Lectures on Zariski-type structures
Part I

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1 Axioms for Zariski structures

Let M be a set and let $\mathcal{C}$ be a distinguished sub-collection of the subsets of M^n, $n = 1, 2, \ldots$. The sets in $\mathcal{C}$ will be called **closed**. The relations corresponding to the sets are the basic (primitive) relations of the language we will work with.
$\langle M, \mathcal{C}\rangle$, or M, is **a topological structure** if it satisfies axioms (L).

(L) **Language and Topology**

1. finite intersections and unions of closed sets are closed;
2. M is closed;
3. the graph of equality is closed;
4. any singleton of the domain is closed;
5. Cartesian products of closed sets are closed;
6. for $a \in M^k$ and S a closed subset of M^{k+l} defined by a predicate $S(x, y)$ ($x = \langle x_1, \ldots, x_k\rangle$, $y = \langle y_1, \ldots, y_l\rangle$), the set $S(a, M^l)$ (the *fiber over* a) is closed;

7. the inverse image of a closed set under a projection $\mathrm{pr}_{i_1,\dots,i_m}$ is closed.

Constructible sets are by definition the Boolean combinations of closed sets.

We continue with the list of axioms.

(SP) **semi-Properness** of projection mappings:
the image $\mathrm{pr}_{i_1,\dots,i_m}(S)$ of a closed subset $S \subseteq M^n$ is constructible.

A topological structure is said to be **complete** if

(P) the image $\mathrm{pr}_{i_1,\dots,i_m}(S)$ of a closed subset $S \subseteq M^n$ is closed.

A topological structure is called Noetherian if it also satisfies

(DCC) **Descending chain condition** for closed subsets: for any closed

$$S_1 \supseteq S_2 \supseteq \dots S_i \supseteq \dots$$

there is i such that for all $j \geq i$, $S_j = S_i$.

A closed S is called **irreducible** if there are no closed sets S_1 and S_2 such that $S_1 \subsetneq S_2$, $S_2 \subsetneq S_1$ and $S = S_1 \cup S_2$.
It follows from (DCC), that for any closed S there are distinct closed irreducible $S_1, \dots, S_k$ such that

$$S = S_1 \cup \dots \cup S_k.$$

These S_i will be called **irreducible components of** S. They are defined up to a numeration uniquely.
We can also consider a decomposition $S = S_1 \cup S_2$ for S constructible and S_1, S_2 closed in S. If there is no proper such a decomposition of a constructible S, we say that S is irreducible.

To any closed subset $S \subseteq M^n$ a natural number called **the dimension of** S, $\dim S$ is attached
(DP) **Dimension of a point** is 0;

(DU) **Dimension of unions:** $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$;

(FC) **Fiber condition:** for any $\mathrm{pr} = \mathrm{pr}_{i_1,\ldots,i_m}$ and a closed irreducible $S \subseteq M^n$ the set

$$\mathcal{P}^{\mathrm{pr}}(S,k) = \{a \in \mathrm{pr}S : \dim(S \cap \mathrm{pr}^{-1}(a)) > k\}$$

is closed in $\mathrm{pr}S$.

(AF) **Addition formula:**

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in \mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

for any closed irreducible S.

(SI) **Strong irreducibility**: $\dim S_1 < \dim S$ for irreducible S and $S_1 \subset S$, $S_1 \neq S$;

Noetherian topological structures satisfying (SP) and (DP)-(SI) will be called **Zariski structures,** sometimes with adjective *Noetherian,* to distinguish from analytic Zariski structures introduced later.
In main cases we assume that a Zariski structure satisfies also

(EU) **Essential uncountability** If a closed $S \subseteq M^n$ is a union of countably many closed subsets, then there are finitely many among the subsets, the union of which is S.

The following is an assumption crucial for developing a rich theory for Zariski structures

(PS) **Pre-smoothness** For any closed irreducible $S_1, S_2 \subseteq M^n$, the dimension of any irreducible component of $S_1 \cap S_2$ is not less than

$$\dim S_1 + \dim S_2 - \dim M^n.$$

For simplicity, we add also the extra assumption that M **itself is irreducible.** However, most of the arguments in the first half of the notes hold without this assumption.

2 Basic examples

2.1 Algebraic varieties over algebraically closed fields

Let K be an algebraically closed field and M the set of K-points of an algebraic variety over K. We are going to consider a structure on M :
The natural language for algebraic varieties M is the language the basic n-ary relations $\mathcal{C}$ of which are the Zariski closed subsets of M^n.

Theorem 2.1 *Any algebraic variety M over an algebraically closed field in the natural language and the dimension notion as that of algebraic geometry is a Zariski structure. The Zariski structure is complete if the variety is complete. It satisfies* (PS) *if the algebraic variety is smooth. It satisfies* (EU) *iff the field is uncountable.*

Proof. Use a book on algebraic geometry. (L) and (DCC) follows immediately from the definition of Zariski geometry and the Noetherianity of polynomial rings. (P) is a canonical property (completeness) of projective varieties. The Fiber Condition (FC) along with the addition formula (AF) is given by *Dimension of Fibers Theorem.* (PS) is being discussed in the context of smoothness. $\square$

2.2 Compact complex manifolds

The natural language for a compact complex manifold M has the analytic subsets of M^n as basic n-ary relations $\mathcal{C}$.

Theorem 2.2 *Any compact complex manifolds M in a natural language and dimension given as complex analytic dimension is a complete Zariski structure and satisfies assumptions* (PS) *(pre-smoothness) and* (EU) *(essentially uncountable).*

Remark 2.3 In fact, the theorem holds for compact analytic spaces, except for the pre-smoothness condition.

Remark 2.4 Let

$$T = \mathbb{C}^n/\Lambda$$

Λ an additive subgroup of $\mathbb{C}^n$ on $2n$ $\mathbb{R}$-independent generators (lattice).

Then T is a compact complex manifold, a torus.
When $n > 1$, for most lattices $(T, \mathcal{C})$ is just a group structure, that is, is locally modular.

2.3 Proper varieties of rigid analytic geometry

(S.Bosch, U.Guentzer, and R.Remmert, **Non-Archimedean Analysis**)

It is built over a completion of a non-Archimedean valued algebraically closed field K. The main objects are analytic varieties over K.
The natural language for an analytic variety M is again the language with analytic subsets of M^n as basic relations.
The definition of a neghbourhood and so of an analytic subset is much more involved than in the complex case. The main obstacle for an immediate analogy is the fact that the non-Archimedean topology on K is highly disconnected.

Theorem 2.5 *Let M be a proper (rigid) analytic variety. Then M, with respect to the natural language, is a complete Zariski structure satisfying* (EU). *It is pre-smooth if the variety is smooth.*

Theorem 2.6 *Any Zariski structure M admits elimination of quantifiers, i.e. any definable subset $Q \subseteq M^n$ is constructible.*

Theorem 2.7 *Any Zariski structure M satisfying (EU) is of finite Morley rank. More precisely,* $\operatorname{rk} Q \leq \dim Q$ *for any definable set* Q.

In particular

Corollary 2.8 *A compact complex space is of finite Morley rank.*

Proposition 2.9 *For any essentially uncountable Zariski structure M and $M' \succeq M$, the natural extension $\mathcal{C}' \supseteq \mathcal{C}$ of the topology and dimension notions determine a Zariski structure on M'. If we choose M' to be saturated enough then it satisfies (EU). If M is presmooth then so is M'.*

Remark 2.10 Notice that the Proposition fails in regard to (DCC) without assuming (EU) for M.

3 Specialisations

This notion has analogues both in model theory and algebraic geometry. In the latter the notion under the same name has been used by A.Weil, namely, if K is an algebraically closed field and $\bar{a}$ a tuple in an extension K' of K, then a mapping $K[\bar{a}] \to K$ is called a specialization if it preserves all equations with coefficients in K.
In the same setting a specialisation is often called a *place* and is closely related to valuations of K' with residue field K.

The model-theoretic source of the notion is A.Robinson's *standard-part map* from an elementary extension of $\mathbb{R}$ (or $\mathbb{C}$) onto the compactification of the structure. More involved and very essential way the concept emerges in model theory is in the context of *atomic compactness* introduced by J. Mycielski [My] and studied by B. Weglorz[We] and others.

A structure M is said to be **atomic (positive) compact** if any finitely consistent set of atomic (positive) formulas is realised in M.

Theorem 3.1 (B. Weglorz) *The following are equivalent for any structure M:*
(i) M is atomic compact;
(ii) M is positive compact;
(iii) M is a retract of any $M' \succ M$, i.e. there is a homomorphism $\pi : M' \to M$, fixing M pointwise,

We assume in this section that M is just a topological structure.

Definition 3.2 Let ${}^*M \succeq M$ be an elementary extension of M and $M \subseteq A \subseteq {}^*M$. A map $\pi : A \to M$ will be called a **(partial) specialisation,** if for every a from A and an n-ary M-closed S, if $a \in S({}^*M)$ then $\pi(a) \in S$.

Remark 3.3 By the definition a specialisation is an identity on M, since any singleton $\{s\}$ is closed.

Definition 3.4 A topological structure M will be called **quasi-compact** (or just **compact**) if it is complete, that is satisfies (P) and

(QC) For any finitely consistent family $\{C_t : t \in T\}$ of closed subsets of M^n

$$\bigcap_{t \in T} C_t \text{ is non-empty.}$$

By (DCC), every complete Zariski structure is quasi-compact.

Proposition 3.5 *Suppose M is a quasi-compact structure, $M^* \succeq M$. Then there is a specialisation $\pi : M^* \to M$. Moreover, any partial specialisation can be extended to a total one.*

Proposition 3.6 *Let M be a topological structure and suppose that the topology is compact and Hausdorff. Then there is unique specialisation $\pi : {}^*M \to M$.*

3.1 Universal specialisations

Definition 3.7 For a (partial) specialisation $\pi : M^* \to M$, we say that the pair (M^*, π) is **universal (over** M**)** if
for any $M' \succeq M^* \succeq M$, any finite subset $A \subset M'$ and a specialisation $\pi' : A \cup M^* \to M$ extending π, there is an elementary isomorphism $\alpha : A \to M^*$, over $M \cup (A \cap M^*)$, such that

$$\pi' = \pi \circ \alpha \text{ on } A.$$

Proposition 3.8 *For any structure M there exists an universal pair (M^*, π). If M is quasi-compact, then π is total.*

3.2 Infinitesimal neighborhoods

Definition 3.9 For a point $a \in M^n$ we call **an infinitesimal neighborhood of** a the subset in $(M^*)^n$ given as

$$\mathcal{V}_a = \pi^{-1}(a).$$

Clearly then, for $a, b \in M$ we have $\mathcal{V}_{(a,b)} = \mathcal{V}_a \times \mathcal{V}_b$.

Definition 3.10 Given $b \in M^n$ denote the n-type over M^* :

$\mathrm{Nbd}_b(y) = \{\neg Q(c', y) : \ M \models \neg Q(c, b), \ Q$ is closed, 0-definable, $c' \in \mathcal{V}_c, c \in M^k\}$.

As usual $\mathrm{Nbd}_b(M^*)$ will stand for the set of realisations of the type in M^* and Dom π the domain of π in M^*.

Lemma 3.11 *(i)*

$$\mathcal{V}_b = \mathrm{Nbd}_b({}^*M) \cap \mathrm{Dom}\ \pi.$$

(ii) Given a finite a' in ${}^*M$ and a quantifier-free type $F(a', y)$ over Ma', there exists $b' \in \mathcal{V}_b$ satisfying $F(a', b')$ provided the type

$$\mathrm{Nbd}_b(y) \cup \{F(a', y)\}$$

is consistent.

Example 3.12 (Topological and Zariski groups) Let G be a topological structure with a basic ternary relation P defining a group structure on G with the operation

$$x \cdot y = z \equiv P(x, y, z).$$

Suppose that G is compact. Consider ${}^*G \succ G$, a universal specialisation $\pi : G^* \to G$ and the infinitesimal neighborhood $\mathcal{V} \subseteq G^*$ of the unit. Then $\mathcal{V}$

is a nontrivial normal subgroup of G^*.

Let M be a complete Zariski structure and $G \subseteq_{open} M^n$. Take a universal total specialisation

$$\pi : {}^*M \to M.$$

Then $\mathcal{V} \cap G({}^*M)$ is an infinitesimal subgroup of $G({}^*M)$.

Problem Algebraicity conjecture for simple Zariski groups G (without presmoothness) is not known.

3.3 Non-standard analysis in Zariski geometries

Definition 3.13 Assume $F \subseteq D \times M^k$ is irreducible closed in $D \times M^k$ and $\mathrm{pr}(F) = D$. We say then F **is an (irreducible) cover of** D.
Let F be a cover of D and assume that $\dim F(a', y) = r$ for generic $a' \in D$ (we call it *the dimension of a generic fiber*). $a \in D$ will be called **regular for** F if $\dim F(a, y) = r$. The set of points regular for F will be denoted $\mathrm{reg}(F/D)$.
The cover is said to be **(generically) finite** if $r = 0$, that is generic fibers are finite.

Lemma 3.14 $\dim(D \setminus \mathrm{reg}(F/D)) \leq \dim D - 2$.

Proof. The set $\{\langle a, b\rangle \in F : a \in (D \setminus \mathrm{reg}(F/D))\}$ is a proper closed subset of F. Thus the estimate follows from (SI) and (AF).□

Corollary 3.15 *Suppose F is a cover of an irreducible D, $\dim D = 1$. Then every $a \in D$ is regular.*

Theorem 3.16 (Implicit Function Theorem) *Let $F \subseteq D \times M^k$ be an irreducible cover of a presmooth set D, $\langle a, b\rangle \in F$ and assume that $a \in D$ is regular for F. Then for every $a' \in \mathcal{V}_a \cap D(^*M)$ there exists $b' \in \mathcal{V}_b$, such that $\langle a', b'\rangle \in F(^*M)$.*
If the cover is finite then $a' \mapsto b'$ is a function $\mathcal{V}_a \to \mathcal{V}_b$.

4 Multiplicities

Lemma 4.1 *Let $F \subseteq D \times M^k$ be an irreducible finite cover of D in a, D presmooth. If $F(a, b)$ and $a' \in \mathcal{V}_a \cap D(^*M)$ is generic in D then*

$$\#(F(a', {}^*M) \cap \mathcal{V}_b) \geq \#(F(a'', {}^*M) \cap \mathcal{V}_b), \quad \textit{for all } a'' \in \mathcal{V}_a \cap D(^*M)$$

Definition 4.2 Let $\langle a, b\rangle \in F$ and F be a finite covering of D in $\langle a, b\rangle$. Define

$$\mathrm{mult}_b(a, F/D) = \#F(a', {}^*M^k) \cap \mathcal{V}_b, \quad \text{for } a' \in \mathcal{V}_a \text{ generic in } D \text{ over } M.$$

By Lemma 4.1, this is a well-defined notion, independent on the choice of generic a'. Moreover, the proof of Lemma 4.1 contains also the proof of the following

Lemma 4.3 $m \geq \mathrm{mult}_b(a, F/D)$ *iff there is an irreducible cover* $F^{(m)} \subseteq D \times M^{mk}$ *of* D, *finite at* a, *such that for any generic* $a' \in \mathcal{V}_a \cap D({}^*M)$ *there are distinct* $b'_1, \ldots, b'_m \in \mathcal{V}_b$ *with* $\langle a', b'_1, \ldots, b'_m\rangle \in F^{(m)}$.

Call a finite covering **unramified at** $\langle a, b\rangle$ if $\mathrm{mult}_b(a, F/D) = 1$ and let

$$\mathrm{unr}(F/D) = \{\langle a, b\rangle \in F : \mathrm{mult}_b(a, F/D) = 1\}.$$

Let

$$\mathrm{mult}(a, F/D) = \sum_{b \in F(a, M^k)} \mathrm{mult}_b(a, F/D).$$

Proposition 4.4 (Multiplicity Properties) *Suppose* D *is pre-smooth. Then*

(i) the definitions above do not depend on the choice of M^* *and* π;

(ii)

$$\mathrm{mult}(a, F/D) = \#F(a', M^{*k})$$

for $a' \in D(M^*)$ *generic over* M *(not necessarily in* $\mathcal{V}_a$*) and the number does not depend on the choice of* a *in* D;

(iii) the set

$$j_m(F/D) = \{\langle a, b\rangle : a \in \mathrm{reg}(F/D) \ \&\ \mathrm{mult}_b(a, F/D) \geq m\}$$

is definable and relatively closed in the set $\mathrm{reg}(F/D) \times M^k$. *Moreover, there is* m *such that for every* $a \in \mathrm{reg}(F/D)$ *we have* $\mathrm{mult}_b(a, F/D) \leq m$.

(iv) $\mathrm{unr}(F/D)$ *is open in* F *and the set*

$$D_1 = \{a \in \mathrm{reg}(F/D) : \forall b(F(a, b) \to \langle a, b\rangle \in \mathrm{unr}(F/D))\}$$

is dense in D.

5 Elements of intersection theory.

Definition 5.1 Let P and L be constructible irreducible sets in M and $I \subseteq P \times L$ be closed in $P \times L$ and irreducible, $\mathrm{pr}_2 I = L$. We call such an I **a family of closed subsets of** P. One can think of $l \in L$ as the parameter for a closed subset $\{p \in P : \ pIl\}$.

Any $l \in L$ identifies a subset of those points of P, that are incident to l, though we allow two distinct l's of L represent the same set.
As a rule we write simply $p \in l$ instead of pIl, thus the mentioning of I is omitted and we simply refer to L as a family of closed subsets of P.

Definition 5.2 Let L_1 and L_2 be irreducible families of closed subsets of an irreducible set P. In this situation for $p \in P$ and $l_1 \in L_1$, $l_2 \in L_2$ define **the index of intersection of the two sets at the point p with respect to** L_1, L_2 as

$$\mathrm{ind}_p(l_1, l_2/L_1, L_2) = \# l_1' \cap l_2' \cap \mathcal{V}_p,$$

where $\langle l_1', l_2' \rangle \in \mathcal{V}_{l_1,l_2} \cap L_1({}^*M) \times L_2({}^*M)$ is generic over M.

Definition 5.3 The index of intersection of the two families as above is

$$\mathrm{ind}(L_1, L_2) = \#(l_1' \cap l_2')$$

where $\langle l_1', l_2' \rangle \in L_1({}^*M) \times L_2({}^*M)$ is generic over M.

Proposition 5.4 *Assume $L_1 \times L_2$ and $P \times L_1 \times L_2$ are pre-smooth, irreducible and for some generic $\langle l_1, l_2 \rangle \in L_1 \times L_2$ the intersection $l_1 \cap l_2$ is finite. Then*

(i) the definition of the index at a point does not depend on the choice of ${}^*M$, π and generic l_1', l_2';

(ii)

$$\sum_{p \in l_1 \cap l_2} \mathrm{ind}_p(l_1, l_2/L_1, L_2) = \mathrm{ind}(L_1, L_2);$$

(iii) for generic $\langle l_1, l_2\rangle \in L_1 \times L_2$ *and* $p \in l_1 \cap l_2$

$$\mathrm{ind}_p(l_1, l_2/L_1, L_2) = 1;$$

(iv) the set

$$\{\langle p, l_1, l_2\rangle \in P \times L_1 \times L_2 : \mathrm{ind}_p(l_1, l_2/L_1, L_2) \geq k\}$$

is closed.

Proof. This is contained in the properties of multiplicities for finite coverings, since $F = \{\langle p, l_1, l_2\rangle : p \in l_1 \& p \in l_2 \& l_1 \in L_1 \& l_2 \in L_2\}$ is a covering (maybe reducible) of $L_1 \times L_2$. To apply Proposition 4.4 notice that any component F_i of F is of the same dimension, hence the projection of F_i on $L_1 \times L_2$ is dense in $L_1 \times L_2$ and F_i is finite in $\langle p, l_1, l_2\rangle$. Evidently,

$$\mathrm{ind}_p(l_1, l_2/L_1, L_2) = \sum_i \mathrm{mult}_p(\langle l_1, l_2\rangle, F_i/L_1 \times L_2).$$

□

Remark 5.5 The Proposition effectively states that closed subsets from the same presmooth family are numerically equivalent (see [Ha]).

Problem Develop a theory of intersection and of numerical equivalence of closed sets in presmooth Zariski structures.

Definition 5.6 Suppose for some $\langle l_1, l_2\rangle \in L_1 \times L_2 \quad l_1 \cap l_2$ is finite. Two closed sets l_1, l_2 from families L_1, L_2, respectively are called **simply tangent at the point** p **with respect to** L_1, L_2 if there is an infinite irreducible component of $l_1 \cap l_2$, containing p or

$$\mathrm{ind}_p(l_1, l_2/L_1, L_2) \geq 2.$$

6 Curves and their branches (very technical section)

We assume here that C is an one-dimensional irreducible presmooth Zariski set in a Zariski structure M.
We also assume that the Zariski structure on C is **non-linear** (equivalently, non-locally modular). It is equivalent that the Zariski geometry on C is **ample**:

(AMP) There is a 2-dimensional irreducible faithful family L of curves on C^2.
L itself is locally (infinitesimally) isomorphic to an open subset of C^2.

Definition 6.1 Let $\langle a, b\rangle$ be a point in C^2. A subset $\gamma \subseteq \mathcal{V}_{\langle a,b\rangle}$ is said to be **a branch of a curve at** $\langle a, b\rangle$ if there are $m \geq 2$, $c \in C^{m-2}$, an irreducible smooth family G of curves through $\langle a, b\rangle^\frown c$ with an incidence relation I and a curve $g \in G$ such that the cover I of $G \times C$,

$$\langle u, \langle x, y\rangle^\frown z\rangle \mapsto \langle u, x\rangle,$$

is regular (hence finite) and unramified at $\langle g, \langle a, b\rangle^\frown c\rangle$, and

$$\gamma = \{\langle x, y\rangle \in \mathcal{V}_{\langle a,b\rangle} : \exists z \in \mathcal{V}_c \ \langle g', \langle x, y\rangle^\frown z\rangle \in I\}$$

for a $g' \in \mathcal{V}_g \cap G(M^*)$.

The definition assumes that g' represents a possibly 'nonstandard' curve in the neighborhood of g passing through a standard point $\langle a, b\rangle^\frown c$. We usuall denote γ by $\tilde{g'}$.

It follows from the definition and Proposition 3.16 that $\tilde{g}'$ is a graph of a function from $\mathcal{V}_a$ onto $\mathcal{V}_b$. We call the corresponding object **the associated (local) function** $\tilde{g} : \mathcal{V}_a \to \mathcal{V}_b$ **(from** a **to** b **) from a family** G **with trajectory** c.

Definition 6.2 Let I_1 and I_2 be two families of local functions from a to b, with trajectories c_1 and c_2. We say that the correspondent **branches defined by $g_1 \in G_1$ and $g_2 \in G_2$ are tangent at $\langle a, b \rangle$**, and write

$$g_1 \, T \, g_2,$$

if there is an irreducible component $S = S_{(I_1, I_2, a, b, c_1, c_2)}$ of the set

$$\begin{aligned} \{\langle u_1, u_2, x, y, z_1, z_2 \rangle \in G_1 \times G_2 \times C^2 \times C^{m_1 - 2} \times C^{m_2 - 2} : \\ \langle u_1, x, y, z_1 \rangle \in I_1 \;\&\; \langle u_2, x, y, z_2 \rangle \in I_2 \} \end{aligned} \tag{1}$$

such that

1. $\langle g_1, g_2, a, b, c_1, c_2 \rangle \in S$;

2. the image of the natural projections of S into $G_1 \times G_2$

$$\langle u_1, u_2, x, y, z_1, z_2 \rangle \mapsto \langle u_1, u_2 \rangle$$

is dense in $G_1 \times G_2$.

3. for $i = 1$ and $i = 2$ the images of the maps

$$\langle u_1, u_2, x, y, z_1, z_2 \rangle \mapsto \langle x, y, z_i, u_i \rangle$$

are dense in I_i and the corresponding covers by S are regular at the points $\langle a, b, c_i, g_i \rangle$.

Remark 6.3 Once I_1, I_2, a, b, c_1, c_2 have been fixed one has finitely many choices for the irreducible component S.

Remark 6.4 We can write item 3 as a first-order formula

$$\langle a, b, c_i, g_i \rangle \in \operatorname{reg}(S/I_i).$$

Corollary 6.5 *The formula*

$$T := \bigcup_S S(u_1, u_2, a, b, c_1, c_2) \;\&\; \langle a, b, c_1, u_1 \rangle \in \operatorname{reg}(S/I_1) \;\&\; \langle a, b, c_2, u_2 \rangle \in \operatorname{reg}(S/I_2)$$

(with parameters a, b, c_1, c_2) defines the tangency relation between $u_1 \in G_1$ and $u_2 \in G_2$.

7 Getting a group and a field

The following is crucial.

Proposition 7.1 (Jets of curves) *The tangency relation is generically a definable equivalence relation on the families of branches of curves passing through a given point $\langle a, b\rangle \in C^2$.*
The set J of **jets** *(classes under the equivalence) of curves through $\langle a, a\rangle \in C^2$ is a presmooth Zariski curve and has a definable* **pre-group** *structure induced by the composition of the associated local functions.*

In a standard way one derives from this

Lemma 7.2 *There is an irreducible presmooth one-dimensional Abelian group G definable in C, moreover G is locally isomorphic to C and the group operation is Zariski continuous.*

One can now assume that C itself has a group structure $(C, +)$ and consider curves passing through $\langle 0, 0\rangle$. It turns out that the associated local functions are preserving the additive structure on C, modulo tangency. This leads to the following

Proposition 7.3 *There is an irreducible presmooth one-dimensional K in C with a field structure on it.*
The field operations on K are Zariski continuous.

8 Intersection theory in projective spaces over K and the Purity Theorem

We study first how algebraic curves in $\mathbf{P}^2(K)$ intersect with general curves.

L^d stands for the family of algebraic curves on $\mathbf{P}^2(K)$, so curves from L^1 are called *straight lines.*

Lemma 8.1 *Let c be a general irreducible curve. There is a finite subset c_s of c such that for any $d > 0$ and any line $l \in L^d$ tangent to c at a point $p \in c \setminus c_s$ there is a straight line $l_p \in L^1$ which is tangent to both l and c.*

Proof. (Assuming $\mathbf{P}^2(K)$ is complete). By definition of tangency there are distinct points $p', p'' \in \mathcal{V}_p \cap l'_1 \cap l'_2$ for generic $l' \in L^d \cap \mathcal{V}_l$. Obviously, $\langle p', p'' \rangle$ is generic in $c \times c$. Take now the straight line l' passing through p', p''. Set $l_p = \pi(l')$.$\Box$

Lemma 8.2 *Let $l_1 + \cdots + l_d$ denote a curve of degree d, which is a union of d distinct straight lines with no three of them passing through a common point. Then a straight line l is tangent to $l_1 + \cdots + l_d$ with respect to L^1, L^d iff it coincides with one of the lines $l_1, \ldots, l_d$.*

Proof. If l is tangent to $l_1 + \cdots + l_d$, then they intersect in less than d points or have an infinite intersection. In our case only the latter is possible. $\Box$

Definition 8.3 For a family L of curves call **degree of curves of** L the number

$$\deg(L) = \mathrm{ind}(L, L^1),$$

that is the number of points in the intersection of a generic member of L with a generic straight line.

For a single curve c we write $\deg^*(c)$ for $\deg(\{c\})$, that is for the number of points in the intesection of c with a generic straight line.

Theorem 8.4 (The generalised Bezout theorem) *For any curve c on $\mathbf{P}^2(K)$*

$$\mathrm{ind}(\{c\}, L^d) = d \cdot \deg^* c,$$

in particular, for an algebraic curve a

$$\# c \cap a \leq \deg^* c \cdot \deg a.$$

Proof. Assume $a \in L^d$ and take $l_1 + \cdots + l_d$ as above such that none of the straght lines is tangent to c.
Claim. c and $l_1 + \cdots + l_d$ are not tangent.
By 8.1 the tangency would imply that there is an l tangent to c and tangent to $l_1 + \cdots + l_d$. Lemma 8.2 says this not the case.
The claim implies that the intersection indices of the curves c and $l_1 + \cdots + l_d$ are equal to 1 for any point in the intersection, so by formula 5.4(ii)

$$\mathrm{ind}(\{c\}, L^d) = \#c \cap (l_1 + \cdots + l_d) = d \cdot \deg^* c.$$

On the other hand

$$\#c \cap a \leq \mathrm{ind}(\{c\}, L^d)$$

since point multiplicities are minimal for generic intersections, by 5.4(iii). $\square$

Lemma 8.5 *If a curve c is a subset of an algebraic curve a, then c is algebraic.*

Theorem 8.6 (The generalised Chow theorem) *Any closed subset of $\mathbf{P}^n$ is an algebraic subvariety of $\mathbf{P}^n$.*

Proof. (For $n = 2$.) Let c be a closed subset of $\mathbf{P}^2$. W.l.o.g. we may assume c is an irreducible curve. Let $q = \deg^* c$. Now choose d such that $(d-1)/2 > q$. Fix a subset X of c, containing exactly $d \cdot q + 1$ points. Then by dimension considerations there is a curve $a \in L^d$ containing X. By the generalised Bezout Theorem $\#(c \cap a) \leq d \cdot q$ or the intersection is infinite. Since the former is excluded by construction, c has an infinite intersection with the algebraic curve a. Thus c coincides with an irreducible component of a, which is also algebraic by Lemma 8.5.$\square$

Theorem 8.7 (The purity theorem) *Any relation R induced on K from M is definable in the natural language and so is constructible.*

Proof. By elimination of quantifiers for Zariski structures it suffices to prove the statement for closed $R \subseteq K^n$. Consider the canonical (algebraic) embedding of K^n into $\mathbf{P}^n$ and the closure $\bar{R} \subseteq \mathbf{P}^m$ of R. By the generalised Chow theorem $\bar{R}$ is an algebraic subset of $\mathbf{P}^n$. But $R = \bar{R} \cap K^n$.$\Box$

9 Main Theorem

Theorem 9.1 (Main Theorem) *Let M be a a Zariski structure satisfying (EU) and C a presmooth Zariski curve in M. Assume that C is non-linear (equivalently C is ample in the sense of section 6). Then there is a nonconstant continuous map*

$$f : C \to \mathbf{P}^1(K).$$

Moreover, f is a finite map ($f^{-1}(x)$ is finite for every $x \in C$), and for any n, for any definable subset $S \subseteq C^n$, the image $f(S)$ is a constructible subset (in the sense of algebraic geometry) of $[\mathbf{P}^1(K)]^n$.

Proof. This is an easy corollary of theorems in the previous sections. $\Box$

Definition 9.2 For a given Zariski set N and a field K a continuous function $g : N \to K$ with the domain containing an open subset of N will be called **Z-meromorphic on** N.

Notice that the sum and the product of two meromorphic functions on N are Z-meromorphic. Moreover, if g is Z-meromorphic and nonzero then $1/g$ is a meromorphic function. In other words the set of meromorphic functions on N forms a field.

We denote $K_Z(N)$ the **field of Z-meromorphic functions on** N.

Remark 9.3 Notice that if the characteristic of K is $p > 0$ then with any Z-meromorphic function f one can associate distinct Z-meromorphic functions $\phi^n \circ f$, $n \in \mathbb{Z}$, where ϕ is the Frobenious automorphism of the field $x \mapsto x^p$.

More refined version of the main theorem is the following.

Proposition 9.4 *Under the assumptions of 9.1 there exists a smooth algebraic curve X over K and a Zariski epimorphism*

$$\psi : \ C \to X$$

with the universality property: for any algebraic curve Y over K and a Zariski epimorphism $\tau : \ C \to Y$ there exists a Zariski epimorphism $\sigma : X \to Y$ such that $\sigma \circ \tau = \psi$.
The field $K(X)$ of rational functions is isomorphic over K to a subfield of $K_Z(C)$ and $K_Z(C)$ is equal to the unseparable closure of the field $K(X)$.

Remark 9.5 In general τ is not a bijection, that is C is not isomorphic to an algebraic curve.

The main theorem is crucial to prove the Algebraicity Conjecture for groups definable in presmooth Zariski structures.

Theorem 9.6 (Algebraicity of groups) *Let G be a simple Zariski group satisfying (EU) and such that some one-dimensional irreducible Z-subset C in G is presmooth. Then G is Zariski isomorphic to an algebraic group $\hat{G}(K)$, for some algebraically closed field K.*

Proof. We start with a general statement.
Claim 1. Let G be a simple group of finite Morley rank. Then $\mathrm{Th}(G)$ is categorical in uncountable cardinals (in the language of groups). Moreover, G is almost strongly minimal.
This is a direct consequence of the Indecomposability Theorem on finite Morley rank groups.
Claim 2. Given a strongly minimal set C definable in G, there is a a definable relation $F \subseteq G \times C^m$, $m = \mathrm{rk}\, G$, establishing a finite-to-finite correspondence between a subset $R \subseteq G$ and a subset $D \subseteq C^m$ such that $\dim(G \setminus R) < m$ and $\dim(C^m \setminus D) < m$.
This is a consequence of the proof of the above statement.
Claim 3. For G as in the condition of the theorem, there exists a a nonconstant meromorphic function $G \to K$.

To prove the claim first notice that C in Claim 2 can be replaced by K because there is a finite-to-finite correspondence between the two. Now apply the argument with symmetric functions as in the proof of the Claim in Main Theorem. This proves the present claim.
Now consider the field the field $K_Z(G)$ of meromorphic functions $G \to K$. Each $g \in G$ acts on $K_Z(G)$ by $f(x) \mapsto f(g \cdot x)$. This gives a representation of G as the group of automorphisms of $K_Z(G)$. This action can also be seen as the K-linear action on the K-vector space $K_Z(G)$. As is standard in the theory of algebraic groups (Rosenlicht's Theorem) using the Purity Theorem one can see that there is a G-invariant finite dimensional K-subspace V of $K_Z(G)$. Hence G can be represented as a definable subsgroup $\hat{G}(K)$ of $\mathrm{GL}(V)$, and by the Purity Theorem again this subgroup is algebraic. This representation is an isomorphism since G is simple.□

Notice that presmoothness is paramount for this proof. In the case of Zariski groups without presmoothness (which, of course, still are of finite Morley rank by Theorem 2.7) the Algebraicity Conjecture remains open.

Lectures on Zariski-type structures
Part II

10 Analytic Zariski structures

(Based on joint works and discussions with N.Peatfield, L.Smith, A.Hasson, M.Gavrilovich and J.Kirby)

We introduce analytic-Zariski structures as (non-Noetherian) topological structures with good dimension notion, that is the *fiber condition* (FC) and *the addition formula* (AF) hold in the same form as in section 1.

We change the condition of semi-properness (SP) to a more general form consistent with its previous use.

We also generalise (DU), (SI) and (EU) and add an important assumption (AS), the analytic stratification of closed sets.

The intersection of a family of basic closed sets is called closed. The union of a family of basic open sets is called open.

Warning. Closed and open in general are not first-order definable.

We write $U \subseteq_{op} M^n$ to say that U is open in M^n and $S \subseteq_{cl} M^n$ to say 'closed'.

Dimension.

To any nonempty definable S a non-negative integer called **the dimension of** S, $\dim S$, is attached.
In addition to (DP) we assume also.

(CU) **countable unions** If $S = \bigcup_{i \in \mathbb{N}} S_i$ then $\dim S = \max_{i \in \mathbb{N}} \dim S_i$;

(WP) (**weak properness**). Let $D \subseteq_{cl} U \subseteq_{op} M^n$, $F \subseteq_{cl} D \times V$, $V \subseteq_{op} M^k$ and $\mathrm{pr} : D \times V \to D$ be the projection map.
Suppose $\dim \mathrm{pr}(F) = \dim D$. Then there are closed subsets $D_1, D_2 \subseteq_{cl} D$ such that $D_1 \setminus D_2 \subseteq \mathrm{pr}(F)$ and $\dim(D_1 \setminus D_2) = \dim D$.

Obviously, (SP) implies (WP).

Exercise 10.1 *Show that* (CU) *in the presence of* (DCC) *implies both* (DU) *and* (EU) *of section 1.*

Irreducibility.

Definition 10.2 A definable set $S \subseteq M^n$ is said to be **strongly irreducible** if, for every $S' \subsetneq S$, closed in S, we have $\dim S' < \dim S$.

Remark 10.3 For Zariski structures we postulated by (SI) that irreduciblity is equivalent to strong irreducibility. Now this is true for *analytic sets* only (see below).

We postulate as before for S closed in an open set and strongly irreducible:

(AF)
$$\dim \mathrm{pr}(S) = \dim S - \min_{u \in \mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}(u) \cap S);$$

(FC)
$$\{a \in \mathrm{pr}(S) : \dim \mathrm{pr}^{-1}(a) \cap S \geq k\}$$

is closed in $\mathrm{pr}(S)$.

Analytic subsets.

Definition 10.4 A subset $S \subseteq U \subseteq_{op} M^n$ of an open set is called **analytic in** U if S is closed in U and for every $a \in S$ there is an open $V_a \subseteq_{op} U$ such that $S \cap V_a$ can be decomposed into finitely many strongly irreducible subsets.

We postulate the following properties

(INT) (Intersections) If $S_1, S_2 \subseteq_{an} U$ are irreducible then $S_1 \cap S_2$ is analytic in U;

(CMP) (Components) If $S \subseteq_{an} U$ and $a \in S$ then there is $S_a \subseteq_{an} U$, a finite union of irreducible analytic subsets of U, and some $S'_a \subseteq_{an} U$ such that $a \in S_a \setminus S'_a$ and $S = S_a \cup S'_a$;

Each of the irreducible sets whose union is S_a above is called an **irreducible component of S containing a.**

(CC) (Countability of the number of components) Any $S \subseteq_{an} U$ is the union of at most countably many irreducible components.

Exercise 10.5 *For S analytic and $a \in \mathrm{pr}(S)$, the fiber $S(a, M)$ is analytic.*

Exercise 10.6 *If $S \subseteq_{an} U$ is irreducible, V open, then $S \cap V$ is an irreducible analytic subset of V and, if non-empty,* $\dim S \cap V = \dim S$.

Exercise 10.7 *(i) $\emptyset$, any singleton and U are analytic in U;*

(ii) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cup S_2$ is analytic in U;

(iii) If $S_1 \subseteq_{an} U_1$ and $S_2 \subseteq_{an} U_2$, then $S_1 \times S_2$ is analytic in $U_1 \times U_2$;

(iv) If $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$;

(v) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cap S_2$ is analytic in U.

Definition 10.8 Given a subset $S \subseteq_{cl} U \subseteq_{op} M^n$ we define the notion of the **analytic rank** of S in U, $\mathrm{ark}_U(S)$, which is a natural number satisfying

1. $\mathrm{ark}_U(S) = 0$ iff $S = \emptyset$;

2. $\mathrm{ark}_U(S) \leq k+1$ iff there is a set $S' \subseteq_{cl} S$ such that $\mathrm{ark}_U(S') \leq k$ and with the set $S^0 = S \setminus S'$ being analytic in $U \setminus S'$.

The next assumptions guarantees that the class of analytic subsets explicitly determines the class of closed subsets in .

(AS) [**Analytic stratification**] Any closed subset $S \subseteq M^n$ is of finite analytic rank.

Obviously, any nonempty analytic subset of U has analytic rank 1.

Example 10.9 Let $F \subseteq \mathbb{C}^2$ be a graph of an entire analytic function and $\bar{F}$ its closure in $[\mathbf{P}^1(\mathbb{C})]^2$. It follows from Picar's Theorem that

$$\bar{F} = F \cup (\{\infty\} \times \mathbf{P}^1(\mathbb{C})),$$

in particular $\bar{F}$ has analytic rank 2.

Generalised analytic sets are defined as the subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ for all n, obtained from classical (algebraic) Zariski closed subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and $\bar{F}$ by applying the positive operations: cartesian products, finite intersections, unions and projections.
This definition and the following theorem are valid for an arbitrary algebraically closed complete valued field instead of $\mathbb{C}$.

Theorem (1996) *Any generalised analytic set in an algebraically closed complete valued field is of finite analytic rank.*

So, the prototype of *closed sets* are the generalised analytic sets.

A topological structure with good dimension satisfying axioms (INT)-(AS) will be called an **analytic Zariski structure**.

We also are going to use the following special properties.

(PS) **Presmoothness** If $S_1, S_2 \subseteq_{an} U \subseteq_{op} M^n$ both S_1, S_2 irreducible, then for any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

(MLT) **Multiplicities**
Let $S \subseteq_{an} U \times V$, $U \subseteq_{op} M^k$, $V \subseteq_{op} M^m$, $(a, b) \in S$ and suppose S is given by a relation $\mathrm{S}(u, v)$ such that all the fibers of the natural projection $(u, v) \mapsto v$ restricted to S are of dimension 0. Then there exist open subsets $U_0 \subseteq U$, $V_0 \subseteq V$ and an $\mu \in \mathbb{N}$ such that $(a, b) \in U_0 \times V_0$ and if $S^{(\mu)}$ is the analytic subset of $U_0 \times V_0^\mu$ given by $\mathrm{S}(u, v_1) \& \ldots \& \mathrm{S}(u, v_\mu)$, $(a', b'_1, \ldots, b'_\mu) \in S^{(\mu)}$ then $b'_i = b'_j$ for some $i < j \leq \mu$.

Notice that (MLT) is automatic for Noetherian presmooth Zariski structures, see section 4.

11 Compact analytic Zariski structures

We assume in this section that M is compact, that is projections of closed sets are closed (P) and any finitely consistent family of closed sets has a nonempty intersection (QC).

The following theorem from a joint paper with N.Peatfield is an abstract analogue of theorems about complex and rigid analytic manifolds.

Theorem 11.1 *Let $(M, \mathcal{C})$ be a compact analytic Zariski structure and $\mathcal{C}^0$ be the subfamily of $\mathcal{C}$ consisting of subsets analytic in M^n, all n. Then $(M, \mathcal{C}^0)$ is a (Noetherian) Zariski structure.*

12 Non-elementary model theory of analytic Zariski structures

We discuss a QE issues in this section. Presmoothness is not assumed.

Definition 12.1 We call a countable structure M_0 with a countable family $\mathcal{C}_0$ of basic relations **skeletal** in $(M, \mathcal{C})$ if

1. $M_0 \subseteq M,\ \mathcal{C}_0 \subseteq \mathcal{C},\ (M_0, \mathcal{C}_0) \prec (M, \mathcal{C}_0)$;
2. $(M_0, \mathcal{C}_0)$ is a topological structure with dimension induced from M and satisfies analytic stratification (AS);
3. for any $S \subseteq_{an} U \subseteq_{op} M^n$ such that S is M_0-definable, every irreducible irreducible component S_i of S is M_0-definable.

Exercise 12.2 *Given a countable $\mathcal{C}^0 \subseteq \mathcal{C}$, there exists a countable skeletal $(M_0, \mathcal{C}_0)$ such that $\mathcal{C}^0 \subseteq \mathcal{C}_0$.*

We fix below a countable skeletal $(M_0, \mathcal{C}_0)$.

Definition 12.3 For finite $X \subseteq M$ we define the $\mathcal{C}_0$-predimension

$$\delta(X) = \min\{\dim S :\ X \in S,\ S \subseteq_{an} U \subseteq_{op} M^n,\ S \text{ is } \mathcal{C}_0\text{-definable}\}$$

and dimension

$$\partial(X) = \min\{\delta(XY) :\ Y \subseteq M\}.$$

For $X \subseteq M$ finite, we say that X is **self-sufficient** and write $X \leq M$, if $\partial(X) = \delta(X)$.

Definition 12.4 A subset $P \subseteq M^n$ will be called **projective** if $P = \mathrm{pr}(S)$, for some $S \subseteq_{an} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$.

We work now under assumption that $\dim M = 1$ and M is irreducible.

Lemma 12.5 *Given a projective M_0-definable $P \subseteq M^n$,*

$$\dim P = \max\{\partial(x) :\ x \in P(M)\}.$$

Lemma 12.6 *Suppose $X \leq M$, $X' \leq M$ and the quantifier-free M_0-type of X is equal to that of X'. Then*

$$X \equiv_{\omega_1,\omega} X'.$$

Proof. We show that for any Y such that $XY \leq M$ there is Y' with $X'Y' \leq M$ and $XY \equiv_{q-free} X'Y'$. This wins the Ehrenfeucht-Fráisse game for $X \equiv X'$.$\square$

Definition 12.7 For $x \in M^n$, the **projective type of** x **over** M is

$$\{P : \ x \in P(M), \ P \text{ is a projective set over } M_0\} \cup$$

$$\cup \{\neg P : \ x \notin P(M), \ P \text{ is a projective set over } M_0\}.$$

Lemma 12.8 *Suppose, for finite $X, X' \subseteq M$, the projective M_0-types of X and X' coincide. Then*

$$X \equiv_{\omega_1,\omega} X'.$$

Proof. We can extend $X \subseteq \tilde{X} \leq M$ and $X' \subseteq \tilde{X}' \leq M$ so that

$$\tilde{X} \equiv_{q-free} \tilde{X}'.$$

Now apply the Lemma above.

Corollary 12.9 (Non-elementary near model completeness.) *Every M_0-$L_{\omega_1,\omega}$-type realised in M is equivalent to a projective type.*

13 Non-standard analysis in analytic Zariski structures

We study here universal specialisations $\pi : {}^*M \to M$ from an elementary extension ${}^*M \succ M$ onto an analytic Zariski structure M.

Important change in the definition. It is useful to restrict the choice of possible ${}^*M$ to the ones that preserve (countable) irreducible decompositions. So, in particular, for a descrete (that is dimension 0) analytic S, we will have

$$S({}^*M) = S(M).$$

Proposition 13.1 *Let $D \subseteq_{an} U \subseteq_{op} M^m$ be a presmooth irreducible set and $F \subseteq_{an} D \times V$ be a finite cover of D, $V \subseteq_{op} M^k$, $a \in D$.*
Then, for every $a' \in \mathcal{V}_a \cap D({}^*M)$ there exists $b' \in \mathcal{V}_b$, such that $\langle a', b' \rangle \in F$.

Corollary 13.2 *The Implicit Function Theorem holds in presmooth analytic Zariski structures.*

Corollary 13.3 *Let G be a presmooth analytic Zariski group, ${}^*G \succ G$ and $\pi : {}^*G \to G$ a universal specialisation. Then the infinitesimal neighborhood $\mathcal{V}$ of the unit of the group is a subgroup of ${}^*G$.*

14 Analytic Zariski structures. Examples

14.1 Easy examples.

Simplified exponentiation

Let $(V, \exp, F)$ be the two-sorted structure with

$$V = (\mathbb{C}, +), \quad F = (\mathbb{C}, +, \cdot), \quad \exp : V \to F.$$

Theorem 14.1 *The first order theory of* $(V, \exp, F)$ *allows quantifier-elimination in the natural language and is superstable.*
$(V, \exp, F)$ *is presmooth analytic Zariski, all closed sets are analytic.*

$(V, \exp, F)$ is not compact and not complete.

Problem Describe possible completions and compactifications of

$$(V^n, \exp, (F^\times)^n)$$

such that V gets a linear completion $C(V^n)$ and the map exp extends to

$$C(V^n) \to C(F^\times)^n).$$

This problem leads to *toric geometry* and corresponding combinatorics of *cones and fans.*

Because of weak elimination of imaginaries in linear structures "cones-fans" method is almost the only possible way of completing $(V^n, \exp, (F^\times)^n)$.

Theorem 14.2 (L.Smith) *The completion is presmooth if and only if the corresponding fan is simplicial. In particular, the corresponding mirror orbifold (in sort F) is presmooth.*

Analytic tori

We want to bring a reacher language into the class of complex tori, which in the natural (Zariski) language of compact complex manifolds in generic cases reduces to the language of commutative groups.

$$T = \mathbb{C}^g/\Lambda$$

admits no (global) analytic subsets $S \subseteq T^k$ other than those definable in the group structure, if Λ does not satisfy certain conditions.

Lemma 14.3 *For any $2n$-lattice Λ (with $2n$ generators) there is a n-sublattice $\Lambda_0 \subseteq \Lambda$ such that*

$$\mathbb{C}^g/\Lambda_0 \cong (\mathbb{C}^\times)^g \quad \textit{as complex manifolds}$$

and, setting $\Omega = \exp(\Lambda)$, one has

$$T \cong (\mathbb{C}^\times)^g/\Omega \quad \textit{as complex manifolds.}$$

In other words, T is the image under the holomorphic group homomorphism $\mathfrak{p} : (\mathbb{C}^\times)^g \to T$.

Similar construction makes sense in the context of rigid analytic geometry. Namely, for an algebraically closed field K, complete with respect to a non-Archimedean valuation, one considers a discrete subgroup $\Omega \subset (K^\times)^g$ (with respect to multiplication) of rank g and $T = (K^\times)^g/\Omega$.

Let K be an algebraically closed field, $g \geq 1$ and $\Omega \subset K^g$, a g-generators free subgroup of the multiplicative group $(K^\times)^g$. Let

$$T = (K^\times)^g/\Omega \ \text{ and } \ \mathfrak{p}: \ (K^\times)^g \to T \text{ the canonical group homomorphism.}$$

We write $\dot{K}$ for the structure on $K \setminus \{0\}$ in the language with all the algebraic Zariski closed relations as basic.
Let $\dot{K}_\Omega$ denote the expansion of $\dot{K}$ by the g-ary predicate Ω for the subgroup.

Theorem 14.4 *(i) The theory of $\dot{K}_\Omega$ is near model complete (every definable subset is the Boolean combination of projections of quantifier-free).*
(ii) The theory of $\dot{K}_\Omega$ is superstable.

(iii) Any definable subset of Ω^k is a coset of a subgroup Γ of the form

$$\Gamma = \Omega^k \cap \mathrm{Tor}, \quad \mathrm{Tor} \text{ an algebraic subgroup of } \dot{K}^{gk}.$$

(iv) $\dot{K}_\Omega$ is a presmooth analytic Zariski structure.

Proof. Use the Lang property of Ω in $\dot{K}$.$\square$

Obviously $\dot{K}_\Omega$ is definably equivalent to the two-sorted structure $\left(\dot{K}, \mathtt{p}, T\right)$.

Theorem 14.5 $\left(\dot{K}, \mathtt{p}, T\right)$ *is presmooth analytic Zariski in both sorts. It is compact in sort T.*

By the Implicit Function Theorem we have a nontrivial K-algebra of (infinitesimally local) functions

$$f : \mathcal{V}_0 \subseteq^* T \to \mathcal{V}_0 \subseteq^* K.$$

15 Hard examples

Complex exponentiation and the pseudo-exponentiation

We want to endow the field structure $(K, +, \cdot)$ with a new function ex so that

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

and $(K, +, \cdot, \mathrm{ex})$ is as stable as possible
Following Hrushovski's method we introduce the predimension function, for $X \subseteq_{\mathrm{fin}} K$,

$$\delta(X) = \mathrm{tr.deg}_{\mathbb{Q}}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}_{\mathbb{Q}}(X)$$

and postulate first requirement for ex

$$\delta(X) \geq 0 \quad \text{all } X.$$

The Hrushovski inequality, in the case of the complex numbers and ex = exp, is equivalent to

$$\text{tr.deg}_{\mathbb{Q}}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent.
This is **the Schanuel conjecture.**

We then carry out Frásse amalgamation in the class of all structures $K_{\text{ex}} = (K, +, \cdot, \text{ex})$ satisfying

$$\ker \text{ex} \cong \mathbb{Z}.$$

The resulting structures satisfy the Schanuel condition (Sch) and are existentially closed (EC) in the class.

Reminder. Hrushovski's dimension

$$\partial(X) := \min\{\delta(XY) : Y \subseteq_{fin} K\}.$$

$$\partial(Y/X) = \partial(X \cup Y) - \partial(X).$$

$\partial(y/X) = 0$ is a dependence relation.

Theorem 15.1 (Categoricity) *For any uncountable κ there exists unique K_{ex} with* card, $K = \kappa$ *satisfying (Sch), (EC) and (CCP), the* **countable closure property***: for any $X \subseteq_{fin} K$ there is at most countably many $y \in K$ dependent on X.*

Proof. We use
(1) non-elementary model theory (Shelah's excellent classes);
(2) Galois and Kummer's theory;
(3) the theory of linearly disjoint extensions of fields.□

Existential closedness (EC) is the condition:

Let $P_a(x_1, \ldots x_n, y_1, \ldots, y_n)$ be an irreducible system of polynomials with coefficients $\bar{a}$ and $(x_1^0, \ldots x_n^0, y_1^0, \ldots, y_n^0)$ its generic zero. Assume

(normality)

$$\mathrm{tr.deg}_{\mathbb{Q}_a}(x_{i_1}^0, \ldots x_{i_m}^0, y_{i_1}^0, \ldots, y_{i_m}^0) \geq m$$

for any distinct $i_1, \ldots, i_m$ after any *admissible* transformation of variables

(freeness) $x_i^0 \notin \mathrm{acl}\mathbb{Q}(a)$ and $y_i^0 \notin \mathrm{acl}\mathbb{Q}(a)$ for all i after any *admissible* transformation of variables.

Then there is a generic zero of P_a such that

$$y_i^0 = \mathrm{ex}(x_i^0), \quad i = 1, \ldots, n.$$

Theorem 15.2 *Assume* Schanuel's conjecture *as well as (EC) holds for* $\mathbb{C}_{\exp}$. *Then* $\mathbb{C}_{\exp}$ *is isomorphic to the unique* K_{ex} *of cardinality continuum.*

Proposition 15.3 *The normality and freeness of* P_a *is an algebraic (constructible) condition on* $\bar{a}$.

Corollary 15.4 *An uncountable* K_{ex} *is a topological structure with good dimension satisfying semi-properness of projection condition.*

Theorem 15.5 *An uncountable field with pseudo-exponentiation is a pres-smooth analytic Zariski structure.*

Lectures on Zariski-type structures
Part III

Elements of non-commutative Zariski geometry

16 Non-algebraic presmooth Noetherian Zariski curve

Let $G = \mathbb{Z}$ act nontrivially on $\mathbb{Z}/4\mathbb{Z}$, and let $\tilde{G}$ be the semidirect product: f, h are the generators of $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}$, respectively,

$$f^4 = 1; \quad fh = hf^{-1} \tag{2}$$

are the defining relations.
Notice, that h^2 commutes with f, and hence, the centre $Z(G)$ of G is generated by h^2 and f^2, so has index 4 in G.
Fix $a \in \mathbb{C}$, not a root of unity. We claim that there is a strongly minimal double cover $p : D \to \{\mathbb{C}^\times\}$ in the language containing the relations induced by the field structure on $\mathbb{C}$ and unary operations f, h, defining a faithful action of $\tilde{G}$ on D such that:
f, h respect the kernel of p (that is $p(x) = p(x')$ implies $p(f(x)) = p(f(x'))$ and $p(h(x)) = p(h(x'))$ and induce the (unfaithful) action on $\{\mathbb{C}^\times\}$:

$$p(h(x)) = a^2 p(x), \quad p(f(x)) = -p(x). \tag{3}$$

The final example is a structure with the universe D and an equivalence relation E given by the kernel of p on D;

f, h are operations, acting on D as described above;
for any Zariski-closed relation S on $\mathbb{C}^\times$ we have $p^{-1}(S)$ as a relation on D.

Call the relations introduced above closed and introduce the dimension notion in D as induced from $\mathbb{C}$ by p.

Then using elimination of quantifiers one gets

Proposition 16.1 *D is an irreducible 1-dimensional presmooth Zariski structure. Its field of meromorphic functions is $\mathbb{C}(t)$, same as for the affine line $\mathbb{C}$.*

Proposition 16.2 *D is not definable in an algebraically closed field.*

Proof. Suppose to the contrary that D is an algebraic curve.

Claim 1. D is algebraically irreducible.
Use the fact that $\tilde{G}$ is non-splitting.

Claim 2. D can not have a subgroup of regular automorphisms isomorphic to $\tilde{G}$.$\Box$

Interpretation in the reals.
Denote

$$Q_0 = \{x \in \mathbb{C} : \mathrm{Re}(x) \geq 0 \ \& \ \mathrm{Im}(x) > 0 \quad \vee \quad \mathrm{Re}(x) \leq 0 \ \& \ \mathrm{Im}(x) < 0\},$$

$$Q_1 = iQ_0,$$

$$D = Q_0 \cup Q_1$$

$p(x) = x^2$, $f(x) = ix$,

$$h(x) = \begin{cases} ax, & \text{if } x \in Q_0 \\ -ax, & \text{if } x \in Q_1 \end{cases} \tag{4}$$

Notice that $f^4(x) = x$, $f(h(x)) = h(f^{-1}(x))$ for any $x \in \mathbb{C}$ and $p(h(x)) = a^2p(x)$, $p(f(x)) = -p(x)$. It follows that the construction is isomorphic to the Example above.

Coordinate algebra of D

Lemma 16.3 *We can define*

$$x : D \to \mathbb{C} \text{ and } \breve{x} : D \to \mathbb{C}$$

so that inside any orbit $\tilde{G} \cdot e$, $e \in D$, it agrees with f and h :

$$x(t) \cdot x(t) = p(t) = \check{x}(t) \cdot \check{x}(t)$$
$$x(f(t)) = ix(t),\ \check{x}(f(t)) = -i\check{x}(t)$$
$$x(h(t)) = a\check{x}(t),\ \check{x}(h(t)) = ax(t).$$

Let $\mathcal{H}$ be the $\mathbb{C}$-algebra generated by x and $\check{x}$.
f and h act on $\mathcal{H}$ as linear operators

$$F : \psi \mapsto \psi \circ f,$$

$$H : \psi \mapsto \psi \circ h,$$

and so do

$$X : \psi \mapsto x \cdot \psi \ \text{ and } \ \check{X} : \psi \mapsto \check{x} \cdot \psi.$$

We get an algebra $\mathrm{A} = \mathrm{A}(D)$ of linear operators that reflects the structure of D.

Lemma 16.4 *The isomorphism type of* $\mathrm{A}(D)$ *does not depend on the concrete choice of x and $\check{x}$ as long as the defining relations above hold.*

Define the sign function

$$\mathrm{sgn}(t) := \frac{\check{x}(t)}{x(t)}.$$

By defining relations the value of $\mathrm{sgn}(t)$ is either 1 or -1.
Set for $\lambda \in \mathbb{C}^\times$,

$$\mathrm{sgn}(\lambda) = \mathrm{sgn}(t),\ \text{if } \lambda = p(t).$$

Lemma 16.5 *The function*

$$\mathrm{sgn} : \mathbb{C} \to \{-1, 1\}$$

is well-defined and satisfies the properties

$$\mathrm{sgn}(-\lambda) = -\mathrm{sgn}(\lambda),$$

$$\mathrm{sgn}(a^2\lambda) = \mathrm{sgn}(\lambda).$$

Irreducible representations.
Every point $t \in D$ gives rise to an irreducible representation, or an A-module M_t. The operators f and h act naturally on

$$M_t \cong M_s \text{ as A-modules iff } p(t) = p(s).$$

So this is uninteresting (but recovers $\mathbb{C}^\times$).

We choose a *leading eigenvector* in each M_t and define the *orientation* of M_t to be 1 or -1 using sgn.

Proposition 16.6 *The space $\mathcal{D}$ of irreducible modules of orientation* 1 *are in a one-to-one correspondence with points of D. The operators f and h act naturally on the space of modules preserving the orientation. $\mathcal{D}$ is isomorphic to D as Zariski structures.*

17 A generalisation. Quantum torus at the root of unity.

Let $\epsilon \in K$ be a primitive root of unity of prime order n, K an algebraically closed field.
We define the K-algebra Q_n,

$$Q_n = \langle U, U^{-1}, V, V^{-1} \rangle$$

as the algebra given by the generators and relations .

$$UU^{-1} = 1, \quad VV^{-1} = 1,$$
$$U^n = V^n,$$
$$UV = \epsilon VU.$$

We consider the n-sign function

$$\mathrm{sgn} : F \to \{1, \epsilon, \dots, \epsilon^{n-1}\}$$

satisfying, for a fixed $\alpha \in F$,

$$\mathrm{sgn}(\epsilon \cdot \lambda) = \epsilon \mathrm{sgn}(\lambda),$$
$$\mathrm{sgn}(\alpha\lambda) = \mathrm{sgn}(\lambda).$$

We consider the space $\mathcal{Q}_n$ of all orientation 1 irreducible Q_n-modules with the action of natural operators from Q_n on it.

Proposition 17.1 *$\mathcal{Q}_n$ is a 1-dimensional non-algebraic smooth Zariski structure covering the algebraic variety $F^\times$.*
In particular,

$$\mathcal{Q}_2 = \mathcal{D}$$

from the original example.

If we start with 'Manin's quantum plane'

$$UV = \epsilon VU$$

then we have a non-algebraic covering of $F \times F$.

18 Poizat's bad field

Definition 18.1 **A bad field** is a structure $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = N > 1$ and $G < K^\times$, a multiplicative subgroup, $\mathrm{MR}(G) = 1$.

Problem (197?) Do bad field exist?

Theorem (Baldwin and Holland, 2002) Yes, for each $N > 1$, if we drop the requirement that G is a group.

Theorem (B.Poizat, 2000) There exists an almost bad fields $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = \omega \times N$ and
$G < K^\times$, a multiplicative subgroup, $\mathrm{MR}(G) = \omega$.

Proof. Use Hrushovski's construction.

Problem. Explain these examples analytically.

Solutions, for $N = 2$:

Take $K = \mathbb{C}$.
Let h be an irrational real number and

$$G = \{\exp((1 + ih)t + q) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

Theorem 18.2 *Assume Schanuel's conjecture. Then* $(\mathbb{C}, +, \cdot, G)$ *is a model of Poizat's theory.*

Let

$$G = \{\exp((1 + ih)t + m) : t \in \mathbb{R},\ m \in \mathbb{Z}\}$$

Theorem 18.3 *Assume Schanuel's conjecture. Then*

1. $Th(\mathbb{C}, +, \cdot, G)$ *is near model complete and superstable,*

$U(\mathbb{C}) = \omega \cdot 2,\ U(G) = \omega.$

2. The spiral $G^0 = \exp[(1 + ih)\mathbb{R}]$ *is type-definable in* $(\mathbb{C}, +, \cdot, G)$ *as the* connected component *of* G.

3. The field of reals is $L_{\omega_1,\omega}$*-definable in* $(\mathbb{C}, +, \cdot, G)$.

In terms of Hrushovski dimension

$$\dim(\mathbb{C}) = 2, \quad \dim(G) = 1$$

in this theory.

Problem Is $(\mathbb{C}, +, \cdot, G)$ in a natural language an analytic Zariski structure?

Claim The structure

$$T_h^2 := \mathbb{C}^\times / G$$

in the induced language is a form of the non-commutative torus.

Definition 18.4 (A.Connes) The noncommutative (quantum) torus is

$$T_h^2 = (S \times S)/F_h$$

where

$$S \cong \{x \in \mathbb{C} : |x| = 1\}$$

and

$$F_h = \{\langle \exp(it), \exp(iht) \rangle : t \in \mathbb{R}\}, \quad \text{the Kronecker foliation.}$$

Indeed,

$$T_h^2 := \mathbb{C}^\times / G$$

$$G = G^0 \cdot \Gamma, \quad \Gamma = \exp(\mathbb{Z}),$$

$$\mathbb{C}^\times / \Gamma \cong S \times S \quad \text{topologically}$$

the image of the spiral G^0 in $\mathbb{C}^\times / \Gamma$ corresponds to F_h.

Problem Is $\mathbb{C}^\times / G$ in the induced language analytic Zariski? Compare it to $(S \times S)/F_h$.

Alternative representations of the quantum torus.

Let $E(\mathbb{C}) \subseteq \mathbf{P}^2(\mathbb{C})$ be a (generic) elliptic curve. Let $F \subseteq E(\mathbb{C})$ be a Kronecker foliation on $E(\mathbb{C})$.

Feasible Theorem $E(\mathbb{C})$ in the language of Zariski closed relations together with F is near model complete and superstable. Hrushovski dimensions are

$$\dim E = 2, \quad \dim F = 1.$$

Feasible corollary

$$E(\mathbb{C})/F$$

in the induced language is a representation of T_h^2.

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