# Approximation of structures: local and global

Draft

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## 1 Introduction

**1.1** This work along with some earlier ones cited below is motivated by attempts to understand the procedures of approximation in physics, in particular by the question of how infinite and continuous emerges from finite and discrete. It is an age-old question entertained by Greek philosophers and reformulated by Hilbert in his problem 6 where he proposed developing "mathematically the limiting process ... which leads from the atomistic view to the laws of motion of continua", see [SRaymond09].

**1.2** We continue the study of structural approximation initiated in [Z14]. In [Z25-2] and [Z25-3] we realised that along with its initial *global* version there might be another useful version which we identified in a few specific cases and called *local* approximation.

Here we furnish a mathematically rigorous definition of local approximation and show how it is applicable to some interesting classes of structures. This includes the approximation of the Lorentz-invariant Minkowski space-time of [Z25-2] by finite G-invariant lattices as well as the continuous model theory approximation of quantum mechanics from [Z25-3].

Also, addressing a question posed in [Z14], we show that local approximation of compact simple Lie groups by finite groups does exist, while [NST18] proved that a global approximation is not possible.

Our definition of local approximation reduces to the definition of the *local ultraproduct*, the ultraproduct in the class of *emerging-mertic* structures. This generalises the ultraproduct of metric structures. The construction is based on the idea that in some classes of algebraic structures such as groups or fields, in sufficiently large finite structures there is a length-of-words based metric which is well-defined "near the origin" but not necessarily globally. Under certain assumptions, in the first-order ultraproduct, the near-the-origin region will form an algebraic structure with a pseudo-metric, which then is reduced into a structure with a complete metric.

Our context includes both first order topological structures as in [Z14] and [Z25-1] and metric structures as in [BBHU08] as well as continuous logic structures as [Z25-3]. This becomes possible in the setting of *general structures* developed by Kiesler, [Kies20], which allows any predicates, not necessarily continuous, with values in the real interval $[0, 1]$. However, the role of the distance predicate is slightly different. In fact, our emerging metric is a generalisation of the pre-metric expansion of Kiesler.

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# 2 Main definitions

Our context includes both first order topological structures as in [Z14] and [Z25-1] and metric and continuous logic structures as in [BBHU08] and [Z25-3]. This becomes possible in the setting of *general structures* developed by Kiesler, [Kies20], which allows any predicates, not necessarily continuous, with values in the real interval $[0, 1]$.

**2.1 The formalism of general structures** (following [Kies20]). The space of truth values will be $[0, 1]$, with 0 representing truth. A vocabulary consists of a set of predicate symbols $P$ of finite arity, a set of function symbols $f$ of finite arity, and a set of constant symbols $c$. A general structure $\mathbf{M}$ consists of a vocabulary $V$, a non-empty universe set $M$, an element $c_M \in M$ for each constant symbol $c$, a mapping $P^M : M^n \to [0, 1]$ for each predicate symbol P of arity n, and a mapping $f^M : M^n \to M$ for each function symbol $f$ of arity $n$. A general structure determines a vocabulary, but does not determine a metric signature. The formulas are as in [BBHU08], with the connectives being all continuous functions from finite powers of $[0, 1]$ into $[0, 1]$, and the quantifiers $\sup x$, $\inf x$. The truth value of a formula $\varphi(\bar{x})$ at a tuple $\bar{a} \in M^n$ in a general structure $\mathbf{M}$ is an element of $[0, 1]$ denoted by $\varphi^{\mathbf{M}}(\bar{x})$. It is defined in the usual way by induction on the complexity of formulas.

We are going to apply these definitions in a many-sorted version, assuming that our sorts $S_m$, $m \in \mathbb{N}$, are nested $S_m \subseteq S_{m+1}$. The diameters of the sorts can grow which is not consistent with the distance predicate $\mathsf{d} : M^2 \to [0,1]$. This is easily remedied by allowing a separate distance predicate $\mathsf{d}_m$ for each sort $S_m$ and then postulating the scaling relation between $\mathsf{d}_m$.

**2.2 Global approximation of topological structures.** Recall the definition in [Z14] and [Z25-1].

Let $\mathbf{M}$, $\mathbf{M}_{i\in I}$ be topological $L$-structures in a coarse topology determined by $L$. The basic closed subsets of this topology are subsets defined by basic $L$-predicates.

Let $\mathcal{D}$ be an ultrafilter on $I$ and

$$^*\mathbf{M} = \prod\nolimits_{\mathcal{D}} \mathbf{M}_i$$

the first-order ultraproduct of $L$-structures. Then, following [Z14], $\mathbf{M}$ is (structurally) approximated by the family $\mathbf{M}_{i\in I}$ along the ultrafilter $\mathcal{D}$ on $I$ if there is a surjective $L$-homomorphism

$$\mathsf{lm} : {}^*\mathbf{M} \twoheadrightarrow \mathbf{M}. \tag{1}$$

In this paper we call such an approximation **global** and write (1) as

$$\mathsf{lm}^{\mathrm{glob}} \mathbf{M}_i = \mathbf{M} \tag{2}$$

Here and below we omit reference to the ultrafilter $\mathcal{D}$.

**2.3** In notation of 2.2, suppose the initial $L$ is extended to the vocabulary $L^+$ by unary predicates $S_m$, $m \in \mathbb{N}$, defining new basic closed sets in $\mathbf{M}$ such that

$$\bigwedge_{m\in\mathbb{N}} S_m \subseteq S_{m+1} \tag{3}$$

and a distance predicate $\mathsf{d} : M^2 \to \mathbb{R}$, so that the usual properties hold:

$$\mathsf{d}(x_1, x_2) \leq r_{12} \ \& \ \mathsf{d}(x_2, x_3) \leq r_{23} \to \mathsf{d}(x_1, x_3) \leq r_{12} + r_{23} \tag{4}$$

$$\bigwedge_{n\in\mathbb{N}} \mathsf{d}(x_1, x_2) \leq \frac{1}{n} \ \to \ x_1 = x_2 \tag{5}$$

and, for each basic $n$-ary function $L$-symbol $f(x_1...x_n)$, for each $m$ there is a number $k = k(m, f)$ such that

$$x_1...x_n \in S_m \to f(x_1...x_n) \in S_k \tag{6}$$

Also we assume that the finer topology on $\mathbf{M}$ is determined by the distance predicate. This means that:

$$x_1, x_2 \in S_m \to \mathsf{d}(x_1, x_2) \leq m \tag{7}$$

for each $\epsilon \in \mathbb{R}_{>0}$ there is $\delta \in \mathbb{R}_{>0}$ such that

$$x_1...x_n, y_1...y_n \in S_m \ \& \ \bigwedge_i \mathsf{d}(x_i, y_i) < \delta \to \mathsf{d}(f(x_1...x_n), f(y_1...y_n)) < \epsilon \tag{8}$$

and for each basic $n$-ary predicate $P$ a point $(x_1^0...x_n^0) \in S_m^n \setminus P$ there is $\delta \in \mathbb{R}_{>0}$ such that

$$\bigwedge_i \mathsf{d}(x_i^0, x_i) < \delta \to (x_1...x_n) \notin P \tag{9}$$

**2.4 Local ultraproduct and local approximation.** Let $\mathbf{M}_{i\in I}$ be a family of $L$-structures that can be expanded to $L^+$ structures satisfying (3). We say that this is a family with **emerging metric** if

$$\bigcup_{m\in\mathbb{N}} S_m(M_i) = M_i : \quad i \in I \tag{10}$$

and its first-order ultraproduct ${}^*\mathbf{M} = \prod_{\mathcal{D}} \mathbf{M}_i$ satisfies (4), (6), (7), (8) and (9). .

Define the local part of ${}^*\mathbf{M}$

$${}^*\mathbf{M}_{\text{loc}} = \prod\nolimits_{\mathcal{D}}^{\text{loc}} \mathbf{M}_i$$

to be the substructure of ${}^*\mathbf{M}$ with the universe

$${}^*M_{\text{loc}} := {}^*M \cap \bigcup_{m\in\mathbb{N}} S_m({}^*M).$$

The definition implies that for any two points $x_1, x_2$ in ${}^*\mathbf{M}_{\text{loc}}$ there is a standard natural number $m$ such that $\mathsf{d}(x_1, x_2) \leq m$.

The **local ultraproduct** of $\mathbf{M}_{i\in I}$ is the $L^+$-structure $\mathbf{M}_{\text{loc}}$ obtained as the quotient

$$\mathbf{M}_{\text{loc}} := {}^*\mathbf{M}_{\text{loc}}/\approx$$

where

$$x_1 \approx x_2 \Leftrightarrow \forall n \in \mathbb{N}\ \mathsf{d}(x_1, x_2) \leq \frac{1}{n}.$$

Since (4) - (9) holds in $\mathbf{M}_{\text{loc}}$. We obtain:

**2.5 Proposition.** $\mathbf{M}_{\text{loc}}$ *is a complete metric structure with L-predicates defining closed subsets and L-function symbols defining continuous operations on the metric space.*

*There is an L-homomorphism*

$$\mathsf{lm} : {}^*\mathbf{M}_{\text{loc}} \twoheadrightarrow \mathbf{M}_{\text{loc}} \tag{11}$$

*onto a metric structure* $\mathbf{M}_{\text{loc}}$ *which is the standard part map on values of distance.*

**2.6** Accordingly, in analogy with (2), we write

$$\mathsf{lm}^{\text{loc}}\mathbf{M}_i = \mathbf{M}_{\text{loc}} \tag{12}$$

local approximation by $\mathbf{M}_{i\in I}$, determined by a metric and the standard part map.

**2.7 Remark.** From the point of view represented in 2.2 the map $\mathsf{lm}$ in (11) is a partial homomorphism ${}^*\mathbf{M} \to \mathbf{M}$ and so, according to the general theory in [Z14], it can be extended to a global homomorphism $\mathsf{lm}$ as in (2).

Thus, *a local approximation could be extended to a global one.*

**2.8 Theorem.** *Let, for each $i \in I$, $\sigma_{a(i)}$ be a general structure formula in vocabulary $L^+$ with parameter $a(i) \in \mathbf{M}_i$ and quantifiers restricted to $S_m$. Then*

$$\mathsf{lm}_{\mathcal{D}}\, \sigma_{a(i)}^{\mathbf{M}_i} = \sigma_a^{\mathbf{M}_{\text{loc}}}, \quad a = \mathsf{lm}_{\mathcal{D}} a(i)$$

*where $\mathsf{lm}_{\mathcal{D}}$ is the ultraproduct limit of $\mathbf{M}_i$, $i \in I$ along the ultrafilter $\mathcal{D}$ in the sense of general structures [Kies20].*

**Proof.** This is a direct consequence of [Kies20], 2.2.2. $\square$

# 3 Fields and rings

**3.1** We may assume, using an argument in [Z25-1], that non-standard numbers $\mathcal{N}, \mathfrak{q} \in {}^*\mathbb{Z}_{>0}$ are such that there is a model $\mathbb{Z}^f$ of arithmetic

$$\mathbb{Z} \prec \mathbb{Z}^f \prec {}^*\mathbb{Z} \text{ and } \mathcal{N} \geq \mathfrak{q} > \mathbb{Z}^f \tag{13}$$

($\mathbb{Z}^f$ is the "feasible" part of ${}^*\mathbb{Z}$). Let $\mathfrak{l} \in \mathbb{Z}^f_{>0}$, and so $\mathfrak{l}^n < \mathfrak{q}$ for all $n \in \mathbb{N}$.

Let

$$\mathbb{Z}^f_{/\mathfrak{l}}(n) := \{k \in {}^*\mathbb{Z} : \ |k| < \mathfrak{l}^n\}, \ \ \mathbb{Z}^f_{/\mathfrak{l}} := \bigcup_{n\in\mathbb{N}} \mathbb{Z}^f_{/\mathfrak{l}}(n).$$

**3.2** Let

$$\mathrm{F} = {}^*\mathbb{Z}_{\mathfrak{q}} = \prod_{\mathcal{D}} \mathrm{F}_q$$

be a pseudofinite field, an ultraproduct of finite prime fields $\mathrm{F}_q$

By assumptions, there is an injective map, for all $n$,

$$i_F : \mathbb{Z}^f_{/\mathfrak{l}}(n) \subset \mathrm{F}; \quad k \mapsto k \bmod \mathfrak{q} \tag{14}$$

and we will assume that it is actual embedding.

Consider also the subsets of F, for $m \in \mathbb{N}$,

$$S_m(\mathrm{F}) := \{z \in \mathrm{F} : \ \exists k_1, k_2 \in \mathbb{Z}^f_{/\mathfrak{l}}(m) \ z = k_1 \cdot k_2^{-1} \ \& \ \frac{|k_1|}{|k_2|} \leq m\}$$

$$\mathrm{F}_{/\mathfrak{l}} := \bigcup_{m\in\mathbb{N}} S_m(\mathrm{F}). \tag{15}$$

**Claim.** $\mathrm{F}_{/\mathfrak{l}}$ is a subfield of F.

Indeed, $z_1, z_2 \in S_m(\mathrm{F})$ implies $z_1 \cdot z_2 \in S_{m^2}(\mathrm{F})$ and $z_1 + z_2 \in S_{2m}(\mathrm{F})$.

**3.3** For an element $z = z(k_1, k_2) \in S_m(\mathrm{F})$ as above define

$$||z|| := \mathrm{st}(\frac{|k_1|}{|k_2|}) \in \mathbb{R}_{\geq 0}$$

where $k_1, k_2$ is the minimal pair such that $z = k_1 \cdot k_2^{-1}$.

This is a $\mathbb{R}$-valued norm on the subfield $\mathrm{F}_{/\mathfrak{l}}$ of F. Define respectively

$$\mathsf{d}(z_1, z_2) = ||z_1 - z_2||.$$

The same calculation as in the proof of the claim in 3.2 show that the norm and the distance are well-defined for the infinite pseudo-finite F with $\mathfrak{l}$ as in 3.1.

In each finite field $\mathrm{F}_q$ the name $\mathfrak{l}$ corresponds to some (non-unique) element $\mathfrak{l}(q) < q$ so that $\mathfrak{l}$ in the ultraproduct corresponds to the sequence $\mathfrak{l}(q)$, $q \in \mathbb{N}$ prime.

For each $n \in \mathbb{N}$ the embedding (14) holds for almost all $q$ with respect to $\mathcal{D}$ and we can formally apply the above definitions to $\mathrm{F} = \mathrm{F}_q$ interpreting $\mathfrak{l}$ as $\mathfrak{l}(q)$. In this setting the union (15) is equivalent to a finite union and $\mathrm{F}_{/\mathfrak{l}} = \mathrm{F}$. Note that the distance satisfies (4) only for $x_1, x_2 \in S_m(\mathrm{F})$ with $\mathfrak{l}(q)^m$ sufficiently small compared to $q$.

**3.4 Proposition.** *Finite fields* $\mathrm{F}_q$ *locally approximate the field of real numbers:*

$$\mathsf{Im}^{\mathrm{loc}}\mathrm{F}_q = \mathbb{R}. \tag{16}$$

**Proof.** The local ultraproduct of finite fields by the construction is the image of $\mathrm{F}_{/\mathfrak{l}}$, which consists of elements $k_1 \cdot k_2^{-1}$ which represent non-standard finite rational numbers embedded into the first-order ultraproduct of fields. Factoring by the equivalence $\approx$ corresponds to application of the standard part map, which gives us $\mathbb{R}$. □

**Remark.** In [Z14], Theorem 5.2 states that the only structure globally approximated by finite fields $\mathrm{F}_q$ is the compactification $\bar{\mathrm{E}}$ of an algebraically closed field of characteritic 0. And if E is a field with metric, then $\mathrm{E} = \mathbb{C}$, the field of complex numbers. Thus, the local approximation (16) can be extended to a global one

$$\mathsf{Im}^{\mathrm{glob}}\mathrm{F}_q = \bar{\mathbb{C}} \tag{17}$$

and the limit object here is unique up to isomorphism.

Note that the construction of $\mathsf{Im}^{\mathrm{glob}}$ is different from $\mathsf{Im}$ in [Z25-1] as the latter uses a different emergent metric on $\mathrm{F}_q$.

**3.5 Rings.** Consider a pseudo-finite ring

$$\mathrm{K} = \mathrm{K}_{\mathcal{N}} = {}^*\mathbb{Z}_{\mathcal{N}} \cong \prod \mathbb{Z}_n,$$

the ultraproduct of finite residue rings $\mathbb{Z}_n$, as topological structure in Zariski topology.

We will assume that the $\mathfrak{q}|\mathcal{N}$ and thus there is a surjective ring homomorphism

$$s_{K,F} : \mathrm{K} \to \mathrm{F}; \quad k \mapsto k \bmod \mathfrak{q}.$$

On the other hand, like in (14) there an embedding of rings

$$i_K : \mathbb{Z}^f_{/\mathfrak{l}} \hookrightarrow \mathrm{K}$$

and this agrees with $i_F$ in such a way that $s_{K,F} \circ i_K = i_F$.

Define

$$K_{/\mathfrak{l}} := s^{-1}_{K,F} \mathrm{F}_{/\mathfrak{l}}$$

and use pseudo-metric $\mathsf{d} = \mathsf{d}_F$ on $\mathrm{F}_{/\mathfrak{l}}$ to define a pseudo-metric on $\mathrm{K}_{/\mathfrak{l}}$

$$\mathsf{d}_K(x, y) := \mathsf{d}_F(s_{K,F}(x), s_{K,F}(y)).$$

Set, for $m \in \mathbb{N}$,

$$\mathrm{K}_{/\mathfrak{l}}(m) := \{z \in \mathrm{K}_{/\mathfrak{l}} : \mathsf{d}(0, z) \leq m\}.$$

With these notations we can proceed with the same construction and arguments as in the case of fields and obtain

$$\mathsf{lm}^{\mathrm{loc}} \mathrm{K}_n = \mathbb{R}. \tag{18}$$

In the global setting we obtain again

$$\mathsf{lm}^{\mathrm{glob}} \mathrm{K}_n = \bar{\mathbb{C}} \tag{19}$$

due to the fact that $\mathsf{lm}^{\mathrm{loc}}$ filters through $\mathrm{K} \twoheadrightarrow \mathrm{F}$,

**3.6** Note that (18) is written [Z25-2] as $\mathrm{Lm}\, \mathrm{K}_{\mathrm{Loc}} = \mathbb{R}$ and was used therein to prove Theorem 3.7 (19), which in the current notation is

$$\mathsf{lm}^{\mathrm{loc}} \left( \mathbf{M}(\mathrm{K}_n), \mathrm{SL}(2, \mathrm{K}^{(2)}_n) \right) = (\mathbf{M}(\mathbb{R}), \mathrm{SL}(2, \mathbb{C})) \, .$$

Recalling that the centre $C$ of $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbf{M}(\mathbb{R})$ trivially and that $\mathrm{SL}(2, \mathbb{C})/C \cong \mathrm{SO}(1, 3)$, we establish the equivalent statement in terms of the Lorentz group $\mathrm{SO}(1, 3)$ :

$$\mathsf{lm}^{\mathrm{loc}} \left( \mathbf{M}(\mathrm{K}_n), \mathrm{SL}(2, \mathrm{K}^{(2)}_n)/C \right) = (\mathbf{M}(\mathbb{R}), \mathrm{SO}(1, 3)) \, .$$

where $C$ on the left-hand side is the 2-element centre of group $\mathrm{SL}(2, \mathrm{K}^{(2)}_n)$.

**3.7 Varieties over fields and rings.** Let $V \subseteq \mathrm{A}^n$ be an affine variety defined over $\mathbb{Z}$, $V(\mathrm{F}_q)$ its set of $\mathrm{F}_q$-points. We want to calculate $\mathsf{Im}^{\mathrm{loc}} V(\mathrm{F}_q)$, where $\mathsf{Im}^{\mathrm{loc}}$ on $\mathrm{F}_q$-points is the same as in (16).

**Proposition.**

$$\mathsf{Im}^{\mathrm{loc}} V(\mathrm{F}_q) = \overline{V(\mathbb{Q})} \subseteq V(\mathbb{R})$$

*the set of limit points of the set of* $\mathbb{Q}$*-ponits in* $V(\mathbb{R})$.

**Proof.** For the pseudofnite $\mathrm{F}_\mathsf{q} = \mathrm{F}$ we need to consider the image of $V(\mathrm{F}_{/\mathrm{I}})$. The argument in the proof of 3.4 tells us that this is the same as the image of $V({}^*\mathbb{Q}_{\mathrm{fin}})$ under the standard part map, which gives us $\overline{V(\mathbb{Q})}$. □

**3.8 Corollary.** With the same notation, for finite rings $\mathrm{K}_n$ as above,

$$\mathsf{Im}^{\mathrm{loc}} V(\mathrm{K}_n) = \overline{V(\mathbb{Q})} \subseteq V(\mathbb{R}).$$

**3.9 Simple Lie groups.**

1. $\mathrm{SO}_n(\mathbb{R})$, $n > 2$. It is well-known, see [Borel91], 18.2, that rational points form a dense subgroup $\mathrm{SO}_n(\mathbb{Q})$ in each such group. Using 3.7 we get local approximation by finite algebraic groups

$$\mathsf{Im}^{\mathrm{loc}} \mathrm{SO}_n(\mathrm{F}_q) = \mathrm{SO}_n(\mathbb{R}).$$

2. $\mathrm{SU}_n(\mathbb{C})$, $n \geq 2$, It follows from the general theory, see [Pl-Rap93], Chapter 6, that $\mathrm{SU}_n$ is rational over $\mathbb{Q}[i]$ and so $\mathrm{SU}_n(\mathbb{Q}[i])$ is dense in $\mathrm{SU}_n(\mathbb{C})$.

Consider the rings $\mathrm{F}_q^{(2)} = \mathrm{F}_q \times \mathrm{F}_q$ with operations

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2),\\ (a_1, b_1) \cdot (a_2, b_2) &= (a_1 \cdot a_2 - b_1 \cdot b_2, a_1 \cdot b_2 + b_1 \cdot a_2)\end{aligned}$$

It is clear that using the same emerging metric on $\mathrm{F}_q$ that

$$\mathsf{Im}^{\mathrm{loc}} \mathrm{F}_q^{(2)} = \mathbb{R}[i] = \mathbb{C}.$$

and so

$$\mathsf{Im}^{\mathrm{loc}} \mathrm{SU}_n(\mathrm{F}_q^{(2)}) = \mathrm{SU}_n(\mathbb{C}).$$

3. It was noted in [Z25-2] and above in that the Lorentz group $\mathrm{SO}^+(1,3)$ is locally approximated by finite groups.

Note that this is in contrast with the fact proved in [NST18] that the global approximation of simple Lie groups by finite groups is not possible.

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