Geometric stability and Zariski geometries

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July 28, 2010
Lecture I

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- **Zariski geometries** is the class of structures discovered in this exploration.
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- **Zariski geometries** is the class of structures discovered in this exploration.

- Zariski geometries are on the very top of stability hierarchy, so, in the very centre of mathematics.
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Noetherian Zariski structures: The idea

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**Example.** Algebraic Geometry is a model theory of (algebraically closed) fields with the emphasis on positively quantifier-free definable sets (**Zariski-closed** sets).
Noetherian Zariski structures: Definition and Axioms

Let $\mathbf{M}$ be a structure and let $\mathcal{C}$ be a distinguished sub-collection of the definable subsets of $M^n$, $n = 1, 2, \ldots$. The sets in $\mathcal{C}$ are called (definable) **closed**. The relations corresponding to the sets are the basic (primitive) relations of the language we will work with. $\langle \mathbf{M}, \mathcal{C} \rangle$, or $\mathbf{M}$, is a **topological structure** if it satisfies axioms:
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(L) Topological **Language**: The primitive $n$-ary relations of the language are exactly the ones that distinguish definable closed subsets of $M^n$, all $n$ (that is the ones in $\mathcal{C}$), and every quantifier-free positive formula in the language defines a closed set (so is equivalent to an atomic one).
Noetherian Zariski structures: Definition and Axioms

More precisely:

1. the intersection of a finite family of closed sets is closed;
2. finite unions of closed sets are closed;
3. the domain of the structure is closed;
4. the graph of equality is closed;
5. any singleton of the domain is closed;
6. Cartesian products of closed sets are closed;
7. the image of a closed $S \subseteq M^n$ under a permutation of coordinates is closed;
8. for $a \in M^k$ and $S$ a closed subset of $M^{k+l}$ defined by a predicate $S(x, y)$ the fibre over $a$

$$S(a, M^l) = \{ b \in M^l : M \models S(a, b) \}$$

is closed.
Noetherian Zariski structures: Definition and Axioms

Remarks
L6 assumes that, for \( S_1 \subseteq M^n \) and \( S_2 \subseteq M^m \) closed, \( S_1 \times S_2 \) is canonically identified with a subset of \( M^{n+m} \) which is closed in the latter.

The canonical identification is

\[
\langle \langle x_1, \ldots, x_k \rangle, \langle y_1, \ldots, y_m \rangle \rangle \mapsto \langle x_1, \ldots, x_k, y_1, \ldots, y_m \rangle.
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A projection

$$pr_{i_1, \ldots, i_m} : \langle x_1, \ldots, x_n \rangle \mapsto \langle x_{i_1}, \ldots, x_{i_m} \rangle, \quad i_1, \ldots, i_m \in \{1, \ldots, n\}.$$ 

is a continuous map, by L6: the inverse image of a closed set $S$ is closed. Indeed,

$$pr_{i_1,\ldots,i_m}^{-1} S = S \times M^{n-m}$$

up to the order of coordinates.
Noetherian Zariski structures: Definition and Axioms

**Constructible sets** are the Boolean combinations of members of \( C \).

A subset of \( M_n \) will be called projective if it is a finite union of sets of the form \( \text{pr} S_i \), for some \( S_i \subseteq \text{cl} U_i \subseteq \text{op} M_n + k_i \) and projections \( \text{pr} (i) : M_n + k_i \to M_n \).

Note that any constructible set is projective with trivial projections in its definition.

A topological structure is said to be complete if the image \( \text{pr}_{i_1}, \ldots, i_m S \) of a closed subset \( S \subseteq \text{cl} M_n \) is closed.

A topological structure \( M \) will be called quasi-compact (or just compact) if it is complete and satisfies

\[ \bigcap_{t \in T} C_t \text{ is non-empty.} \]
Noetherian Zariski structures: Definition and Axioms

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(P) **Properness** of projections condition holds:
the image $\text{pr}_{i_1,...,i_m} S$ of a closed subset $S \subseteq \text{cl } M^n$ is closed.
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(QC) For any finitely consistent family $\{C_t : t \in T\}$ of closed subsets

$$\bigcap_{t \in T} C_t$$

is non-empty.
Noetherian Zariski structures: Definition and Axioms

A topological structure is called **Noetherian** if it also satisfies: (DCC) **Descending chain condition** for closed subsets: for any closed

\[ S_1 \supseteq S_2 \supseteq \ldots \supseteq S_i \supseteq \ldots \]

there is \( i \) such that for all \( j \geq i \), \( S_j = S_i \).
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A definable set \( S \) is called **irreducible** if there are no relatively closed subsets \( S_1 \subseteq_{cl} S \) and \( S_2 \subseteq_{cl} S \) such that \( S_1 \not\subset S_2 \), \( S_2 \not\subset S_1 \) and \( S = S_1 \cup S_2 \).
Noetherian Zariski structures: Definition and Axioms

Good dimension

We assume that to any non-empty projective $S$ a non-negative integer called the dimension of $S$, $\dim S$, is attached. We postulate the following properties of a good dimension notion:

(DP) **Dim of a point** is 0;

(DU) **Dim of unions:** $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$;

(SI) **Strong irreducibility:** For any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and its closed subset $S_1 \subseteq_{cl} S$, if $S_1 \neq S$ then $\dim S_1 < \dim S$;

(AF) **Addition formula:** For any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection map $\text{pr} : M^n \to M^m$,

$$\dim S = \dim \text{pr}(S) + \min_{a \in \text{pr}(S)} \dim(\text{pr}^{-1}(a) \cap S).$$

(FC) **Fibre condition:** For any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection map $\text{pr} : M^n \to M^m$ there exists $V \subseteq_{op} \text{pr} S$ (relatively open) such that

$$\min_{a \in \text{pr}(S)} \dim(\text{pr}^{-1}(a) \cap S) = \dim(\text{pr}^{-1}(v) \cap S), \text{ for any } v \in V \cap \text{pr}(S).$$
Complete Noetherian topological structures with good dimension will be called **complete (Noetherian) Zariski structures**.

More generally we replace $(P)$ by $(SP)$ semi-Properness of projection mappings: given a closed irreducible subset $S \subseteq \overline{M}$ and the projection map $pr: M^n \to M^k$, there is a proper closed subset $F \subset pr(S)$ such that $pr(S) \setminus F \subseteq pr(S)$.

Noetherian topological structures with good dimension and satisfying $(SP)$ will be called **(Noetherian) Zariski structures**.
Noetherian Zariski structures: Definition and Axioms

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Noetherian Zariski structures: Definition and Axioms

In many cases we assume that a Zariski structure satisfies also (EU) **Essential uncountability:** If a closed $S \subseteq M^n$ is a union of countably many closed subsets, then there are finitely many among the subsets, the union of which is $S$. 

1-dimensional presmooth Noetherian Zariski structures satisfying (EU) are called (1-dim Noetherian) Zariski geometry. This can be generalised to a definition of a ($n$-dim Noetherian) Zariski geometry.
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The following is an extra condition crucial for developing a rich theory for Zariski structures.

(PS) **Presmoothness:** For any closed irreducible $S_1, S_2 \subseteq M^n$, for any irreducible component $S_0$ of $S_1 \cap S_2$,

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim M^n.$$
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Noetherian Zariski geometries: Examples


"Uncountable" needed to satisfy (EU).
Natural language: $\mathcal{C}$ consists of Zariski-closed subsets of $M^n$. 


3. Definable substructures of $\text{DCF}_0(n)$ of finite Morley rank. (2001)

4. "Quantum geometries."
Noetherian Zariski geometries: Examples


Natural language: $\mathcal{C}$ consists of analytic subsets of $M^n$. 

3. Definable substructures of DCF$_0(\mathbb{R})$ of finite Morley rank.

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3. Definable substructures of $\text{DCF}_0(n)$ of finite Morley rank. (2001)

More precisely: every definable substructure of finite Morley rank can be made Zariski in a natural language by removing a subset of smaller rank.


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Model theory of Noetherian Zariski structures

Let \( M = (M, C) \) be a Noetherian Zariski structure.
Model theory of Noetherian Zariski structures

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**Theorem 3** Assume $M$ satisfies (EU). Given $M' \succeq M$ one can naturally extend the topology to $M'$ so that $M'$ becomes a Noetherian Zariski structure satisfying (EU).
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**Proof.**
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Using axioms, $\dim \text{pr} S_2 < \dim \text{pr}(S_1 \setminus S_2)$ and so the above can be understood by induction hypothesis on dimension.
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Using axioms, $\dim \text{pr} S_2 < \dim \text{pr}(S_1 \setminus S_2)$ and so the above can be understood by induction hypothesis on dimension. All axioms are needed.
Model theory of Noetherian Zariski structures

**Theorem 2** The theory of $M$ is $\omega$-stable of finite Morley rank, assuming $M$ satisfies (EU).
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Proof. Use Theorem 1 to show by induction on $\dim Q$, constructible $Q$, that $\text{Mrk } Q \leq \dim Q$.

(EU) provides $\aleph_0$-saturation for countable fragments of the language.
Theorem 3 Assume $\mathcal{M}$ satisfies (EU). Given $\mathcal{M}' \succeq \mathcal{M}$ one can naturally extend the topology to $\mathcal{M}'$ so that $\mathcal{M}'$ becomes a Noetherian Zariski structure satisfying (EU).
Model theory of Noetherian Zariski structures

**Theorem 3** Assume $M$ satisfies (EU). Given $M' \supseteq M$ one can naturally extend the topology to $M'$ so that $M'$ becomes a Noetherian Zariski structure satisfying (EU).

**Proof.** We declare subsets of the form $P(a, M')$ in $M'$ closed if $P$ is positive quantifier free.
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Proof. We declare subsets of the form \( P(a, M') \) in \( M' \) closed if \( P \) is positive quantifier free.
Define \( \dim P(a, M') \geq k \) if \( a \) satisfies the formula that says so (given by (FC)).
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The main difficulties are in checking axioms (SI: strong irreducibility) and (DCC: descending chain condition).
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Define $\dim P(a, M') \geq k$ if $a$ satisfies the formula that says so (given by (FC)).
The main difficulties are in checking axioms (SI: strong irreducibility) and (DCC: descending chain condition).
Again, (EU) is essential in providing a saturation.
Generalities:

▶ Zariski Geometry is a geometry.
Lecture II

Generalities:

- Zariski Geometry is a geometry.
- Zariski Geometry is a "logical completion" of Algebraic Geometry.
Specialisations and infinitesimal calculus

Given a topological structure $\mathbf{M}$ and $\mathbf{M}' \succeq \mathbf{M}$, a specialisation is a surjective homomorphism

$$\pi : \mathbf{M}' \rightarrow \mathbf{M}.$$
Specialisations and infinitesimal calculus

Given a topological structure $\mathbf{M}$ and $\mathbf{M}' \preceq \mathbf{M}$, a **specialisation** is a surjective homomorphism

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Note:
Specialisations and infinitesimal calculus

Given a topological structure $M$ and $M' \succeq M$, a specialisation is a surjective homomorphism

$$\pi : M' \to M.$$ 

Note:
$\pi$ preserves closed subsets.
Specialisations and infinitesimal calculus

Given a topological structure $M$ and $M' \supseteq M$, a specialisation is a surjective homomorphism

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$\pi$ is the identity on $M$, since every element of $M$ is named.
Specialisations and infinitesimal calculus

Given a topological structure $M$ and $M' \simeq M$, a **specialisation** is a surjective homomorphism

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Note:

$\pi$ preserves closed subsets.

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**Example.** The field of reals $\mathbb{R}$ is a topological structure in a natural language and, for $\mathbb{R}' \simeq \mathbb{R}$ a specialisation, $\pi : \mathbb{R}' \rightarrow \mathbb{R}$ is the **standard part map**.
Specialisations and infinitesimal calculus

Given a topological structure $\mathbf{M}$ and $\mathbf{M}^\prime \succeq \mathbf{M}$, a specialisation is a surjective homomorphism

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**Example.** The field of reals $\mathbb{R}$ is a topological structure in a natural language and, for $\mathbb{R}^\prime \succeq \mathbb{R}$ a specialisation, $\pi : \mathbb{R}^\prime \rightarrow \mathbb{R}$ is the *standard part map*.

**Proposition.** Suppose $\mathbf{M}$ is a quasi-compact structure, $\mathbf{M}^\prime \succeq \mathbf{M}$. Then there is a total specialisation $\pi : \mathbf{M}^\prime \rightarrow \mathbf{M}$. Moreover, any partial specialisation can be extended to a total one.
Specialisations and infinitesimal calculus

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Example. The field of reals $\mathbb{R}$ is a topological structure in a natural language and, for $\mathbb{R}' \succeq \mathbb{R}$ a specialisation, $\pi : \mathbb{R}' \rightarrow \mathbb{R}$ is the standard part map.

Proposition. Suppose $\mathbf{M}$ is a quasi-compact structure, $\mathbf{M}' \succeq \mathbf{M}$. Then there is a total specialisation $\pi : \mathbf{M}' \rightarrow \mathbf{M}$. Moreover, any partial specialisation can be extended to a total one. The inverse also holds for a right choice of topology on $\mathbf{M}$. 
Specialisations and infinitesimal calculus

Given $a \in M^n$ we call $\pi^{-1}(a)$ the infinitesimal neighbourhood of $a$ (in $M'$). Also denoted $\mathcal{V}_a(M')$ or just $\mathcal{V}_a$. 

Proposition. Every specialisation $\pi_0: M_0 \to M$ can be extended to a universal one $\pi: \ast M \to M$.

Proof. Straightforward Fraissé argument. Assuming $\pi$ is universal, the geometric properties of $\mathcal{V}_a$ are independent on $\pi$ and $\ast M$. 

Specialisations and infinitesimal calculus

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Specialisations and infinitesimal calculus

Given $a \in M^n$ we call $\pi^{-1}(a)$ the infinitesimal neighbourhood of $a$ (in $M'$). Also denoted $\nu_a(M')$ or just $\nu_a$.
This depends strongly on $M'$ and $\pi$.

A specialisation $\pi : *M \to M$, for $*M \succeq M$, is said to be universal if:

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Given \( a \in M^n \) we call \( \pi^{-1}(a) \) the infinitesimal neighbourhood of \( a \) (in \( M' \)). Also denoted \( \nu_a(M') \) or just \( \nu_a \).

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A specialisation \( \pi : *M \to M \), for \( *M \succeq M \), is said to be **universal** if:

for any \( M' \succeq *M \succeq M \), any finite subset \( A \subset M' \) and a specialisation \( \pi' : A \cup *M \to M \) extending \( \pi \), there is an elementary embedding \( \alpha : A \to *M \), over \( A \cap *M \), such that

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Specialisations and infinitesimal calculus

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**Proposition.** Every specialisation $\pi^0 : M^0 \rightarrow M$ can be extended to a universal one $\pi : *M \rightarrow M$. 

Specialisations and infinitesimal calculus

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Assuming $\pi$ is universal, the geometric properties of $\nu_a$ are independent on $\pi$ and $*M$. 

Proposition. Given irreducible $S \subseteq_{\text{cl}} M^n$ and $a \in S$, the intersection $S(\ast M) \cap \mathcal{V}_a$ contains a generic point.
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Specialisations and infinitesimal calculus

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**Theorem** (Implicit Function Theorem) Given a Zariski geometry $M$ and an irreducible constructible presmooth $D \subseteq M^n$ suppose an irreducible $F \subseteq_{\text{cl}} D \times M^k$ projects onto $D$ with finite fibres (finite covering of $D$).

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The maximal number of possible such $b'$ for a given $a' \in \mathcal{V}_a$ will be called **the multiplicity of** $F$ **at** $a$: $\text{mult}_a(F/D)$. 
Specialisations and infinitesimal calculus

**Proposition.** Given irreducible \( S \subseteq \text{cl} \ M^n \) and \( a \in S \), the intersection \( S(\ast M) \cap \mathcal{V}_a \) contains a generic point.

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2. There is an open subset \( \text{Reg} F/D \subseteq_{\text{op}} D \) such that \( \text{mult}_a(F/D) = 1 \) iff \( a \in \text{Reg} F/D \).
Specialisations and infinitesimal calculus

**Proposition.** Given irreducible $S \subseteq_{\text{cl}} M^n$ and $a \in S$, the intersection $S(*)M) \cap \mathcal{V}_a$ contains a generic point.

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Corollary. For $a \in \text{Reg } F/D$ and $\langle a, b \rangle \in F$ the set $F \cap (\mathcal{V}_a \times \mathcal{V}_b)$ is the graph of a function $\varphi : \mathcal{V}_a \to \mathcal{V}_b$ (local function).
Specialisations and infinitesimal calculus

Let $L_1, L_2$ and $P$ be constructible irreducible presmooth sets and $I_i \subseteq \text{cl} L_i \times P$, $i = 1, 2$, irreducible. We will call a curve coded by $\ell \in L_i$ the set

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$$T(p, \ell_1, \ell_2) := \ell_1 \text{ and } \ell_2 \text{ are tangent at point } p \in \hat{\ell}_1 \cap \hat{\ell}_2$$

As a corollary we can define the jet of curves from $L_1$ passing through $p \in P$ and tangent to generic $\ell \in L_2$:

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Lemma. Given a family of curves $L$ on $P$ as above, the set of jets $[L]_p$ through $p$ is definable (interpretable) and under certain assumptions can be identified with a Zariski constructible set.
Proposition. Non-local modularity implies: some irreducible $P \subseteq \text{op} M \times M$, some Zariski irreducible presmooth set $L$ in $M$ and $I \subseteq \text{cl} L \times P$ define a 2-dimensional family of curves on $P$.

At a generic point $\langle a, b \rangle \in M^2$ a generic curve $\ell_1$ locally (i.e. in infinitesimal neighbourhood) is the graph of a local function $\lambda_1: V_a \to V_b$.

Given $\ell_1$ and $\ell_2$, the local function $\lambda_2^{-1}/\lambda_1: V_a \to V_a$ corresponds to a new curve through $\langle a, a \rangle$ (rather a branch of a curve).
Classification of 1-dim non-locally modular Noetherian Zariski geometries

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The set $\Gamma$ of all local functions $\gamma : \mathcal{V}_a \rightarrow \mathcal{V}_a$ obtained in this way is definable.
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Corollary. There is a group structure $(G, \cdot)$ definable by Zariski-closed predicates on a 1-dim irreducible Zariski set. (Copy the proof of Weil's group chunk theorem).
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- For any generic pair $\gamma_1, \gamma_2 \in \Gamma$ there is a generic $\gamma \in \Gamma$ such that
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With more work one obtains

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Classification of 1-dim non-locally modular Noetherian Zariski geometries

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- The theory of multiplicities can be applied to get an intersection theory in projective spaces. In particular, the following generalisation of Bezout’s theorem holds: given in \(\mathbb{P}^2(K)\) a curve \(\ell\) and an algebraic curve \(\ell_{\text{alg}}\)

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\#_{\text{mult}}(\ell \cap \ell_{\text{alg}}) = \deg \ell \cdot \deg \ell_{\text{alg}},
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- The latter implies that any \(S \subseteq_{\text{cl}} \mathbb{P}^n(K)\) must be algebraic (generalisation of Chow’s theorem).
Classification of 1-dim non-locally modular Noetherian Zariski geometries

Since $M$ is not orthogonal to $K$, there is a finite-to-finite correspondence between $M$ and $K$. 

This can be converted into a non-constant partial map $f: M \to K$ (meromorphic map) and to a total Zariski-continuous function $\bar{f}: M \to \mathbb{P}^1(K)$. 

In general, such functions can be seen as coordinate functions and given $f = \langle \bar{f}_1, \ldots, \bar{f}_n \rangle$ we obtain a map $f: M \to \mathbb{P}^1(K)^n \subseteq \mathbb{P}^N(K)$. 

$f(M)$ is a quasi-projective curve $C \subseteq \mathbb{P}^N(K)$ and $f: M \to C$ is a Zariski-continuous finite covering of the algebraic curve $C$. 

The latter classifies $M$ up to the finite fibres $f^{-1}(a)$, $a \in C$. 

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Lecture III

Generalities:

The classification of 1-dimensional Zariski geometries found its application in e.g. Diophantine Geometry.
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But even more interesting is that it lead to the discovery of a class of **new geometric objects**.
New geometric objects

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There exists 1-dimensional $\mathbf{M}$ such that no covering $f : \mathbf{M} \to C$ is bijective ($C$ an algebraic curve). In other words, 1-dimensional Zariski geometry can be different from an algebraic curve.
New geometric objects

**Example.** Let $M$ be the set

$$\{\langle x, \epsilon \rangle : x, \epsilon \in K, \epsilon^2 = 1\}$$

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So, the set $K = M/E$ is definable and we have all polynomially defined relations on $K$, lifted to relations on $M$, in our language. Let $R \subseteq K \setminus \{0\}$ be a subset with the property:

$$y \in R \text{ iff } -y \notin R.$$

Introduce a new ternary relation $A \in \mathcal{C}, A \subseteq M \times M \times K$:

$$A(\langle x_1, \epsilon_1 \rangle, \langle x_2, \epsilon_2 \rangle, y) \text{ iff } x_2 = x_1 + 1 \land y^2 = x_1^2 \land$$

$$\land ((y \in R \land \epsilon_1 = \epsilon_2) \lor (y \notin R \land y \neq 0 \land \epsilon_1 \neq \epsilon_2) \lor y = 0)$$
New geometric objects

**Proposition.** (i) \(M\) is a 1-dimensiona Noetherian Zariski geometry which (ii) can not be identified with an algebraic curve. Moreover, \(M\) is not definable (not interpretable) in an algebraically closed field.
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**Proof.** (i)
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- every formula is a Boolean combination of $\exists$-formulas.
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Proof. (i)

- every formula is a Boolean combination of $\exists$-formulas.
- Closed sets are defined as given by positive $\exists$-formulas of a certain form.

(ii) Use the well-known fact: If an ACF $K$ is interpretable in an ACF $F$, then $K$ is definably isomorphic to $F$.

Consider Galois theory of $(K(x,\epsilon))_K$ and prove that one cannot interpret $⟨x,\epsilon⟩$ as a tuple in a field extension of $K$. 
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- every formula is a Boolean combination of $\exists$-formulas.
- Closed sets are defined as given by positive $\exists$-formulas of a certain form.
- With some work, check all the Zariski axioms.
New geometric objects

**Proposition.** (i) $M$ is a 1-dimensional Noetherian Zariski geometry which (ii) can not be identified with an algebraic curve. Moreover, $M$ is not definable (not interpretable) in an algebraically closed field.

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- Use the well-known fact: If an $\text{ACF}_0$ $K$ is interpretable in an $\text{ACF}_p$ $F$, then $K$ is definably isomorphic to $F$. 
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- Use the well-known fact: If an ACF$_0$ $K$ is interpretable in an ACF$_p$ $F$, then $K$ is definably isomorphic to $F$.
- Consider *Galois theory* of $(K(\langle x, \epsilon \rangle) : K)$ and prove that one can not interprete $\langle x, \epsilon \rangle$ as a tuple in a field extension of $K$. 
New geometric objects
Reinterpretation.

Think of \( \langle x, 1 \rangle \) and \( \langle x, -1 \rangle \) as "vectors" \( e^x \) and \( -e^x \), a pair for each value of \( x \in K \).
The 1-dimensional space generated by \( e^x \) consists of formal pairs \( y \cdot e^x \), for \( y \in K \), equivalently, \( z \cdot (-e^x) \), \( z \in K \), with assumption \( y \cdot e^x = (-y) \cdot (e^x) \).
Given \( e^x \) we will have, by assumptions, a \( y = \sqrt{x} \) such that \( A(e^x, e^x+1, y) \) and \( A(e^x, -e^x+1, -y) \) hold.
Interpret this as a map \( a: e^x \mapsto y \cdot e^x+1 \) or a linear operator
\( a: z \cdot e^x \mapsto yz \cdot e^x \).
The same \( A(e^x, e^x+1, y) \) can be given the interpretation
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We have two linear operators \( a \) and \( a^* \) acting in the linear space generated by the \( e^x \) which satisfy
\( (a^*a - aa^*) e^x = e^x \).
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Reinterpretation. Think of $\langle x, 1 \rangle$ and $\langle x, -1 \rangle$ as "vectors" $e_x$ and $-e_x$, a pair for each value of $x \in K$. 
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Interpret this as a map $a : e_x \mapsto y \cdot e_{x+1}$ or a *linear operator* on 1-dimensional spaces:

$$ a : z \cdot e_x \mapsto yz \cdot e_{x+1}. $$
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Co-ordinate algebra for $M$.

1. Want to explain a geometric object $M$ in terms of co-ordinates in $K$. 
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2. For "non-classical" $M$ the algebra $K[M]$ of Zariski-continuous functions can not separate points in $M$ : $K[M] = K[C_M]$ (same as for the algebraic curve).
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4. Consider the algebra $A(\mathbf{M})$ of linear operators on $\mathcal{H}[\mathbf{M}]$ generated by ones of the form $\psi(t) \to f(t) \cdot \psi(bt)$,
   $\psi, f \in \mathcal{H}[\mathbf{M}]$,
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5. $(A(\mathbf{M}), *)$ does not depend on $\mathcal{H}(\mathbf{M})$, only on $\mathbf{M}$.
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   We also define formal adjoint \( X^* \) for operators \( X \) in \( \mathcal{A}(\textbf{M}) \), depending on the structure of \( \textbf{M} \).
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Co-ordinate algebras: Quantum algebras at roots of unity

A canonical correspondence

\[ M \leftrightarrow A(M) \]

is well-established only for special class of algebras \( A \) and structures \( M \).
Co-ordinate algebras: Quantum algebras at roots of unity

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is well-established only for special class of algebras \( \mathcal{A} \) and structures \( M \).

A \( K \)-algebra \( \mathcal{A} \) will be called an algebra at root of unity if it satisfies:

- \( \mathcal{A} \) is finitely generated Noetherian.
- \( \mathcal{A} \) is a finite-dimensional module over its centre \( Z(\mathcal{A}) \).
- Further assumptions (that might be redundant).

Examples

- The algebra \( T_2^q \) generated by \( U \) and \( V \) with defining relation \( UV = qVU \), in case \( q \in \mathbb{N} = 1 \).
- Many other algebras, e.g. quantum groups \( SL(2, K)^q \), \( Usl_q(2, K) \).
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- The algebra \( T_q^2 \) generated by \( U \) and \( V \) with defining relation

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- Many other algebras, e.g. quantum groups \( SL(2, K)_q \), \( Usl_q(2, K) \).
Theorem. There is a canonical procedure that puts in correspondence to any $K$-algebra $\mathcal{A}$ at root of unity, $K$ algebraically closed, a Zariski geometry $\mathcal{M}$, so that $\mathcal{A}$ can be canonically recovered from $\mathcal{M}$. 
Co-ordinate algebras: Quantum algebras at roots of unity

**Theorem.** There is a canonical procedure that puts in correspondence to any $K$-algebra $A$ at root of unity, $K$ algebraically closed, a Zariski geometry $M$, so that $A$ can be canonically recovered from $M$.

**Construction.** Consider the affine variety $V = V(A)$ corresponding to the affine commutative algebra $Z(A)$. To each point of $V$ corresponds a unique, up to isomorphism, $N$-dimensional $A$-module. The bundle of such modules over $V$ is $M(A)$. 
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The procedure extends the classical duality between an affine algebraic variety and its co-ordinate algebra.
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Question. What to do for a general value of $q$?
Trichotomy conjecture and Hrushovski counterexamples

Classical first-order $\lambda$-categorical structures for uncountable $\lambda$:

1. Structures with trivial geometry
2. Linear (locally-modular) structures: (Abelian divisible torsion-free groups; Abelian groups of prime exponent; Vector spaces over a given division ring ...)
3. Algebraically closed fields.

Trichotomy Conjecture: Every strongly minimal structure is reducible to 1, 2 or 3.
False in general (Hrushovski, 1988).
Almost true for Zariski geometries (HZ, 1993).
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Hrushovski counterexamples: construction

Given a class of structures $M$ with dimension notions $d_1$ and $d_2$, we want to consider a new function $f$ on $M$.

On $(M, f)$, introduce a predimension $\delta(X) = d_1(X \cup f(X)) - d_2(X)$.

Consider structures $(M, f)$ which satisfy the Hrushovski inequality: $\delta(X) \geq 0$ for any finite $X \subset M$.

Amalgamate all such structures to get a universal and homogeneous structure in the class. The resulting structure $(\tilde{M}, f)$ will have a good dimension notion and a nice geometry.
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- Given a class of structures $\mathcal{M}$ with a dimension notions $d_1$, and $d_2$ we want to consider a *new function* $f$ on $\mathcal{M}$. 

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- *The resulting structure* $(\tilde{\mathbf{M}}, f)$ *will have a good dimension notion and a nice geometry.*
Example of Hrushovski’s construction

Given a class of fields \((K, +, \cdot)\) we want to consider a new function \(f\) on \(K\).

Introduce a predimension \(\delta(X) = \text{tr.d}.(X \cup f(X)) - |X|\).

Consider structures \((K, f)\) which satisfy the Hrushovski inequality:

\[ \delta(X) \geq 0 \]

for any finite \(X \subset K\).

Amalgamate all such structures to get a universal and homogeneous structure in the class.

The resulting structure \((\tilde{K}, f)\) is \(\omega\)-stable and with some extra work (collapse) one can get a new uncountably categorical structure from \((\tilde{K}, f)\).
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- Amalgamate all such structures to get a universal and homogeneous structure in the class.
Example of Hrushovski’s construction

- Given a class of fields $(K, +, \cdot)$ we want to consider a new function $f$ on $K$.
- On $(K, f)$ introduce a predimension
  \[
  \delta(X) = \text{tr.d.}(X \cup f(X)) - |X|.
  \]
- Consider structures $(K, f)$ which satisfy the Hrushovski inequality:
  \[
  \delta(X) \geq 0 \text{ for any finite } X \subset K.
  \]
- Amalgamate all such structures to get a universal and homogeneous structure in the class.
- The resulting structure $(\tilde{K}, f)$ is $\omega$-stable and with some extra work (collapse) one can get a new uncountably categorical structure from $(\tilde{K}, f)$. 
Are Hrushovski structures mathematical pathologies?

Observation: If $K$ is a field and we want $f = e^x$ to be a group homomorphism, then the predimension must be $\delta(X) = \text{tr} \cdot d(X \cup e^X) - \text{lin} \cdot d(Q(X)) \geq 0$.

The Hrushovski inequality, in the case of the complex numbers, $e^x = \exp$, is equivalent to:

$\text{tr} \cdot d(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$ assuming that $x_1, \ldots, x_n$ are linearly independent.

This is the Schanuel conjecture.
Are Hrushovski structures mathematical pathologies?

Observation: If $K$ is a field and we want $f = \exp$ to be a group homomorphism

$$
\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)
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The Hrushovski inequality, in the case of the complex numbers, $\exp = \exp$, is equivalent to:

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\text{tr} \cdot \text{dim}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n
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Are Hrushovski structures mathematical pathologies?

Observation: If $K$ is a field and we want $f = \text{ex}$ to be a group homomorphism

$$\text{ex}(x_1 + x_2) = \text{ex}(x_1) \cdot \text{ex}(x_2)$$

then the predimension \textbf{must be}

$$\delta(X) = \text{tr.d.}(X \cup \text{ex}(X)) - \text{lin.d.}_\mathbb{Q}(X) \geq 0.$$
Are Hrushovski structures mathematical pathologies?

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assuming that $x_1, \ldots, x_n$ are linearly independent.

This is the Schanuel conjecture.
Pseudo-exponentiation

Consider the class of fields of characteristic 0 with a function \( \text{ex} : K \rightarrow (K, +, \cdot, \text{ex}) \) satisfying

**EXP1:** \( \text{ex}(x_1 + x_2) = \text{ex}(x_1) \cdot \text{ex}(x_2) \)

**EXP2:** \( \ker \text{ex} = \pi \mathbb{Z}, \) some \( \pi \in K. \)

Consider the subclass satisfying the Schanuel condition **SCH:**

\( \text{tr}.d.(X \cup \text{ex}(X)) - \text{lin}.d.(X) \geq 0. \)

Amalgamation process produces an algebraically-exponentially closed field with pseudo-exponentiation, \( K^{\text{ex}}(\lambda), \) of cardinality \( \lambda. \)
Consider the class of fields of characteristic 0 with a function \( \text{ex} \): \( K_{\text{ex}} = (K, +, \cdot, \text{ex}) \) satisfying

\[
\text{EXP1: } \text{ex}(x_1 + x_2) = \text{ex}(x_1) \cdot \text{ex}(x_2)
\]

\[
\text{EXP2: ker ex} = \pi \mathbb{Z}, \text{some } \pi \in K
\]

Consider the subclass satisfying the Schanuel condition \( \text{SCH} \):

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Consider the class of fields of characteristic 0 with a function \( \text{ex} : K_{\text{ex}} = (K, +, \cdot, \text{ex}) \) satisfying

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Consider the class of fields of characteristic 0 with a function $\text{ex}$: $K_{\text{ex}} = (K, +, \cdot, \text{ex})$ satisfying

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\end{align*}

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\[ \text{SCH: } \text{tr.d.}(X \cup \text{ex}(X)) - \text{lin.d.}(X) \geq 0. \]
Pseudo-exponentiation

Consider the class of fields of characteristic 0 with a function $ex: K_{ex} = (K, +, \cdot, ex)$ satisfying

**EXP1:** $ex(x_1 + x_2) = ex(x_1) \cdot ex(x_2)$

**EXP2:** $\ker ex = \pi \mathbb{Z}$, some $\pi \in K$.

Consider the subclass satisfying the Schanuel condition

$$\text{SCH} : \quad \text{tr.d.}(X \cup ex(X)) - \text{lin.d.}(X) \geq 0.$$ 

Amalgamation process produces an *algebraically-exponentially closed field with pseudo-exponentiation*, $K_{ex}(\lambda)$, of cardinality $\lambda$. 
Pseudo-exponentiation

Algebraic-exponential closedness (existential closedness) takes the form: $EC$: For any non-overdetermined irreducible system of polynomial equations $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$ there exists a generic solution satisfying $y_i = e^{x_i}$ for $i = 1, \ldots, n$.

Also we have the Countable Closure property: $CC$: Analytic subsets of $n$ of dimension 0 are countable. $ACF_0$: Axioms for algebraically closed fields of characteristic 0.
Pseudo-exponentiation

*Algebraic-exponential closedness* (existential closedness) takes the form:

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**ACF\(_0\):** Axioms for algebraically closed fields of characteristic 0.
Main Theorem  Given an uncountable cardinal $\lambda$, there is a unique, up to isomorphism, structure $K_{ex}$ of cardinality $\lambda$ satisfying

$$ACF_0 + EXP + SCH + EC + CC$$
Pseudo-exponentiation

**Main Theorem** Given an uncountable cardinal \( \lambda \), there is a unique, up to isomorphism, structure \( K_{ex} \) of cardinality \( \lambda \) satisfying

\[
ACF_0 + \text{EXP} + \text{SCH} + \text{EC} + \text{CC}
\]

**Conjecture** The field of complex numbers \( \mathbb{C}_{exp} \) is isomorphic to the unique field with exponentiation \( K_{ex} \) of cardinality \( 2^{\aleph_0} \).
Main Theorem  Given an uncountable cardinal $\lambda$, there is a unique, up to isomorphism, structure $K_{ex}$ of cardinality $\lambda$ satisfying

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Conjecture  The field of complex numbers $\mathbb{C}_{exp}$ is isomorphic to the unique field with exponentiation $K_{ex}$ of cardinality $2^{\aleph_0}$.

Equivalently: $\mathbb{C}_{exp}$ satisfies $SCH + EC$. 
Pseudo-exponentiation
The Main Theorem is a consequence of:

Theorem A

\[ \text{The } \omega, \omega(\mathbb{Q}) \text{-sentence ACF}_0 + \text{EXP} + \text{SCH} + \text{EC} + \text{CC} \text{ is axiomatising a quasi-minimal excellent class.} \]

Theorem B

(Essentially S. Shelah 1983)

A quasi-minimal excellent class is categorical in any uncountable cardinality.

The proof of Theorem A uses:

1. The Galois and Kummer theory.
2. The structure of the multiplicative group \( F^* \) for global fields \( F \).
3. The new fact (with M. Bays): Let \( L_1, \ldots, L_n \) be algebraically closed fields mutually linearly disjoint over their intersections. Then, for the multiplicative group of their composite, \( (L_1 \cdot \ldots \cdot L_n)^* \sim = L_1^* \cdot \ldots \cdot L_n^* \times A \), for a free abelian group \( A \).
Pseudo-exponentiation

The Main Theorem is a consequence of:

**Theorem A**  The $L_{\omega_1,\omega}(Q)$-sentence

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is axiomatising a **quasi-minimal excellent class**.
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The proof of Theorem A uses:

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3. The new fact (with M. Bays): Let $L_1, \ldots, L_n$ be algebraically closed fields *mutually linearly disjoint over their intersections*. Then, for the multiplicative group of their composite, 

$$(L_1 \cdot \ldots \cdot L_n)^* \cong L_1^* \cdot \ldots \cdot L_n^* \times A,$$

for a free abelian group $A$. 
Conclusion

Hrushovski’s counter-examples are not pathologies.
Generalities:

- Noetherian Zariski Geometry is an extension of Algebraic Geometry (into a non-commutative domain).
Lecture V

Generalities:

- Noetherian Zariski Geometry is an extension of Algebraic Geometry (into a non-commutative domain).
- Some interesting mathematics may lie outside the narrow context of Noetherian Zariski geometries.
Analytic Zariski geometries

Definition. We say that $\mathbf{M} = (\mathcal{M}, \mathcal{C})$ is a pre-analytic Zariski structure if:

- $\mathbf{M} = (\mathcal{M}, \mathcal{C})$ is a topological structure with good dimension notion.

- For every $S \subseteq \text{cl} \mathcal{U} \subseteq \text{op} \mathcal{M}^n$ there are at most countably many constructible irreducible sets $\mathcal{S}_i \subseteq \mathcal{M}^n$, $I \in \mathbb{N}$, with $S = \bigcup \mathcal{S}_i.$
Analytic Zariski geometries

Definition. We say that $\mathbf{M} = (M, C)$ is a pre-analytic Zariski structure if:

1. $\mathbf{M} = (M, C)$ is a topological structure with good dimension notion.
2. (case $\dim M = 1$) given $F \subseteq_{\text{cl}} V \subseteq_{\text{op}} M^{n+k}$ with the projection $\text{pr} : M^{n+k} \to M^n$ such that $\dim \text{pr} F = n$, there exists $D \subseteq_{\text{op}} M^n$ such that $D \subseteq \text{pr} F$. 
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- For every $S \subseteq_{cl} U \subseteq_{op} M^n$ there are at most countably many constructible irreducible sets $S_i \subseteq M^n$, $i \in \mathbb{N}$, with
  \[ S = \bigcup S_i. \]
Definition (continued) A pre-analytic Zariski $\mathbf{M}$ is said to be analytic if

- Given a subset $S \subseteq \text{cl} \ U \subseteq \text{op} \ M^n$ the natural number $U(S)$, (analytic rank) is well-defined by:

  1. $U(S) = 0$ iff $S = \emptyset$;
  2. $U(S) \leq k + 1$ iff there is a set $S' \subseteq \text{cl} S$ such that $U(S') \leq k$, and the set $S_0 = S \setminus S'$ is a countable union of irreducible closed subsets.
Definition (continued) A pre-analytic Zariski $\mathbf{M}$ is said to be analytic if

- Given a subset $S \subseteq_{\text{cl}} U \subseteq_{\text{op}} \mathbb{M}^n$ the natural number $u(S)$, (analytic rank) is well-defined by:
  1. $u(S) = 0$ iff $S = \emptyset$;
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A subset $S \subseteq_{\text{cl}} U \subseteq_{\text{op}} \mathbb{M}^n$ is said to be analytic if $u(S) = 1$. 
Let $\mathcal{M}$ be an analytic Zariski structure of dimension 1. We choose a large enough countable fragment $\mathcal{C}_0 \subseteq \mathcal{C}$ (including constants) closed under certain properties.
Let $M$ be an analytic Zariski structure of dimension 1. We choose a large enough countable fragment $C_0 \subseteq C$ (including constants) closed under certain properties.

**Theorem 1** Every $L_{\infty,\omega}(C_0)$-type realised in $M$ is equivalent to a type consisting of existential (first-order) formulas and the negations of existential formulas (non-elementary near-model-completeness).
Model theory of pre-analytic Zariski structures

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Theorem 2. There are only countably many $L_{\infty,\omega}(C_0)$-types realised in $\mathbf{M}$ (non-elementary $\omega$-stability).
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Model theory of analytic Zariski structures

How the proof works.
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For finite $X \subseteq M$ we define the $C_0$-predimension

$$\delta(X) = \min \{ \dim S : \bar{X} \in S, S \subseteq_{cl} U \subseteq_{op} M^n, S \text{ is } C_0\text{-definable} \}$$
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and dimension

$$d(X) = \min\{\delta(XY) : \text{finite } Y \subset M\}.$$
Model theory of analytic Zariski structures

How the proof works.

For finite $X \subseteq M$ we define the $C_0$-predimension

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and dimension

$$d(X) = \min\{\delta(XY) : \text{finite } Y \subset M\}.$$ 

For $X \subseteq M$ finite, we say that $X$ is **self-sufficient** and write $X \leq M$, if $d(X) = \delta(X)$. 
How the proof works.

**Lemma 1** For a projective $P \subseteq M^n$

$$\dim P = \max\{d(X) : \tilde{X} \in P\}.$$
How the proof works.

**Lemma 1** For a projective $P \subseteq M^n$

$$\dim P = \max \{d(X) : \vec{X} \in P\}.$$ 

**Lemma 2.** Given $X, X', XY$ all finite self-sufficient, suppose $X \equiv_{qftp} X'$. Then there is $Y'$ such that $XY \equiv_{qftp} X'Y'$.
Model theory of analytic Zariski structures

Pregeometry on $M$

Set, for finite $X \subseteq M$,

$$(X) = \{ y \in M : d(Xy) = d(X) \}.$$
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In other words, $M$ is quasi-minimal $\omega$-homogeneous over submodels.
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3. $z \in (X, y) \setminus (X) \Rightarrow y \in (X, z)$;
4. $(X)$ is countable for a countable $X$;
5. $Y \equiv \exists (X) Y' \Rightarrow$ exists an elementary monomorphism over $(X)$, $(XY) \rightarrow (XY')$. 

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Model theory of analytic Zariski structures

Is M excellent?
Is $M$ excellent?

**Fact.** For all *natural* analytic Zariski $M$, when the answer is known: yes.

**Theorem 4** Suppose $M$ is excellent. Then for every $\kappa > \text{card}M$ there is a (pre)analytic Zariski $M'$ of cardinality $\kappa$,

$$M \leq M'.$$
Is $M$ excellent?

**Fact.** For all *natural* analytic Zariski $M$, when the answer is known: *yes.*

**Theorem 4** Suppose $M$ is excellent. Then for every $\kappa > \text{card}M$ there is a (pre)analytic Zariski $M'$ of cardinality $\kappa$, 

$$M \leq M'.$$

This $M'$ is unique up to isomorphism.
Examples of analytic Zariski geometries

1. Abstract covers of the algebraic torus $K^*$, for an uncountable algebraically closed field $K$, any characteristic.
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2. Universal covers of complex abelian varieties in Gavrilovich’s language.
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3. Some structures obtained via Hrushovski construction, as pre-analytic structures.
Examples of analytic Zariski geometries

1. Abstract covers of the algebraic torus $K^*$, for an uncountable algebraically closed field $K$, any characteristic.
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3. Some structures obtained via Hrushovski construction, as pre-analytic structures.
4. Pseudo-exponentiation, as a pre-analytic structure (?)