we may identify the closure in Zariski topology

$$\bar{\mathcal{M}} := \pi(L) = \{[s : t : x_1 : ..., x_3 : x_4] \ : \ s^2 + t^2 = x_1^2 +, ..., x_3^2 + x_4^2 = 0\}$$

a smooth quadric in $\mathrm{E}\mathbf{P}^5$.

4.3 The complex case. In case $\mathrm{E} = \mathbb{C}$, the field of complex numbers, $\bar{\mathcal{M}}(\mathbb{C})$ by the above formula is a Klein quadric in $\mathbb{C}\mathbf{P}^5$ and is easily identifiable with the complex Grassmanian $\mathrm{Gr}(2, 4) \subset \mathbb{C}\mathbf{P}^5$.

This identifies $\mathbf{P}(\mathbb{C})$ of 3.7 as $\mathbb{C}\mathbf{P}^5$,

4.4 The real case and conformal compactification. In case $\mathrm{E} = \mathbb{R}$ it is possible to define the embedding

$$\psi : [s : t : x_1 : x_2 : x_3 : x_4] \mapsto ([s : t], [x_1 : x_2 : x_3 : x_4]); \ \ \pi(L) \to \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^3$$

Indeed, given $(s, t, x_1, x_2, x_3, x_4) \in L$, non-zero, we have $s^2 + t^2 = x_1^2 + ... + x_4^2$. Thus one of the affine coordinates s, t has to be nonzero because otherwise $s^2 + t^2 = 0 = x_1^2 + ... + x_4^2$ and so $s = t = x_1 = x_2 = x_3 = x_4 = 0$. For the same reason one of the coordinates of (x_1, x_2, x_3, x_4) has to be non-zero. Thus $[s : t] \in \mathbb{R}\mathbf{P}^1$ and $[x_1 : x_2 : x_3 : x_4] \in \mathbb{R}\mathbf{P}^3$ which defines ψ.

The embedding is surjective. Indeed, let $[s : t] \in \mathbb{R}\mathbf{P}^1$ and $[x_1 : ..., x_3 : x_4] \in \mathbb{R}\mathbf{P}^3$. Let

$$\lambda = \sqrt{\frac{x_1^2 + ... + x_4^2}{s^2 + t^2}}$$

Then $[\lambda s : \lambda t] = [s : t]$ and $[\lambda s : \lambda t : x_1 : ..., x_3 : x_4] \in \pi(L)$. By definition

$$\psi : \ [\lambda s : \lambda t : x_1 : ..., x_3 : x_4] \mapsto ([s : t], [x_1 : ..., x_3 : x_4]).$$

Now note that $\mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^3$ is homeomorphic to the quotient $(\mathbb{S}^1 \times \mathbb{S}^3)/\mathbb{Z}_2$ of products of the spheres, where $\mathbb{Z}_2$ acts diagonally. In fact the map

$$\psi' : \left(\frac{(s, t)}{\sqrt{s^2 + t^2}}, \frac{(x_1, x_2, x_3, x_4)}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}} \right) \mapsto ([s : t], [x_1 : x_2 : x_3 : x_4])$$

is a 2-1 map $\mathbb{S}^1 \times \mathbb{S}^3 \twoheadrightarrow \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^3$.

$\mathbb{S}^1 \times \mathbb{S}^3$ is a well-known conformal compactification of $\mathcal{M}(\mathbb{R})$. Since ψ' preserves conformal metric, our algebraic construction provides another known conformal compactification of $\mathcal{M}(\mathbb{R})$.

4.5 Set the boundary of $\mathcal{M}$ to be the difference

$$\mathcal{F} = \pi(L) \setminus \pi\Phi(\mathcal{M}) = \bar{\mathcal{M}} \setminus \mathcal{M}.$$

The entire boundary is described by the single condition $s + x_4 = 0$ together with the quadratic equation $\bar{Q} = 0$ (as in 4.1). That cuts out exactly one quadric in the hyperplane $\{s + x_4 = 0\}$ in $\mathrm{E}\mathbf{P}^5$, The hyperplane $\{s + x_4 = 0\}$ is isomorphic to $\mathrm{E}\mathbf{P}^4$ and does not intersect with $\mathcal{M}$ (that is with $\pi\Phi(\mathcal{M})$). Restricting $\bar{Q}$ to this hyperplane yields a nondegenerate quadric

$$Q^3 : \; -t^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

of dimension 3 inside $\mathrm{E}\mathbf{P}^4$. Thus,

$$\mathcal{F} \cong Q^3.$$

which, for $\mathrm{E} = \mathbb{R}$ is the **light cone at infinity** in the terminology of R.Penrose.

5 The Klein-Gordon equation as an example

5.1 In [Z25] we considered a mathematical model $\mathbb{U}$ of a one-dimensional universe with wave-functions $\phi : \mathbb{U}^n \to \mathrm{F}_\mathfrak{p}$. $\mathbb{U}$ is assumed to be a pseudo-finite residue group ${}^*\mathbb{Z}_\mathcal{N}$ of order $\mathcal{N}$ such that $\mathcal{N}$ is divisible by $\mathfrak{p} - 1$, for $\mathfrak{p}$ a prime number, and $\mathrm{F}_\mathfrak{p} = {}^*\mathbb{Z}_\mathfrak{p}$ the pseudo-finite field with $\mathfrak{p}$ elements. Under the conditions there is a surjective homomorphism of groups

$$\exp_\mathfrak{p} : \mathbb{U} \twoheadrightarrow \mathrm{F}_\mathfrak{p}^\times$$

Now assume that $\mathbb{U}$ is the additive group of $\mathrm{K} = \mathrm{K}_\mathcal{N}$ and $\mathcal{N}$. In particular $\mathbb{U}$ is a K-module and we can identify $\mathbb{U}^4 = \mathrm{K}^4$ with $\mathcal{M}(\mathrm{K})$.

5.2 Klein-Gordon equation

Recall the Klein-Gordon equation (with $\hbar = 1 = c$)

$$-\frac{\partial^2}{\partial t^2}\phi = (-\sum_{j=1,2,3}\frac{\partial^2}{\partial r_j^2} + m^2)\phi$$

which has solutions

$$\phi_{\mathbf{k}}(\bar{r}, t) := \exp(i\sum_{j=1,2,3} k_j r_j - i\omega t)$$

with

$$\omega^2 - k_1^2 - k_2^2 - k_3^2 = m^2 \tag{20}$$

This can be rewritten in variables $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and parameters $\mathbf{k} = (\omega, k_1, k_2, k_3)$ as

$$\phi_{\mathbf{k}}(\mathbf{x}) = \exp(-i\,\mathbf{k}\cdot\mathbf{x}) \tag{21}$$

where

$$\mathbf{k}\cdot\mathbf{x} = \omega x_0 - k_1x_1 - k_2x_2 - k_3x_3$$

is the inner product in Minkowski space. It is well-known that by going through all $\mathbf{k} \in \mathbb{R}^4$ satisfying (20) the solutions (21) generates the space of all solutions to the Klein-Gordon equation.

Now we consider the analogue of solution (21) in $\mathcal{M}(\mathrm{K})$ by assuming $\mathbf{k}, \mathbf{x} \in \mathrm{K}^4$ and replacing $\exp(-ix)$ by $\exp_{\mathfrak{p}}(x)$. Thus (21) becomes

$$\phi_{\mathbf{k}}(\mathbf{x}) = \exp_{\mathfrak{p}}(\mathbf{k}\cdot\mathbf{x}) \tag{22}$$

5.3 Lorentz invariance: Recall that the substitute for the Lorentz group $\mathrm{SO}^+(1,3) = \mathrm{SL}(2,\mathbb{C})/C$ is the K-Lorentz group $\mathrm{SL}(2,\mathrm{K}^2)/C$ as established in 3.4.

For $g \in \mathrm{SL}(2,\mathrm{K}^2)$ consider the Lorentzian action of g on $\mathbf{x} \in \mathcal{M}(\mathrm{K})$

$$\mathbf{x} \mapsto \mathbf{x}^g$$

which transforms $\phi_{\mathbf{k}}$:

$$\phi^g_{\mathbf{k}}(\mathbf{x}) := \phi_{\mathbf{k}}(\mathbf{x}^g)$$

Note that

$$\phi^g_{\mathbf{k}}(\mathbf{x}) = \phi_{g^{-1}\mathbf{k}}(\mathbf{x}) \tag{23}$$

since g preserves the metric on K, and so preserves Minkowski inner product, giving

$$\mathbf{k}\cdot\mathbf{x}^g = \mathbf{k}^{g^{-1}}\cdot\mathbf{x}$$

(23) proves that the action by g on a solution produces just another solution, thus proving that the Klein-Gordon equation over $\mathcal{M}(\mathrm{K})$ is invariant under K-Lorentz action.

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