Vacuum energy in cyclic universe
(Draft)

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1 Cyclic universe

1.1 Cyclic universe. The assumption of huge-finite[1] universe of the form $\mathbb{U}^4$ (instead of Minkowski $\mathbb{R}^4$), with $\mathbb{U}$ of the form $\mathbb{Z}_{\mathcal{N}} = \mathbb{Z}/\mathcal{N}\mathbb{Z}$ (the ring of residues of integer numbers modulo a highly divisible number $\mathcal{N}$) was considered in [Z25-1] and [Z25-2]. In particular, it was argued in [Z25-1] that because of cyclicity the appropriate numerical coordinate system should be a huge size finite field $\mathrm{F}_{\mathfrak{p}}$ ($\mathfrak{p}$ a prime number) or the residue ring $\mathrm{K} = \mathbb{Z}_{\mathcal{N}}$. More specifically, see also [Z25-2], $\mathbb{U}$ should be treated as a 1-dimensional K-module. And there is a useful abstractly defined map

$$\exp_{\mathfrak{p}} : \mathbb{U} \to \mathrm{F}_{\mathfrak{p}}$$

a homomorphism of the additive group of K onto the multiplicative group of $\mathrm{F}_{\mathfrak{p}}$, that is a good analogue of the classical exponentiation map.

This basic model allows to consider wave functions on $\mathbb{U}^n$ with values in $\mathrm{F}_{\mathfrak{p}}$ and Minkowski spacetime structure on $\mathbb{U}^4$ invariant under the action of a K-Lorentz group, along with other related structures. The main statement of the theory (see [Z25-1], [Z25-2], [Z25-3]) is that these discrete structures approximate, in a certain well-defined sense, the standard continuous models of physics.

In section 2 we work out some mathematics over the ring K. In section 3 we apply it to make a statement about the cosmological constant.

[1] formalised as **hyperfinite** in [Alb86] or pseudo-finite, in model theory slang

2 Physics over the ring K

2.1 A crucial point in the analysis of physics over such a cyclic universe is that it can be interpreted by the physicist in two ways - either by a **local** or by a **global** approximation. Conceptually, in local approximation the observer sees only the ininital interval of a line coordinatised by K, and in global approximation one sees the full cycle and K with its algebraic structure as a whole. The definition and a detailed mathematical analysis of these notions are in [Z26] but in order to understand the claims below it will suffice to quote the following:

Fact ([Z26]). Let K be a huge-finite field or ring. Then:

- a global approximation of K is $\bar{\mathbb{C}}$, the field of complex numbers compactified by adding ∞, written

$$\mathsf{Im}^{\mathrm{glob}}(\mathrm{K}) = \bar{\mathbb{C}}$$

- a local approximation of K is $\mathbb{R}$, the field of real numbers, written

$$\mathsf{Im}^{\mathrm{loc}}(\mathrm{K}) = \mathbb{R}.$$

2.2 Numerical system. Once our universe is assumed huge-finite and cyclic, with the underlying 1-dim cycle K of length $\mathcal{N}$, there is some absolute maximum of any *linear* units (length, time, energy,...) in any system. We assume that this maximum, counted in some absolute units, to be $\mathcal{N}$.

Example. Suppose at each point p of a set $\Pi(\mathrm{K})$ underlying a physical system Π there is a particle of energy E_p. How do we estimate the energy of the whole system? We suggest that the natural count is

$$E = \sum_{p \in \Pi(K)} E_p \mod \mathcal{N}$$

because any additions along $\{0, 1, 2, \dots \mathcal{N} - 1\}$ if it reaches the end value has to cyclically start from the beginning. In the addendum, section 5 we support this suggestion by explaining that the broadly accepted method of zeta-regularisation can be reduced to summation modulo $\mathcal{N}$.

But note that if the system is relatively small, then we would expect $\sum_{p \in \Pi(K)} E_p << \mathcal{N}$, that is the calculations can be actually carried out in usual arithmetic, that is locally.

The following is going to be useful.

2.3 Lemma. Let $m \in \mathrm{K}$ is such that $\mathsf{lm}^{\mathrm{loc}}(m) = \mu \in \mathbb{R}_{>0}$, and

$$\Theta(\mathrm{K}) = \{(\omega, \mathbf{k}) \in \mathrm{K}^4 : \ \omega^2 = \mathbf{k}^2 + m^2\},$$

$$\Sigma(\mathrm{K}) := \{\mathbf{k} = (k_1, k_2, k_3) \in \mathrm{K}^3 : \ \exists \omega \in \mathrm{K} \ \omega^2 = \mathbf{k}^2 + m^2\}$$

where $\mathbf{k}^2 = k_1^2 + k_2^2 + k_3^2 + m^2$. And let

$$\Omega(\mathrm{K}) = \{\omega \in \mathrm{K} : \exists \mathbf{k} \in \mathrm{K}^3 \ \omega^2 = \mathbf{k}^2 + m^2\}$$

Then

$$\mathsf{lm}^{\mathrm{loc}}\Sigma(\mathrm{K}) = \mathbb{R}^3 \text{ and } \mathsf{lm}^{\mathrm{loc}}\Omega(\mathrm{K}) = \mathbb{R}_{\geq\mu} \cup -\mathbb{R}_{\geq\mu}. \tag{1}$$

We may assume that in local setting only positive ω are admissible then

$$\mathsf{lm}^{\mathrm{loc}}\Omega(\mathrm{K}) = \mathbb{R}_{\geq\mu}.$$

Proof. Clearly, Σ and Ω are projections of Θ,

$$\Omega(\mathrm{K}) = \mathrm{pr}_1\Theta(\mathrm{K}), \quad \Sigma(\mathrm{K}) = \mathrm{pr}_2\Theta(\mathrm{K}).$$

The approximation map

$$\mathsf{lm}^{\mathrm{loc}} : \ \Theta(\mathrm{K}) \to \Theta(\mathbb{R})$$

is by definition a map with domain $\Theta(\mathrm{K}_{\mathrm{loc}}) \subset \Theta(\mathrm{K})$ which preserves the algebraic relations (Zariki topology) and satisfies $\mathsf{lm}^{\mathrm{loc}}(\mathrm{K}_{\mathrm{loc}}) = \mathbb{R}$ as in 2.1. Such a map commutes with projection maps and hence (1). $\square$

Commentary. $\Sigma(\mathrm{K})$ is the set of 3-momenta $\mathbf{k}$ over which there can be installed a particle of frequency ω such that $\omega^2 = \mathbf{k}^2 + m^2$. The lemma asserts that in local approximation this set looks like $\mathbb{R}^3$ and the set $\Omega(\mathrm{K})$ of possible ω looks like $\mathbb{R}_{\geq\mu}$ (in union with its negative mirror).

2.4 Now we want to introduce a map

$$\mathbf{k} \mapsto \omega_{\mathbf{k}}$$

such that

$$\{(\omega_{\mathbf{k}}, \mathbf{k}) : \ \mathbf{k} \in \Sigma(\mathrm{K})\} \subseteq \Theta(\mathrm{K})$$

which locally behaves like $\mathbf{k} \mapsto \sqrt{\mathbf{k}^2 + m^2}$. We will require that when moving along axis of K^3 the triple $\mathbf{k} = (k_1, k_2, k_3)$ reaches $(\mathcal{N} - k_1, \mathcal{N} - k_2, \mathcal{N} - k_3) = -\mathbf{k}$ we get

$$\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}} \tag{2}$$

This is clearly achievable when $\mathbf{k} \neq -\mathbf{k}$. In case $\mathbf{k} = -\mathbf{k}$ the components k_i of $\mathbf{k} = \mathbf{k}_0$ are either 0 or $-\frac{\mathcal{N}}{2}$ and $k_i^2 = 0$ ($\frac{\mathcal{N}}{4}$ is an integer because $\mathcal{N}$ is highly divisible). In each of the 8 cases

$$\mathbf{k}_0^2 + m^2 = m^2$$

and we assume

$$\omega_{\mathbf{k}_0} = m \tag{3}$$

2.5 As a corollary we have a well-defined notion of the sum

$$\mathrm{E}_0^{\mathrm{tot}}(\mathrm{K}) = \sum_{\mathbf{k} \in \Sigma(\mathrm{K})} \omega_{\mathbf{k}} = 8m \tag{4}$$

the value of which $8m << \mathcal{N}$.

Commentary. It is essential that the summation formula (4) is interpreted as global and K, respectively, is seen as approximating $\bar{\mathbb{C}}$. In this interpretation $\sqrt{\mathbf{k}^2 + m^2}$ is a square root of a complex number and it is natural to assign it both $\pm\sqrt{\mathbf{k}^2 + m^2}$ values.

2.6 Let $\sigma(\mathrm{K})$ be the number such that

$$0 \leq \sigma(\mathrm{K}) < \mathcal{N} \ \ \& \ \ \sigma(\mathrm{K}) = |\Sigma(\mathrm{K})| \bmod \mathcal{N}$$

Proposition.

$$|\Sigma(\mathrm{K})| = \mathrm{O}(\mathcal{N}^3) \tag{5}$$

$$\sigma(\mathrm{K}) = 0. \tag{6}$$

Proof.

3 Vacuum energy

3.1 Vacuum energy for a free global field. For a free field, classically, in convenient units, vacuum energy density

$$\rho_0(\mathbb{R}) = \int_{\mathbf{k} \in \mathbb{R}^3} d\mathbf{k}\, \sqrt{\mathbf{k}^2 + m^2} \tag{7}$$

We replace the definition, in accordance with 2.1 and 2.3 by the density

$$\rho_0(\mathrm{K}) := \frac{\mathrm{E}_0^{\mathrm{tot}}}{|\Sigma(\mathrm{K})|}$$

distributed over the 3-space $\Sigma(\mathrm{K})$

By 2.5

$$\rho_0(\mathrm{K}) = \mathrm{O}(\frac{1}{\mathcal{N}^3}). \tag{8}$$

Let us now look at the possible contribution of Higgs field interaction term ϕ^4. Its groundstate density is a constant $= v$ and we assume that $v \in \mathrm{K}$ and it corresponds to the contribution to energy at a given point $\mathbf{k}$, the whole contribution is

$$|\Sigma(\mathrm{K})| \cdot v = \sigma(\mathrm{K}) \cdot v \mod \mathcal{N}$$

treated as an element of K, that is modulo $\mathcal{N}$.

Thus, by (6) the contribution of interaction to the density is

$$\rho_{\mathrm{int}} = 0$$

and the final estimate

$$\rho_{\mathrm{vac}} = \rho_0(\mathrm{K}) + \rho_{\mathrm{int}}(\mathrm{K}) = \mathrm{O}(\frac{1}{\mathcal{N}^3}).$$

3.2 Now we address the question why it is consistent with the direct calculation, for example for the phonon field, supproted by experimental data, which returns much bigger values.

Phonon field vacuum energy Following [LancBl2014] and others the groundstate energy of periodic 1-dim field is

$$C \cdot \sum_{k=-N}^{N-1} |\sin k\frac{\pi}{N}| = 2C \cdot \sum_{k=0}^{N-1} \sin k\frac{\pi}{N} = 2C \cot\frac{\pi}{2N} \sim 4CN. \tag{9}$$

The answer is in the difference between **local** and **global** versions of models of the free field. The general theory is addressed in [Z26]. The phonon field by its physical nature is local, of some limited size in spacetime. Then the parameter N for this field is much much smaller than $\mathcal{N}$, the parameter for global fields. One can say that operations with numbers $k < N$ never reaches $\mathcal{N}$ and so these numbers should be treated like usual numbers (i.e. feasible numbers of [Z26] and [Z25-1]). This explains the different method of summation.

4 Proofs

We are going to slightly simplify the equation defining Θ to

$$x^2 + y^2 + z^2 + 1 = w^2$$

Note that over a field of characteristic 2 this is equivalent to a linear equation.

4.1 Lemma. Over any field F of characteristic $\neq 2$ the followig rational map

$$\Psi : \mathrm{F}^3 \to \Theta(\mathrm{F}); \quad (u, v, t) \mapsto (x, y, z, w)$$

$$x = \frac{2u}{1 - r^2}, \; y = \frac{2v}{1 - r^2}, \; z = \frac{2t}{1 - r^2}, \; w = \frac{1 + r^2}{1 - r^2}. \tag{10}$$

where $r^2 = u^2 + v^2 + t^2$ is dominant.

Proof. Check that (x, y, z, w) lies on the variety:

$$x^2 + y^2 + z^2 = \frac{4(u^2 + v^2 + t^2)}{(1 - r^2)^2} = \frac{4r^2}{(1 - r^2)^2}.$$

$$w^2 = \frac{(1 + r^2)^2}{(1 - r^2)^2}.$$

- So

$$w^2 - (x^2 + y^2 + z^2) = \frac{(1 + r^2)^2 - 4r^2}{(1 - r^2)^2} = \frac{1 - 2r^2 + r^4}{(1 - r^2)^2} = \frac{(1 - r^2)^2}{(1 - r^2)^2} = 1.$$

Thus

$$x^2 + y^2 + z^2 + 1 = w^2$$

holds identically in u, v, t wherever the denominators are defined (i.e. $r^2 \neq 1$). $\square$

4.2 Lemma. For any $(r.u, v, t)$ such that $1 \neq r^2 = u^2 + v^2 + t^2$ there is unique $(x, y, z) \in \Sigma(\mathrm{F})$, $(x, y, z, w) \in \Theta(\mathrm{F})$.

Proof. Immediate from (10). $\square$

4.3 Lemma. The statements 4.1 and 4.2 could be extended to the ring K.

Proof. Note that

$$\mathrm{K} = \prod_{p^n} \mathbb{Z}_{p^n}.$$

Hensel's Lemma allows to lift the solutions modulo prime p to unique solutions modulo p^n for odd p. For $p = 2$ the statement is trivial since $x^2 = x \mod 2$. □

This implies (5).

4.4 Lemma. $|\Sigma(\mathrm{K})| = 0 \mod \mathcal{N}$.

Proof. This is just a more accurate calculation of $|\Sigma(\mathrm{K})|$ following 4.3.

5 Zeta regularization

Our aim here is to show that in some standard situations in physics the use of zeta-regularisation in evaluating infinite sums can be replaced by evaluating actual sums and products modulo $\mathfrak{p}$. In other words, not only the assumption of numerical system modulo $\mathcal{N}$ is already acceptable in physics, but it also explains and justifies methods already in use.

5.1 Let

$$S_k(N) = \sum_{n=1}^{N} n^k \quad \text{and} \quad S_k^\dagger(N) = \sum_{n=N+1}^{2N} n^k$$

We are interested in $N = \frac{\mathfrak{p}-1}{2}$ for some (large) prime $\mathfrak{p}$ and in

$$S_k(\frac{\mathfrak{p}-1}{2}) \mod \mathfrak{p}$$

and its complement

$$S_k^\dagger(\frac{\mathfrak{p}-1}{2}) \mod \mathfrak{p}$$

5.2 Lemma.

$$S_k(\frac{\mathfrak{p}-1}{2}) + S_k^\dagger(\frac{\mathfrak{p}-1}{2}) = 0 \mod \mathfrak{p}$$

and

$$S_{2m}(\frac{\mathfrak{p}-1}{2}) = 0 \mod \mathfrak{p}$$

Proof. It is easy to see that

$$S_k(\frac{\mathfrak{p}-1}{2}) + S_k^\dagger(\frac{\mathfrak{p}-1}{2}) = S_k(\mathfrak{p}-1) = 0 \mod \mathfrak{p}.$$

□

Note that by the Faulhaber formula, $S_k(N)$ is a polynomial in variable N of degree $k+1$ over $\mathbb{Q}$.

From now on we identify n, N and $S(N)$ with respecive elements with the field $\mathrm{F}_\mathfrak{p}$. In particular, $N = -\frac{1}{2}$ in $\mathrm{F}_\mathfrak{p}$ and

$$S_k(-\frac{1}{2}) = 1^k + 2^k + \ldots (\frac{\mathfrak{p}-1}{2})^k$$

and

$$S^{\dagger}_k(-\frac{1}{2}) = (\frac{1}{2})^k + (\frac{1}{2}+1)^k + (\frac{1}{2}+2)^k + \ldots + (\mathfrak{p}-1)^k$$

$$S^{\dagger}_k(-\frac{1}{2}) = -S_k(-\frac{1}{2}) \tag{11}$$

Hurwitz zeta. Recall that Hurwitz zeta function $\zeta(-s,a)$ is the analytic continuation of the function

$$\zeta(-s,a) = a^s + (a+1)^s + (a+2)^s + \ldots + \ldots$$

5.3 Proposition. For each non-negative integer m

$$S^{\dagger}_m(-\frac{1}{2}) = \zeta(-m, 1/2). \tag{12}$$

Proof. The Hurwitz zeta satisfies

$$\zeta(-m,a) = -\frac{B_{m+1}(a)}{m+1},$$

where $B_{m+1}(x)$ is the $(m+1)$-st Bernoulli polynomial.

On the other hand, the Faulhaber sum has the Bernoulli-polynomial form

$$S_m(N) = \sum_{n=1}^{N} n^m = \frac{B_{m+1}(N+1) - B_{m+1}(0)}{m+1},$$

valid as a polynomial identity in N.

Evaluating at $N = -\frac{1}{2}$

$$S_m\big(-\tfrac{1}{2}\big) = \frac{B_{m+1}(\frac{1}{2}) - B_{m+1}(0)}{m+1} = -\zeta(-m, \tfrac{1}{2}) + \zeta(-m, 1).$$

Since $\zeta(-m,1) = \zeta(-m)$, this is

$$S_m\big(-\tfrac{1}{2}\big) = \zeta(-m) - \zeta(-m, \tfrac{1}{2}).$$

Now use the explicit Bernoulli value at $x = \frac{1}{2}$:

$$B_{m+1}\big(\tfrac{1}{2}\big) = \big(2^{-m} - 2\big) B_{m+1},$$

so

$$\zeta(-m, \tfrac{1}{2}) = -\frac{B_{m+1}(\frac{1}{2})}{m+1} = -\frac{(2^{-m}-2)B_{m+1}}{m+1},$$

and

$$S_m\left(-\tfrac{1}{2}\right) = \frac{(2^{-m}-2)B_{m+1}}{m+1}.$$

Comparing the last two formulas gives the clean identity

$$S_m\left(-\tfrac{1}{2}\right) = -\,\zeta\left(-m, \tfrac{1}{2}\right).$$

and thus we get (12) from (11). □

5.4 Determinans of infinite-dimensional operators Let $\{\lambda_n : n = 0, 1, \ldots\}$ be eigenvalues of an infinite dimensional operator Λ and

$$\zeta_\Lambda(s) = \sum \lambda_n^{-s}$$

the respective ζ-function. In some physics applications, e.g. [Haw77] one considers $\mathrm{e}^{\zeta'_\Lambda(0)}$ to give a meaning to $\det \Lambda$ defining

$$\det \Lambda = \prod \lambda_n := \mathrm{e}^{\zeta'_\Lambda(0)} \tag{13}$$

in Hawking black hole entropy calculation. It is essential in our argument that the physical assumptions here are global, that is at the spacetime scale.

In [Haw77] λ_n depend on temperature parameter $t = \beta^{-1}$ and up to a good approximation[2] have the form

$$\lambda_n = (tn)^2 y_n$$

where $y_n = 1 + O(\frac{1}{n^2})$ and thus

$$Y = \prod_{n\in\mathbb{N}} y_n$$

is well-defined in the standard analysis.

Thus it remains to evaluate

$$P_2(t) = \prod_{n\in\mathbb{N}} (tn)^2.$$

The main result using the above zeta-regularisation method is that $P_2(t)$ does not depend on t and is a finite non-zero.

[2]To be explained

5.5 We use the calculation modulo in $\mathrm{F}_{\mathfrak{p}}$ and set, for $t \in \mathrm{F}_{\mathfrak{p}}^{\times}$ the group of order $\mathfrak{p}-1$,

$$P_2^*(t) := \prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} (tn)^2 = t^{\mathfrak{p}-1} \prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} n^2 = \prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} n^2.$$

Thus, it remains to evaluate the product of n^2. Note that similarly to (11) and since $\frac{\mathfrak{p}-1}{2}$ is even

$$\prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} n = \prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} (n + \frac{\mathfrak{p}-1}{2})$$

and thus

$$\prod_{n=1}^{\frac{\mathfrak{p}-1}{2}} n^2 = \prod_{n=1}^{\mathfrak{p}-1} n = 1.$$

This establishes the value for $P_2(t)$ and eventually for $\det \Lambda$.

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