

Axioms of Quantum Mechanics in light of Continuous Model Theory

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Abstract

The aim of this note is to recast somewhat informal axiom system of quantum mechanics used by physicists (Dirac calculus) in the language of Continuous Logic.

We note an analogy between Tarski's notion of cylindric algebras - developed as a tool of algebraisation of first order logic - and Hilbert spaces which can serve the same purpose for continuous logic of physics.

1 Introduction

1.1 The axiomatic formulation of quantum mechanics was introduced by Paul Dirac in 1930 [1] through a description of Hilbert space, and later developed with greater mathematical rigor in a monograph of 1932 by John von Neumann. Since 1930, Dirac went through several rewritings and new editions to refine his calculus to a level he considered satisfactory. In the 1950s the theory settled with the notion of *rigged Hilbert space*, or Gel'fand triple. Modern books present Dirac's axioms in a succinct form, often omitting much of the technical detail.

In section 2 we survey the axioms of quantum mechanics following [2]. Readers with a background in logic will notice that what physicists call "axioms" is very far from what is a conventional set of axioms in a formal language, even in its early form such as Hilbert's axiomatisation of geometry [3].

In section 3 we argue that the language that Dirac introduced is, in fact, the language of *continuous logic*. We then go further and explain that Dirac's axiomatisation has chosen the formalism termed *algebraic logic* as exemplified e.g. by A.Tarski's notion of cylindric algebras [6]. In fact, Hilbert spaces can be seen as a continuous model theory version of cylindric algebras.

The main contribution of the current paper is the introduction of an analogue of cylindric algebra $\mathfrak{C}(\mathbf{M})$ for general structures \mathbf{M} of continuous logic. Theorem 4.4 proves that under natural assumptions on \mathbf{M} , $\mathfrak{C}(\mathbf{M})$ takes the form of a rigged Hilbert space. Moreover, \mathbf{M} can be recovered from $\mathfrak{C}(\mathbf{M})$.

1.2 Continuous logic and continuous model theory were introduced in the monograph [4] in the 1960s and have since been developed and further generalised for various applications. For readers without a background in continuous model theory the article of E.Hushovski [9] outlines a philosophy behind the mathematical formalism.

The link between physics formalism and continuous logic was proposed and initially explored by the present author in [7]. Paper [14] is devoted to a rigorous interpretation of Dirac - von Neumann axioms as a system of axioms of continuous logic and the study of its models.

2 Dirac's calculus and axiomatisation of quantum mechanics

Below we reproduce a slightly edited version of axioms as presented in [2], 6.3.

2.1 Axiom 1. The state of a quantum system is described by a vector $|\psi\rangle$ belonging to a complex Hilbert space \mathcal{H} . This state is usually called "ket ψ ". A complex Hilbert space \mathcal{H} is a vector space, which can be finite dimensional or infinite dimensional, equipped with the complex scalar product (also called inner product) $\langle\psi|\psi'\rangle$ between any pair of states $|\psi\rangle, |\psi'\rangle$ in \mathcal{H} . The norm, or modulus, of a generic vector $|\psi\rangle \in \mathcal{H}$ is defined as

$$\|\psi\| = |\langle\psi|\psi\rangle|$$

and usually $|\psi\rangle$ is normalized to one, i.e. $\|\psi\| = 1$. The symbol $\langle\psi|$ which appears in the definition of the norm is called "bra ψ " and it can be interpreted

as the function

$$\langle \psi | : \mathcal{H} \rightarrow \mathbb{C}.$$

For any $|\psi'\rangle \in \mathcal{H}$ this function gives a complex number $\langle \psi | \psi' \rangle$ obtained as scalar product of $|\psi\rangle$ and $|\psi'\rangle$. In a complex Hilbert space \mathcal{H} it exists a set of basis vectors $|\phi_\alpha\rangle$ which are orthonormal, i.e. $\langle \phi_\alpha | \phi_\beta \rangle = \delta(\alpha - \beta)$, and such that

$$|\psi\rangle = \sum_{\alpha} c_{\alpha} |\phi_{\alpha}\rangle \quad (1)$$

for any $|\psi\rangle$, where the coefficients c_{α} belong to \mathbb{C} .

Axiom 2. Any observable (measurable quantity) of a quantum system is described by a self-adjoint linear operator $F : \mathcal{H} \rightarrow \mathcal{H}$ acting on the Hilbert space of state vectors.

For any classical observable F it exists a corresponding quantum observable F .

Axioms 3. The possible measurable values of an observable F are its eigenvalues f , such that

$$F|f\rangle = f|f\rangle$$

with $|f\rangle$ the corresponding eigenstate. The observable $|f\rangle$ admits the spectral resolution

$$F = \sum_f f |f\rangle \langle f| \quad (2)$$

where $\{|f\rangle\}$ is the set of orthonormal eigenstates of F , and the mathematical object $\langle f|$, called “bra of f ”, is a linear map that maps into the complex number. This also satisfy the identity

$$\sum_f |f\rangle \langle f| = \mathbf{I}.$$

Axiom 4. The probability P of finding the state $|\psi\rangle$ in the state $|f\rangle$ (both of norm 1) is given by

$$P = |\langle f | \psi \rangle|^2$$

This probability P is also the probability of measuring the value f of the observable F when the system is in the quantum state $|\psi\rangle$.

Axiom 5. The time evolution of states and observables of a quantum system with Hamiltonian H is determined by the unitary operator

$$K^t := \exp(-iHt/\hbar)$$

, such that $|\psi(t)\rangle = K^t|\psi\rangle$ is the time-evolved state $|\psi\rangle$.

2.2 Now we make several comments on the axioms.

The term ‘‘Hilbert space’’ here should actually be read as *the rigged Hilbert space* (see [10]), equivalently, a *Gel'fand triple*. It differs from the standard definition by accommodating along with a Hilbert space \mathcal{H} both a subspace Φ of *test-functions* and the dual space Φ^* with

$$\Phi \subseteq \mathcal{H} \subseteq \Phi^*.$$

The summation formulas like (1) and (2) are presented in a form of an integral if the family $|\psi_\alpha\rangle$ is continuous but seems natural in the summation form when α runs in the discrete spectrum of an operator.

2.3 Remark. Rigged Hilbert spaces provide a powerful mathematical framework to extend quantum mechanics, allowing distributions and generalized eigenfunctions to be rigorously handled. However, it is still contested whether the condition of completeness of \mathcal{H} is necessary. It is almost generally accepted that not every element corresponds to a physically realisable state – some are purely mathematical artifacts, see e.g. [11].

In the more general context of quantum field theories Wightman axioms explicitly postulate that physically meaningful part of the rigged Hilbert space \mathcal{H} is a dense subset $\mathcal{D} \subset \mathcal{H}$.

3 Algebraisation of Logic

3.1 Algebraisation of first order logic and cylindric algebras

The axiomatic description of quantum mechanical theory in the form of rigged Hilbert space may be quite confusing from the logician point of view – there are no logical sentences which can be called axioms. What Axioms 1 – 5 render instead is the topological-algebraic structure of a Hilbert space with operators.

Recall now the *algebraisation of logic* approach, perhaps less popular among model theorists nowadays, versions of which were introduced by A.Lindenbaum, A.Tarski, P.Halmos for the first order setting.

It is quite natural to see the Hilbert space formalism as the form of algebraic logic in the context of the continuous logic.

Recall that, given a first order structure \mathcal{A} in a language \mathcal{L} one can associate with it the cylindric algebra \mathfrak{CA} as follows:

Let, for distinct $i_1, \dots, i_n \in \mathbb{N}$, F_{i_1, \dots, i_n} be the Lindenbaum algebra of \mathcal{L} -formulas in variables x_{i_1}, \dots, x_{i_n} up to equivalence in \mathcal{A} . There is a natural embedding $F_X \subset F_{X'}$, for sets of variables $X \subset X'$. Respectively one defines the Boolean algebra

$$\mathcal{F} := \bigcup_{i_1, \dots, i_n} F_{i_1, \dots, i_n}$$

Now introduce, for each i_k , the quantifier

$$\exists x_{i_k} : F_{X'} \rightarrow F_X$$

for each X and X' such that X' differs from X by variable x_{i_k} .

Cylindric algebra \mathfrak{CA} is the Boolean algebra \mathcal{F} equipped with quantification operators $\exists x_{i_k}$.

The structure \mathcal{A} is an interpretation of \mathfrak{CA} .

3.2 Basic Theorem on Cylindric Algebras (see e.g. [6])

Let \mathcal{A} and \mathcal{B} be two structures in the same first-order language, and $\mathfrak{CA}, \mathfrak{CB}$ the respective cylindric algebras.

Then \mathcal{A} is elementarily equivalent to \mathcal{B} iff $\mathfrak{CA} \cong \mathfrak{CB}$, where the isomorphism identifies sets definable by the same formulas.

4 Algebraic Continuous Logic

In this section we discuss how the above axioms may be interpreted in the framework of Continuous Logic (CL) and Continuous Model Theory.

In general terms, the language of CL consists of predicate symbols, function symbols, a collection of “connectives” continuous functions $\mathbb{C}^n \rightarrow \mathbb{C}$ and “quantifiers”, which are continuous transformations of predicates. Any language contains at list one predicate - the binary predicate of distance $\text{dist}(x_1, x_2)$. A basic CL formula is built from predicate symbols using connectives and quantifiers.

An interpretation begins with a choice of a universe Ω , which is a metric space, sometimes with a measure. An n -ary predicate symbol ψ is interpreted as a uniformly continuous map

$$\psi : \Omega^n \rightarrow D_\psi \subset \mathbb{C},$$

where D_ψ is a compact subset of \mathbb{C} . If Ω is unbounded one represents

$$\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k$$

a union of nested family of metric subspaces of diameter k , and replaces $\text{dist}(x_1, x_2)$ with the collection $\text{dist}_k(x_1, x_2)$, $k \in \mathbb{N}$, restrictions of $\text{dist}(x_1, x_2)$ to Ω_k .

m -ary function symbols f correspond to uniformly continuous maps

$$f : \Omega^m \rightarrow \Omega$$

and are used in the construction of terms and formuli in the same way as in first order logic.

If $\alpha : \mathbb{C}^n \rightarrow \mathbb{C}$ is a connective and $\varphi_1, \dots, \varphi_n$ are formulas, then the formula $\alpha(\varphi_1, \dots, \varphi_n)$ is interpreted as the composition of the maps defined by the φ_i with α .

Quantifiers are interpreted as transformations of formulas in $n + 1$ variables into formulas in n variables. Uniform continuity moduli ensure uniformity of interpretation across structures.

A structure in continuous model theory consists of a universe Ω together with interpretations of the predicate and function symbols. Definable sets are obtained not only by CL formulas but also as limits of families of formula-definable sets; see [4], [5], [13].

4.1 Spaces of predicates Let Ω be a complete metric space and \mathbf{M} a continuous structure with universe Ω and basic n -ary predicates

$$\psi : \Omega^n \rightarrow D_\psi.$$

Since $D_\psi \subset \mathbb{C}$ is compact the following definition of a norm of a predicate ψ

$$\|\psi\| = \sup_{x \in \Omega^n} |\psi(x)|$$

makes sense.

The following is easy but the author have failed to find a reference to any earlier formulation.

4.2 Proposition. *Let Ω and \mathbf{M} be as in 4.1.*

The set of definable predicates on Ω^n is a Banach space $\mathcal{B}(\Omega^n)$ over \mathbb{C} with regards to the norm.

Let

$$f : \Omega^n \mapsto \Omega^m$$

be a definable uniformly continuous map. Then

$$F : \psi \mapsto \psi \circ f; (\psi \circ f)(\bar{x}) = \psi(f(\bar{x}))$$

is a homomorphism of Banach spaces

$$F : \mathcal{B}(\Omega^m) \rightarrow \mathcal{B}(\Omega^n).$$

In particular, if $f : \Omega^n \mapsto \Omega^n$, F is a linear norm preserving operator on $\mathcal{B}(\Omega^n)$.

Proof. By the general continuous model theory definable predicates are uniformly continuous and have bounded codomain. Hence finite linear combinations of definable predicates are definable and have a finite norm and a Cauchy sequence of definable predicates has a limit which is uniformly continuous and definable.

The statement on F is a direct consequence of assumptions. \square

4.3 Continuous linear functionals on $\mathcal{B}(\Omega^n)$.

Fact 1. (Riesz representation theorem). *Every continuous linear functional ϕ on a space of continuous functions comes from integration against a measure:*

$$\phi : \psi \mapsto \int \psi d\mu_\phi, \quad \mu_\phi \in M(\Omega^n) \tag{3}$$

where $M(\Omega^n)$ is the space of complex (or finite signed) regular Borel measures on Ω^n .

Fact 2. Finite atomic measures are weak-* dense in $M(\Omega)$ for compact Ω .

In particular, given μ , for any system of $1/N$ -dense lattices $\Omega_N \subset \Omega$ there is a system μ_N of atomic measures based on points of Ω_N such that for any continuous ψ on Ω

$$\int \psi d\mu = \lim_{N \rightarrow \infty} \sum_{x \in \Omega_N} \psi(x) \mu_N(x) \tag{4}$$

Formulas in (3) and (4) can be seen as formulas $\pi(\psi)$ written in a language that allows predicate variables ψ . We will say that such a formula or relation $\pi(\psi_1, \dots, \psi_n, a_1, \dots, a_m)$ is **definable uniformly in predicates ψ_1, \dots, ψ_n and parameters a_1, \dots, a_m** in a structure \mathbf{M} if it satisfies the ultrapower definability criterion:

$\mathbf{M} \models \pi(\psi_1, \dots, \psi_n, a_1, \dots, a_m)$ if and only if $\mathbf{M}^{\mathcal{D}} \models \pi(\psi_1^{\mathcal{D}}, \dots, \psi_n^{\mathcal{D}}, a_1^{\mathcal{D}}, \dots, a_m^{\mathcal{D}})$

for any ultrafilter \mathcal{D} , predicates ψ_1, \dots, ψ_n of respective arities and parameters $a_1, \dots, a_m \in \Omega$.

Relation $\pi(*, \dots, *, a_1, \dots, a_m)$ satisfying this condition will be called an **imaginary element parametrised by a_1, \dots, a_m** , and the set of all imaginary elements parametrised by $a_1, \dots, a_m \in \Omega$ an **imaginary sort in \mathbf{M}^{eq}** .

4.4 Theorem. *Let \mathbf{M} be a continuous structure with universe Ω .*

A. *To each point $x \in \Omega$ one can associate a linear functional on $\mathcal{B}(\Omega)$*

$$|x\rangle : \psi \rightarrow \psi(x)$$

interpretable in \mathbf{M} over parameter x . The set set of functionals

$$\Omega^* := \{|x\rangle : \psi \mapsto \psi(x) \text{ for } x \in \Omega\}$$

is an imaginary sort in \mathbf{M} .

B. *Assume that for any N there is $1/N$ -dense finite subset $\Omega_N \subset \Omega$ of definable points. Then any continuous linear functional ϕ is interpretable in \mathbf{M} . In other words ϕ can be identified with an imaginary element in \mathbf{M} and the dual Banach space $\mathcal{B}(\Omega^n)^*$ identified with a substructure of \mathbf{M}^{eq} .*

Under the assumption there is $\pi : \Omega \rightarrow \Omega^$, a definable bijection in \mathbf{M}^{eq} .*

C. *Suppose ν is a finite measure on Ω such that $\int \xi d\nu \neq 0$ for any non-zero continuous function ξ with only non-negative values on Ω . Then $\mathcal{B}(\Omega^n)$ embeds into its dual $\mathcal{B}(\Omega^n)^*$ via*

$$\alpha \mapsto \phi_\alpha, \quad \alpha \in \mathcal{B}(\Omega^n), \quad \phi_\alpha \in \mathcal{B}(\Omega^n)^*, \quad \phi_\alpha : \psi \mapsto \int \psi \cdot \bar{\alpha} d\nu.$$

$\mathcal{B}(\Omega^n)$ becomes a pre-Hilbert space with the inner product

$$\langle \psi | \alpha \rangle := \int \psi \cdot \bar{\alpha} d\nu$$

and its completion is the self-dual Hilbert space $\mathcal{H} \cong \mathcal{H}^*$, $\mathcal{H} \subseteq \mathcal{B}(\Omega^n)^*$.

The bijection π above induces an isomorphism from the structure \mathbf{M} to a structure \mathbf{M}^* based on universe Ω^* and predicates defined in terms of $\mathcal{B}(\Omega^n)^*$

Proof. A. Immediate from definitions.

B. By Facts 1 and 2 of 4.3 a linear functional ϕ can be identified with the limit expression in (4). Note that, for any N , by assumptions, $\psi \mapsto \psi(\omega)$ is definable uniformly in $\psi : \Omega \rightarrow \mathbb{C}$ and $\omega \in \Omega_N$. On the other hand μ_N is given by its values $\mu_N(\omega) \in \mathbb{C}$ on points $\omega \in \Omega_N$. Hence

$$\psi \mapsto \sum_{x \in \Omega_N} \psi(x) \mu_N(x)$$

is definable uniformly in ψ .

The limit in (4) is definable by the standard definition. Now formula (4) seen as a formula in variable f defines a map from the family of definable predicates of norm 1 to \mathbb{C} . This is by definition an imaginary element ϕ_μ in \mathbf{M} .

Now consider $\Omega^* \subset \mathbf{M}^{\text{eq}}$. Note that for distinct points $x_1, x_2 \in \Omega$ there is always a predicate ψ such that $\psi(x_1) \neq \psi(x_2)$. Indeed, choose N and a definable point $\omega \in \Omega$ such that $\text{dist}(\omega, x_1) < \frac{1}{N}$ and $\text{dist}(\omega, x_2) > \frac{1}{N}$. Then the definable predicate $\psi(x) = \text{dist}(\omega, x)$ is as required. It follows that $|x_1\rangle \neq |x_2\rangle$.

Definability of the map $x \mapsto |x\rangle$ in \mathbf{M}^{eq} is by definitions.

C. Now suppose ν is a complex measure satisfying the assumption. It induces a finite measure on Ω^n . Then the formula $\langle \psi | \alpha \rangle$ satisfies the assumptions of Hermitian inner product on $\mathcal{B}(\Omega^n)$ and by above is definable uniformly in ψ, α . Definability of $\alpha \mapsto \phi_\alpha$ and the embedding follows.

$\mathcal{H} \cong \mathcal{H}^*$ by the Riesz representation theorem - inner product. Note that \mathcal{H}^* by the theorem consists of linear functionals of the form ϕ_α as in (4). But $\bar{\alpha} d\nu = d\mu$ for some measure $\mu \in M(\Omega^n)$ as in (3). Hence $\phi_\alpha \in \mathcal{B}(\Omega^n)^*$ and $\mathcal{H}^* \subseteq \mathcal{B}(\Omega^n)^*$.

Finally, the definable map $\pi : x \mapsto |x\rangle$ allows to transfer predicate $\psi : \Omega^n \rightarrow \mathbb{C}$ to $\psi^\pi : (\Omega^*)^n \rightarrow \mathbb{C}$ by the uniformly definable rule

$$\psi^\pi(\bar{x}) := \langle \psi | \bar{x} \rangle.$$

In particular, it transfers metric distance predicate dist which defines an isometric metric on Ω^* . Functions are transferred by bijection π in the standard way.

□

Call a continuous structure \mathbf{M} **tame** if it satisfies assumptions B and C of Theorem 4.4.

4.5 Remarks. 1. For a tame \mathbf{M} we get n -Gel'fand triples

$$\mathcal{B}(\Omega^n) \subset \mathcal{H}^{\otimes n} \subset \mathcal{B}(\Omega^n)^*$$

with projections associated with $\Omega^{n+1} \rightarrow \Omega^n$. This system can be taken to be the cylindric algebra $\mathfrak{C}(\mathbf{M})$ of the continuous structure \mathbf{M} .

2. The two assumptions of 4.4 can be satisfied by standard choices in the case when Ω is a finite volume real manifold. For a more general locally compact case requires a special choice of measure ν satisfying assumption C. In particular, for $\Omega = \mathbb{R}$ one can choose the measure $e^{-x^2} dx$ which satisfies most of the requirements of quantum mechanics except for the important condition of translation invariance of the measure.

3. It is easy to see that condition B implies \aleph_0 -categoricity of \mathbf{M} . If also condition C is satisfied $\mathfrak{C}(\mathbf{M})$ allows to reconstruct \mathbf{M} categorically thus providing a stronger version of the first order case – Theorem 3.2.

4. Generally, the classification status of $\mathfrak{C}(\mathbf{M})$ (as a continuous model) is lower than that of \mathbf{M} .

4.6 Conclusion A tame structure fits well the context of quantum mechanics over a finite volume manifold once we include the treatment of general self-adjoint and unitary operators. We develop the general case in [14].

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