Axioms of Quantum Mechanics in light of Continuous Model Theory

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Abstract

The aim of this note is to recast somewhat informal axiom system of quantum mechanics used by physicists (Dirac calculus) in the language of Continuous Logic.

We note an analogy between Tarski's notion of cylindric algebras, as a tool of algebraisation of first order logic, and Hilbert spaces which can serve the same purpose for continuous logic of physics.

1 Dirac's calculus and axiomatisation of quantum mechanics

1.1 Axioms of quantum mechanics

The axiomatic formulation of quantum mechanics was set up by Paul Dirac in 1930 [1] and, in different but equivalent form, by John von Neumann in 1932. Since 1930 it took Dirac several rewritings and new editions to bring his calculus to what he could consider satisfactory form. Modern books present Dirac's axioms in a succint form, in fact, leaving much of the technical detail. We present here the Axioms of Quantum Mechanics following [2], section 6.

The reader with logic background would note that what physicists see as axioms is very far from what is a conventional set of axioms in a formal language even in its early form as presented e.g. by Hilbert's axiomatisation of geometry [3]

1.2 Axiom 1. The state of a quantum system is described by a vector $|\psi\rangle$ belonging to a complex Hilbert space $\mathcal{H}$. This state is usually called "ket ψ". A complex Hilbert space $\mathcal{H}$ is a vector space, which can be finite dimensional or infinite dimensional, equipped with the complex scalar product (also called inner product) $\langle\psi|\psi'\rangle$ between any pair of states $|\psi\rangle$, $|\psi'\rangle$ in $\mathcal{H}$. The norm, or modulus, of a generic vector $|\psi\rangle \in \mathcal{H}$ is defined as

$$||\psi|| = |\langle\psi|\psi\rangle|$$

and usually $|\psi\rangle$ is normalized to one, i.e.$||\psi|| = 1$. The symbol $\langle\psi|$ which appears in the definition of the norm is called "bra ψ" and it can be intepreted as the fuction

$$\langle\psi| : \mathcal{H} \to \mathbb{C}.$$

For any $|\psi'\rangle \in \mathcal{H}$ this function gives a complex number $\langle\psi|\psi'\rangle$ obtained as scalar product of $|\psi\rangle$ and $|\psi'\rangle$. In a complex Hilbert space $\mathcal{H}$ it exists a set of basis vectors $|\phi_\alpha\rangle$ which are orthonormal, i.e. $\langle\phi_\alpha|\phi_\beta\rangle = \delta(\alpha - \beta)$, and such that

$$|\psi\rangle = \sum_\alpha c_\alpha |\phi_\alpha\rangle \tag{1}$$

for any $|\psi\rangle$, where the coefficients c_α belong to $\mathbb{C}$.

Axiom 2. Any observable (measurable quantity) of a quantum system is described by a self-adjoint linear operator $F : \mathcal{H} \to \mathcal{H}$ acting on the Hilbert space of state vectors.

For any classical observable F it exists a corresponding quantum observable F.

Axioms 3.The possible measurable values of an observable F are its eigenvalues f, such that

$$F|f\rangle = f|f\rangle$$

with $|f\rangle$ the corresponding eigenstate. The observable $|f\rangle$ admits the spectral resolution

$$F = \sum_f f|f\rangle\langle f| \tag{2}$$

where $\{|f\rangle\}$ is the set of orthonormal eigenstates of F, and the mathematical object $\langle f|$, called "bra of f", is a linear map that maps into the complex number. This also satisfy the identity

$$\sum_f |f\rangle\langle f| = \mathrm{I}.$$

Axiom 4. The probability P of finding the state $|\psi\rangle$ in the state $|f\rangle$ (both of norm 1) is given by

$$P = |\langle f|\psi\rangle|^2$$

This probability P is also the probability of measuring the value f of the observable F when the system is in the quantum state $|\psi\rangle$.

Axiom 5. The time evolution of states and observables of a quantum system with Hamiltonian H is determined by the unitary operator

$$K^t := \exp(-i\mathrm{H}t/\hbar)$$

, such that $|\psi(t)\rangle = K^t|\psi\rangle$ is the time-evolved state $|\psi\rangle$.

1.3 Now we make several comments on the axioms.

The term "Hilbert space" here should actually be read as *the rigged Hilbert space* (see [4]) because it differs from the standard definition by accommodating both the Hilbert space Φ of ket-vectors and the dual space Φ^* of bra-vectors.

The position states $|x\rangle$ form the orthonormal basis of the space Φ^* of bra-vectors.

The summation formulas like (1) and (2) are presented in a form of an integral if the family $|\psi_\alpha\rangle$ is continuous but seems natural in the summation form when α runs in the discrete spectrum of an operator. The two forms can be represented uniformly by using spectral measure and the integration over a relevant spectral measure.

In fact, when a family $\{|\psi_\alpha\rangle : \alpha \in A\}$, $A \subseteq \mathbb{R}$ is given, it can be assumed that a measure $\mu = \mu_\psi$ on $\mathbb{R}$ is given such that $\mu(\mathbb{R} \setminus A) = 0$ and (1) can be rewritten as

$$|\psi\rangle = \int_{\mathbb{R}} c(\alpha)\, |\psi(\alpha)\rangle\, d\alpha \tag{3}$$

where $d\alpha$ stands for $d\mu$.

2 The axioms in the setting of Continuous Logic

2.1 Similarly to [6] we think in terms of continuous logic (CL) predicates/states on the domain which, for simplicity, is identified with $\mathbb{R}$ in its usual metric.

We assume

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} I_k$$

where I_k are intervals of finite length, $I_k \subset I_{k+1}$.

Similarly to [6] define basic n-ary predicates, also called states, to be maps

$$\psi : \mathbb{R}^n \to \mathbb{C}$$

which are limits of continuous maps

$$\psi_k : I_k^n \to \mathbb{S} \subset \mathbb{C}, \quad k \in \mathbb{N}$$

where $\mathbb{S}$ is the unit circle in $\mathbb{C}$.

Among the states there are the **position** states $\langle x|$, $x \in \mathbb{R}$, which have the form of Dirac's delta functions $\delta_x^{\mathrm{Dir}}(z) := \delta^{\mathrm{Dir}}(x - z)$. These form an orthonormal basis of the space of bra-vectors.

The space of ket-vectors is represented by continuous functions $\psi : \mathbb{R} \to \mathbb{S}$ and the inner product between $|\psi\rangle$ and $\langle x|$

$$\langle x|\psi\rangle := \psi(x)$$

and the inner product of position states is set to be

$$\langle x_1|x_2\rangle = \delta^{\mathrm{Dir}}(x_2 - x_1). \tag{4}$$

The **momentum** states $|p\rangle$ are represented as

$$|p\rangle := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ipx}, \quad p \in \mathbb{R}$$

and form an orthonormal basis of the space of ket-vectors.

The summation formula and the inner product is given in Dirac's interpratation, in particular

$$\langle p_1|p_2\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-ip_1 x} \mathrm{e}^{ip_2 x} dx = \delta^{\mathrm{Dir}}(p_2 - p_1)$$

where one uses the standard Fourier integral result

$$\int_{\mathbb{R}} \mathrm{e}^{-px} dx = 2\pi \delta^{\mathrm{Dir}}(p).$$

Orthogonality here and in (4) is understood in terms of Dirac's delta in place of the Kronecker delta.

Dirac's calculus allows rigorous interpretation of such calculations in terms of finite complex values.

More generally, let $\mathcal{H}_m$ be the set of all m-ary predicates which by definition have structure of $\mathbb{C}$-vector spaces and

$$\mathbb{C} = \mathcal{H}_0 \subset \ldots \subset \mathcal{H}_m \subset \ldots \subset \mathcal{H}_{m+1} \ldots \mathcal{H}.$$

Also, one uses quantifiers, linear maps written as integrals

$$\phi(z_1, \ldots, z_n) \mapsto \int_{\mathbb{R}} \phi(z_1, \ldots, z_n) dz_n.$$

In fact, this is a collection of linear maps

$$\int : \ \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

the rules of calculation of which as defined by Dirac [1] improper integration. In particular,

$$\int_{\mathbb{R}} \phi(z_1, \ldots, z_n) dz_n := \lim_{k \to \infty} \int_{I_k} \phi(z_1, \ldots, z_n) dz_n \tag{5}$$

(which fits with the requirements of continuous model theory).

A special binary operation in the spaces, inner product,

$$\mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C}; \ \ \langle \phi(z_1, \ldots, z_n), \psi(z_1, \ldots, z_n) \rangle = \int_{\mathbb{R}^m} \phi^* \cdot \psi \, dz_1 \ldots dz_m$$

where ϕ^* is the complex comjugate of ϕ and $\int_{\mathbb{R}^m}$ is m-multiple integral. $\langle \phi | \psi \rangle$ can be seen as a continuous predicate of equality $\phi = \psi$.

One restricts the notion of **state** to predicates ϕ such that $\langle \phi | \phi \rangle = 1$.

An important role in the theory is played by a collection of linear maps (operators)

$$L : \mathcal{H}_m \to \mathcal{H}_m$$

with physical meanings. Each of these of the form

$$\phi(\bar{z}_1, \bar{z}_2) \mapsto \int_{\mathbb{R}^k} \alpha(\bar{y}, \bar{z_1}) \cdot \phi(\bar{y}, \bar{z}_2) \ d\bar{y}$$

where $|\bar{y}| = |\bar{z}_1| = k$, $\alpha \in \mathcal{H}_{2k}$.

All of the above together makes the $\mathcal{H}_m$ a collection of Hilbert spaces with linear operators and $\mathcal{H}$ an ambient Hilbert space.

The **time evolution operator** $\exp(-i\mathrm{H}t/\hbar)$ acts on states as a unitary operator determining the evolution of a state in time t with a given Hamiltonian H. A state ϕ_{t_0} determining a system at time t_0 evolves into a state $\phi_t := \exp(-i\mathrm{H}(t-t_0)/\hbar)$ with the *probability amplitude* equal to $\langle\phi_{t_0}|\phi_t\rangle$, which is a complex number of modulus 1. The calculation of the CL-formulae ϕ_t and $\langle\phi_{t_0}|\phi_t\rangle$ (which involve mainly calculations of the application of quantifier $\int$) is the central problem of quantum theory, equivalent to solving the associated Schrödinger equation.

The above (along with further details of the Dirac calculus given in [1]) describes the formulae, the connectives and the quantifiers $\int$ of continuous logic for quantum mechanics.

3 Abstract algebraic logic and the Hilbert space formalism

3.1 The axiomatic description of quantum mechanical theory in the form of rigged Hilbert space may be quite confusing from the logician point of view – there are no logical sentences which can be called axioms.

What Axioms 1 – 5 render instead is the topological-algebraic structure of a Hilbert space with operators.

Recall now the *algebraisation of logic* approach, perhaps less popular among logicians nowadays, versions of which were introduced by A.Lindenbaum, A.Tarski, P.Halmos for the first order setting.

It is quite natural to see the Hilbert space formalism as the form of algebraic logic in the context of the continuous logic of physics.

The qualification 'physics' seems relevant here because of the specific nature of its predicates (states) and quantifiers.

3.2 In drawing an analogy with the first order case, namely Tarski's cylinder algebras, note that the physical theory misses a clear definition of an **interpretation**, that is of a model, an elementary equivalence and related notions.

Recall **the Main Theorem on Cylindric Algebras** (see e.g. [5])

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures in the same first-order language, and $\mathfrak{C}\mathcal{A}, \mathfrak{C}\mathcal{B}$ the respective cylinder algebras.

Then $\mathcal{A}$ is elementarily equivalent to $\mathcal{B}$ iff $\mathfrak{C}\mathcal{A} \cong \mathfrak{C}\mathcal{B}$, where the isomorphism identifies sets definable by the same formulas.

The problem of furnishing a definition of a **structure M** and the interpretation of the language of quantum mechanics in the context of continuous model theory is practically solved by Dirac for the case of the particle quantum mechanics. One chooses a manifold $\mathcal{M}$ for the universe of the structure and sets predicates (states) to be of the form

$$\psi : \mathcal{M}^n \to \mathbb{C}$$

where, for each subdomain $D \subseteq \mathcal{M}^n$ of finite diameter, $\psi(D) \subseteq C_D$ for a compact $C_D \subset \mathbb{C}$.

For an **interpretation** of the language associated with $\mathcal{H}$ one defines the states (predicates) on $\mathcal{M}$ and rules of calculating the *rigged Hilbert space* operations over the states (CL-connectives) and the quantifier $\int_{\mathcal{M}}$. Note that linear operators, including the time evolution operator, are included in the list of Hilbert space operations.

Write the respective structure as

$$(\mathcal{M}; \mathcal{H}).$$

Once the structure of a rigged Hilbert space $\mathcal{H}$ is fixed it identifies a **complete quantum mechanical theory**. The class of structures $(\mathcal{M}; \mathcal{H})$ with a fixed $\mathcal{H}$ will be considered as the class of models of the complete theory represented in the form of the rigged Hilbert space $\mathcal{H}$.

The fragment of QM where such interpretation is well-defined is the theory of a finite number of free particles and more generally "Gaussian" quantum mechanics, determined by a Hamiltonians with quadratic potential (which includes the quantum harmonic oscillator, not a free particle). Such theory, with the choice of operators in $\mathcal{H}$ restricted to unitary operators (the Weyl operators and the time evolution operators) is analysed in [6]. It is noticed that the theory has quantifier elimination under the natural choice of basic predicates. Moreover, the theory has a continuous model $(\mathbb{R}^n; \mathcal{H})$ as well as a family of discrete pseudo-finite models (with the universe $\mathbb{V}_{\mathfrak{v}}$) depending on a choice of parameters $\mathfrak{p}, \mathfrak{l}, \mathbf{i}$, non-standard integers.

To move further from there one needs to include self-adjoint operators (such as P, Q and H) in the definition of $\mathcal{H}$ along with the operation

$$\exp : L \mapsto \mathrm{e}^{iL}$$

defined for self-adjoint L. Note that interpretation of the time evolution operator $\mathrm{e}^{i\frac{\mathrm{H}}{\hbar}t}$ over $\mathcal{M} = \mathbb{R}$ amounts to a path-integral calculation and requires some non-conventional determination of a non-convergent limit even for the case of the quantum harmonic oscillator, see [7], 7.7.4.

More problems arise with including *perturbation methods* into the formalism. These treat the important Planck constant $\hbar$ as an infinitesimal while physics estimates it by a known a real number.

3.3 What is the CL-classification theory status of the theories represented by various $\mathcal{H}$?

3.4 Remark. Rigged Hilbert spaces provide a powerful mathematical framework to extend quantum mechanics, allowing distributions and generalized eigenfunctions to be rigorously handled. However, not every element corresponds to a physically realisable state – some are purely mathematical artifacts. See [9]

3.5 The broader setting of quantum field theory (QFT) is in many regards similar but much more problematic. One of the main sources of difficulties is that $\mathcal{M}$ in this case has to be infinite-dimensional. [8], section 8, contains a discussion of Axioms for QFT.

In the following definition of interpretation (models) of a CL-theories of physics we mimic the first-order setting of models of Hamiltonian mechanics, which has the form of phase space.

3.6 Definition. Let $\mathcal{H}$ be a **rigged Hilbert space**,

$$\mathcal{H} = (H; \mathrm{O})$$

where H is a complete Hermitian space, O a collection of linear operators on $H^n \to H^m$. We write O^e in place of O if we include $\exp : L \to \mathrm{e}^{iL}$ for self-adjoint $L \in \mathrm{O}$. We assume O contains the Weyl operators of the form $\mathrm{e}^{ai\mathrm{P}}$ and $\mathrm{e}^{bi\mathrm{Q}}$ for a, b rational numbers (“rational” Weyl operators).

An $\mathcal{H}$-**structure** $(\mathbb{V};\mathcal{H})$ is given by

- a universe which consists of the **configuration space** $\mathbb{V}$ with a ${}^*\mathbb{R}_+$-valued metric (non-standard reals) on each;

- collections $\mathfrak{F}_n$ $n \in \mathbb{N}$, of **states**, that is maps $\psi : \mathbb{V}^n \to {}^*\mathbb{C}$ (into the non-standard complex numbers) closed under ${}^*\mathbb{C}$-linear combinations;

- a Hermitian inner product $\langle \psi_1 | \psi_2 \rangle;\ \mathfrak{F}_n \times \mathfrak{F}_n \to {}^*\mathbb{C}$ is defined for all n

- a collection of linear operators $\mathrm{O}_F = \{L_F : L \in \mathrm{O}\}$:

$$L_F : \mathfrak{F}_n \to \mathfrak{F}_m \text{ for each } L : H^n \to H^m$$

- an **interpretation functor** is a homomorphism

$$\mathfrak{C} : \begin{array}{l} \mathfrak{F}_n \to \mathcal{H}^{\otimes n};\ n \in \mathbb{N}, \\ \mathrm{O}_F \twoheadrightarrow \mathrm{O}; \\ {}^*\mathbb{C} \twoheadrightarrow \mathbb{C} \cup \{\infty\} \end{array}$$

which respects the Hermitian structure and the algebra of linear operators, coincides with the standard part map on ${}^*\mathbb{C}$ and satisfies the condition:

$$\operatorname{Eig} W_F \twoheadrightarrow \operatorname{Eig} W$$

surjection on the Weyl operators W eigenfunctions-bases, for $W \in \mathrm{O}$.

The respective structure $V = (\mathbb{V}, \mathfrak{C}, \mathcal{H})$ will be called a **model** of $\mathcal{H}$.

3.7 Remarks.

In general, the images $\mathfrak{C}(\mathfrak{F}_n) \subseteq \mathcal{H}^{\otimes n}$ are not uniquely defined by $\mathcal{H}$. This is motivated by Remark 3.4. Respectively, the class of models of $\mathcal{H}$ corresponds in general to a theory which is not necessarily complete.

In analogy with the Main Theorem on Cylindric Algebras, 3.2, we say that structures V and W are CL-equivalent if $\mathfrak{C}\,V \cong \mathfrak{C}\,W$.

3.8 It is not hard to see that $\mathcal{H}$ itself provides a "canonical" universe $\mathbb{V}$ with the trivial interpretation. However, $(\mathbb{V}, \mathfrak{C}, \mathcal{H})$ suggests multitude of other possibilities for $\mathbb{V}$. and $\mathfrak{C}$.

Conceptually one can think of the functor $\mathfrak{C} : V \mapsto \mathcal{H}$ as a **structural approximation** in the sense of [6] and [10] (called lm therein). Indeed, $\mathfrak{C}$ approximates a "rough" model of reality V by a "smooth" $\mathcal{H}$. This is indeed how approximation has been applied in [6] to pass from a family of discrete (pseudo-finite) structures to classical Hilbert space setting.

Note, that in [6] we work in a more general setting where states take their values in a discrete (pseudofinite) field $\psi : \mathbb{V}^n \to \mathrm{F}$. It is then shown that for quantum mechanics one can embed $\mathrm{F} \subset {}^*\mathbb{C}$. This explains our use of ${}^*\mathbb{C}$ in the definition of models, although it might be advisable to use a more abstract F instead.

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