The noncommutative torus and Dirac calculus

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In [1] we considered the noncommutative torus T_q^2 for q a root of unity, and in [5] for generic q, which we showed to be approximated, under certain assumptions, by the structures at roots of unity. Using the idea of structural approximation from [5] we reduce calculations in the generic case to the case when q is a root of unity. This allows a rigorous Dirac calculus in the corresponding structure. We use this to calculate the kernels of the Feynman propagator for the free particle and the simple harmonic oscillator.

1 Preliminaries

Consider the algebra generated by P, Q satisfying the Heisenberg commutation relation

$$\mathrm{QP} - \mathrm{PQ} = i\hbar \qquad (1)$$

(here and below in expressions like QP first P is applied).

This algebra is usually represented by actions on various Hilbert spaces and it generalisations (known also as *rigged* Hilbert spaces). This results in calculations in terms of inner products, eigenvectors and eigenvalues of certain operators expressed in terms of P and Q. These calculations, crucial for physics, termed generally as Dirac calculus, are far from being rigorous, often involve making an educated guess on values of nonconvergent integrals. We present below an alternative approach based on the following:

1.1 We consider Dirac calculus in the context of non-commutative geometry. Crucially, we aim to represent ensuing non-commutative geometry in terms of Zariski geometries, as done e.g. in [1]. These structures are analogues and generalisations of (compact) complex manifolds and the substructure having relevance to physics could be seen as a "real part" of such a Zariski structure, or rather, the Zariski structure should be seen as a complexification of the physicist's one. This approach in general has the two potential advantages:

(i) The physicist's structure (e.g. the system of eigenstates of an operator with a discrete spectrum) is embedded in a rich complex geometric structure.

(ii) The topological nature of a Zariski structure allows an analysis of deformations and approximations.

1.2 In the class of compact Zariski geometries, and more generally, compact topological structures we can use the notion of a *structural approximation,* developed in [5], to replace non-commutaive geometry associated with the Heisenberg algebra by much nicer objects of finitary type, where Dirac calculus gets rigorous meaning.

This Dirac calculus based on structural approximation looks at the first glance quite similar to the usual discrete Dirac calculus based on lattices [2], Ch.12, but as a matter of fact, is essentially different. In particular, it is based on approximation of $\hbar$ by a rational number (in some physical units), and number-theoretic properties of this rational number are crucial for obtaining correct formulas of quantum physics.

1.3 We replace the Heisenberg algebra with the Weyl algebra generated by operators U and V defined as

$$U = \exp 2\pi i\mathrm{Q}, \quad V = \exp i\mathrm{P}$$

and, correspondingly, satisfying the commutation relation

$$VU = qUV, \qquad \text{for } q = \exp ih, \ h = 2\pi\hbar$$

This follows from the Heisenberg commutation relation by the Baker-Campbell-Hausdorff formula. In the same manner as the structure of a Lie algebra can replace the structure of a corresponding Lie group, the calculus in the Weyl algebra (U, V) serve as an adequate enough substitute for the calculus on the Heisenberg algebra (P, Q).

More crucially, we observe that the geometry based on the Weyl algebra (U, V) with generic value of q has an appropriate structural approximation by the geometry with q a root of unity (of order N). This shifts everything into the finitary context, where the usual Hilbert spaces are being replaced by N-dimensional ones in an adequate way. Here Dirac calculus obtains a standard rigorous meaning and we proceed with actual calculations, resulting in a derivation of formulas for Feynman propagators in two simple cases: for the free particle and for the simple harmonic oscillator.

2 Noncommutative 2-torus at root of unity

2.1 This was studied in [1] as one of the basic examples of a quantum Zariski geometry. Below we describe a somewhat different but equivalent (bi-interpretable) structure.

Let q be a root of unity of order N. We consider the algebra $\langle U, V\rangle$ with generators U, U^{-1}, V, V^{-1} and defining relations

$$VU = q\,UV, \quad UU^{-1} = I = VV^{-1}.$$

Fix $u, v \in \mathbb{F}^*$ Consider the irreducible $\langle U, V\rangle$-module $M_{u,v}$ such that $U^N = u^N I$, $V^N = v^N I$ on $M_{u,v}$. This module is generated by the U-eigenvectors $\mathbf{u}(u,v), \mathbf{u}(qu,v), \ldots \mathbf{u}(q^{N-1}u,v)$ (**a canonical basis**) satisfying

$$\begin{array}{l} U : \mathbf{u}(q^k u, v) \mapsto q^k u \mathbf{u}(q^k u, v) \\ V : \mathbf{u}(q^k u, v) \mapsto v \mathbf{u}(q^{k-1} u, v) \end{array} \tag{2}$$

Note, that the isomorphism type of the module is determined by the values u^N and v^N (which we call the **invariants of the module**) but our choice of the "canonical" basis depends on (u, v), see [1]. It is important to note that we use the notation $\mathbf{u}(q^k u, v)$ for convenience in our "metalanguage" but there is no definable correspondence $(u, v) \to \mathbf{u}(q^k u, v)$ in our language.

Lemma. Any two canonical bases $\{\mathbf{u}(u,v), \mathbf{u}(qu,v), \ldots \mathbf{u}(q^{N-1}u,v)\}$ and $\{\mathbf{u}'(u,v), \mathbf{u}'(qu,v), \ldots \mathbf{u}'(q^{N-1}u,v)\}$ given by the condition (2) differ only by a scalar muliplier, that is for some $c \neq 0$,

$$\mathbf{u}'(q^k u, v) = c\,\mathbf{u}(q^k u, v), \quad k = 0, \ldots, N-1.$$

Proof. We will have $\mathbf{u}'(u,v) = c\,\mathbf{u}(u,v)$ for some c since any two U-eigenvectors in an irreducible (U, V)-module are proportional. The rest follows from the definition (2) of the action by V on the bases. $\square$

Define

$$\mathbf{v}(q^m v, u) := \frac{1}{\sqrt{N}} \sum_{0 \le k < N} q^{km} \cdot \mathbf{u}(q^k u, v) \tag{3}$$

Then by (2) and (3),

$$\begin{array}{l} U : \mathbf{v}(q^k v, u) \mapsto u\mathbf{v}(q^{k+1} v, u) \\ V : \mathbf{v}(q^k v, u) \mapsto q^k v\mathbf{v}(q^k v, u) \end{array} \tag{4}$$

We want to think of $\{\mathbf{u}(q^k u, v) : k = 0, \ldots, N-1\}$ as an orthonormal basis of an inner product space. This leads to the definition of the **pairing** (generalised inner product)

$$\mathbf{v}(q^m v, u) * \mathbf{u}(q^k u, v) = \frac{1}{\sqrt{N}} q^{km}$$

and more generally,

$$(\sum_{k=0}^{N-1} a_k \mathbf{u}(q^k u, v)) * (\sum_{k=0}^{N-1} b_k \mathbf{u}(q^k u, v)) = \sum_{k=0}^{N-1} a_k \cdot b_k^{-1}. \tag{5}$$

Remark. In case $\mathbb{F} = \mathbb{C}$, the generalised inner product coincides with the canonically defined inner product if $|a_k| = 1 = |b_k|$, all k.

We can extend this definition to include elements from distinct modules, defining in this case

$$\mathbf{u}(u', v') * \mathbf{u}(u, v) = 0, \text{ if } v' \neq q^m v \text{ or } u' \neq q^k u, \text{ for some } m, k \in \mathbb{Z}.$$

2.2 An (U, V)-system as a structure. Model-theoretically we represent our structure as consisting of 3 sorts, $\mathbb{U}, \mathbb{V}$ and $\mathbb{F}$. $\mathbb{F}$ has the structure of a field and

$$\mathbb{U} = \{a \cdot \mathbf{u}(u, v) : a \in \mathbb{F},\ u, v \in \mathbb{F}^*\},$$

$$\mathbb{V} = \{b \cdot \mathbf{v}(v, u) : b \in \mathbb{F},\ u, v \in \mathbb{F}^*\},$$

with the corresponding pairing

$$\mathbb{U} \times \mathbb{V} \to \mathbb{F}$$

between the sorts.

Note, that in this structure the relation (3) makes sense, as well as the dual to it:

$$\mathbf{u}(q^k u, v) := \frac{1}{\sqrt{N}} \sum_{0\le k<N} q^{-km} \cdot \mathbf{v}(q^m v, u) \tag{6}$$

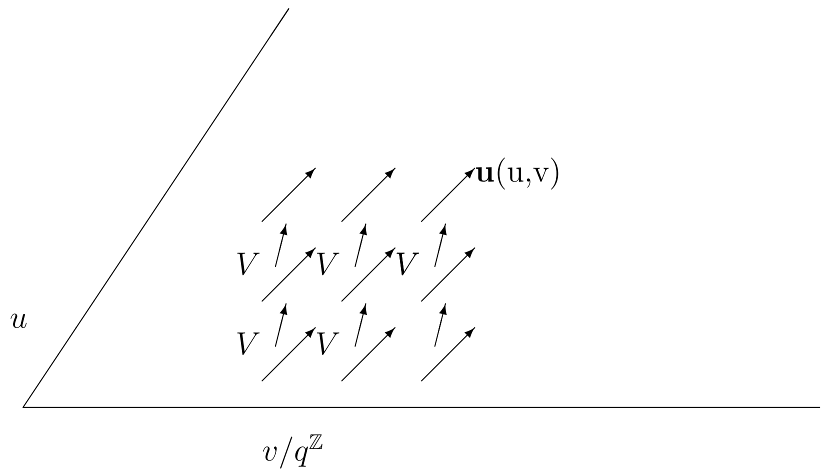
2.3 Geometric interpretation

One may interpret an (U,V)-system over $\mathbb{F}$ as a pair of line bundles with "discrete connections".

Fix $v = v_0$ and consider

$$\{\mathbb{F} \cdot \mathbf{u}(u, v_0) : u \in \mathbb{F}^*\}$$

as the trivial line bundle with the section $(u, v_0) \mapsto \mathbf{u}(u, v_0)$. On this bundle V acts as a discrete connection $x\mathbf{u}(u, v_0) \mapsto v_0 x \mathbf{u}(uq, v_0)$. U acts as the linear map $x\mathbf{u}(u, v_0) \mapsto ux\mathbf{u}(u, v_0)$. As v_0 varies we may wish to consider this as a line bundle over $\mathbb{F}^* \times \mathbb{F}^*$. But on the whole family of line bundles $\{\mathbb{F} \cdot \mathbf{u}(u, v) : u \in \mathbb{F}^*\}$, $v \in \mathbb{F}^*$, there is an extra relations which identifies, for $v' = vq^k$, $k \in \mathbb{Z}$, the vector $\mathbf{u}(u, v)$ with $q^m\mathbf{u}(u, v')$ for a corresponding $m \in \mathbb{Z}$. In other words, the two line bundles $\mathbb{F} \cdot \mathbf{u}(u, v)$ and $\mathbb{F} \cdot \mathbf{u}(u, v')$ are equal but trivialised differently.



Bundle of U-eigenvectors with discrete connection V

Slightly differently we may characterise E as the principal $U^{\mathbb{Z}}$-bundle with $V^{\mathbb{Z}}$ defining the group of connections on the bundle.

Dually there is given a family of line bundles

$$\{\mathbb{F} \cdot \mathbf{v}(u_0, v) : v \in \mathbb{F}^*\}$$

with the discrete connection U (principal $V^{\mathbb{Z}}$-bundles G with the group of connections $U^{\mathbb{Z}}$).

On the top of this structure we defined the pairing $\mathbf{u}(u, v) * \mathbf{v}(v', u')$ providing an interaction between the two families of bundles.

For q a root of unity the pairing organises the family $M_{u,v}$ of (U, V)-modules, with the relations (3) between the basis $\{\mathbf{u}(uq^n, v) : 0 \leq n < N\}$ and the basis $\{\mathbf{v}(vq^k, u) : 0 \leq k < N\}$.

3 From (U, V) to (U^a, V^b).

3.1 From (U, V) to (U, V^m).

We assume below that q is of a finite order N such that the positive integer m divides N.

Consider an algebra with generators U, V^m and the defining relation

$$V^m U = q^m U V^m.$$

This can be naturally identified with the corresponding subalgebra of the algebra generated by U and V discussed above. Clearly in the above (U, V)-system the vectors $\mathbf{v}(v, u)$ can be seen as eigenvectors of V^m with eigenvalue v^m. But, if γ is a root of unity of order m then also $\mathbf{v}(v\gamma, u)$ is an eigenvector of V^m with the same eigenvalue v^m. Moreover, any element of the form

$$\sum_{p=0}^{m-1} a_p \mathbf{v}(v\gamma^p, u)$$

is an eigenvector of V^m with the same eigenvalue v^m.

Lemma. *Let M be an irreducible (U, V)-module with U-eigenvalues $\{uq^k : k = 0, \ldots, N-1\}$ and V-eigenvalues $\{vq^n : n = 0, \ldots, N-1\}$. This may naturally be considered a (U, V^m)-module. Assume m devides N and $p \in \{0, \ldots, m-1\}$.*

Then there is a unique irreducible (U, V^m)*-submodule* $M^{(m)}_{uq^p,v^m}$ *of* M*. Its* V^m*-eigenvalues are* $\{v^m q^{nm} : n = 0, \ldots, \frac{N}{m} - 1\}$*,* U*-eigenvalues are* $\{uq^{p+km} : k = 0, \ldots, \frac{N}{m} - 1\}$ *and its canonical systems of* V^m*-eigenvectors is of the form*

$$\check{\mathbf{v}}(v^m q^{nm}, uq^{km}) := a \sum_{l=0}^{m-1} \mathbf{v}(vq^{n+l\frac{N}{m}}, uq^k), \quad n = 0, \ldots, \frac{N}{m}, \tag{7}$$

that satisfy

$$U : \check{\mathbf{v}}(v^m q^{nm}, uq^{p+km}) \mapsto uq^{p+km} \check{\mathbf{v}}(v^m q^{(n+1)m}, uq^{p+km}),$$

in accordance with (4). The invariants of $M^{(m)}_{uq^p,v^m}$ *are* $(u^{\frac{N}{m}} q^{p\frac{N}{m}}, v^N)$. M is the direct sum of its m-submodules $M^{(m)}_{uq^p,v^m}$, $p = 0, \ldots, m-1$.

Proof. It is easy to check that (7) generates an irreducible module and the dimenion of any irreducible (U, V^m)-module has to be equal to $\frac{N}{m}$.

Suppose a general V^m-eigenvector $\mathbf{v} = \sum_{p=0}^{m-1} a_p \mathbf{v}(vq^{n+l\frac{N}{m}}, u)$ belongs to an irreducible (U, V^m)-submodule M'. Then by the general property (4)

$$U^{\frac{N}{m}}(\mathbf{v}) = u^{\frac{N}{m}} \mathbf{v} \text{ and } U^{\frac{N}{m}} \mathbf{v}(vq^{n+l\frac{N}{m}}, u) = u^{\frac{N}{m}} \mathbf{v}(vq^{n+(l+1)\frac{N}{m}}, u)$$

It follows that $a_l = a_{l+1}$ for all l.

The action of U on the $\check{\mathbf{v}}(v^m q^{nm}, uq^{km})$ satisfies the required property by immediate calculation, which also determines the U-eigenvalues in the submodule. It is easy to see now that for each p

$$\{\mathbf{u}(uq^{p+km}, v) : \; k = 0, \ldots, \frac{N}{m} - 1\}$$

are U-eigenvectors in the correspondent submodule and hence the sum of the m submodules is direct. □

We assume the coefficient a in the definition of $\check{\mathbf{v}}(v^m, u)$ to be $\sqrt{m^{-1}}$, which defines the latter uniquely.

We also rename $\check{\mathbf{u}}(uq^{p+km}, v^m q^{nm}) = \mathbf{u}(uq^{p+km}, vq^n)$ and note that this satisfies (6) with regard to $\check{\mathbf{v}}(v^m q^{nm}, uq^{p+km})$.

3.2 Now we can define a canonical relation between an (U, V)-system and an (U, V^m)-system as (étale) covering maps

$$p_{1,m} : \mathbb{V}_{(U,V)} \to \mathbb{V}_{(U,V^m)}, \;\; p_{(U,V^m)} : \mathbf{v}(v, u) \mapsto \check{\mathbf{v}}(v^m, u)$$

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Lemma 1. *An (U,V^m)-system is definable in an (U,V)-system along with the maps $p_{1,m}$.*

Proof is by the explicit construction above.□

We can conversely consider an extension of an (U,V)-system to an $(U,V^{\frac{1}{n}})$-system by introducing a new operator $V^{\frac{1}{n}}$ with the defining relations

$$V^{\frac{1}{n}}U = q^{\frac{1}{n}}UV^{\frac{1}{n}}, \quad (V^{\frac{1}{n}})^n = V,$$

where the value of $q^{\frac{1}{n}}$ is chosen among possible roots of q of order n.

Lemma 2. *An (U,V^m)-system is definable in an (U,V)-system along with the maps $p_{1,m}$.*

3.3 System of étale coverings

It follows from above that we can conversely consider an extension of an (U,V)-system to an $(U,V^{\frac{1}{n}})$-system by introducing a new operator $V^{\frac{1}{n}}$ with the defining relations

$$V^{\frac{1}{n}}U = q^{\frac{1}{n}}UV^{\frac{1}{n}}, \quad (V^{\frac{1}{n}})^n = V$$

consider a corresponding $(U,V^{\frac{1}{n}})$-system and construct an étale covering as above from the $(U,V^{\frac{1}{n}})$-system onto the (U,V)-system.

Obviously, once $(U,V^{\frac{1}{n}})$-system related (by the covering) to the (U,V)-system is constructed, we can use 3.1 again to add an $(U,V^{\frac{m}{n}})$-system to the structure.

In case q is a root of unity of order N we will always require that $m|N$ (otherwise we can replace m with g.c.d.(m,N)). Note that

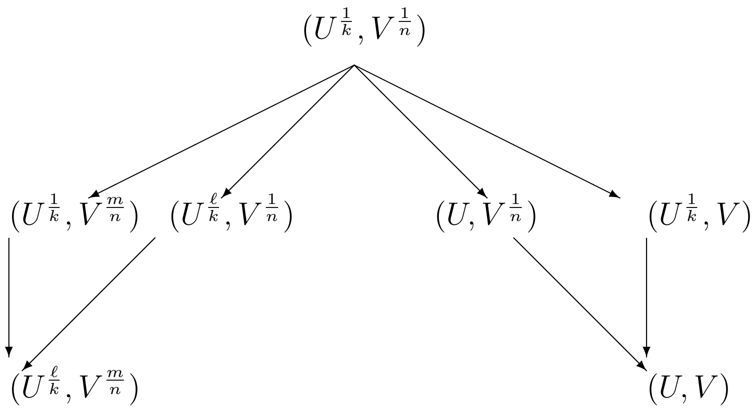
$$\dim M_{u,v^a} = \frac{N}{a}, \text{ where } a = \frac{m}{n}$$

in this case.

Independently we can extend an (U,V^a)-system to an (U^b,V^a)-system, for $b \in \mathbb{Q}$, with defining relation

$$U^bV^a = q^{ab}V^aU^b.$$

This corresponds to a system of étale coverings extending in the two possible directions.



Finally, we have the corresponding formula for the dimension of an irreducible (U^b, V^a)-module at root of unity of order N,

$$\dim M_{u^b, v^a} = \frac{N}{ab} \tag{8}$$

provided N has the divisibility property as above.

3.4 Universal covering

We treat the system of étale coverings of (U^a, V^b)-systems as a multi-sorted structure, its sorts (U^a, V^b)-system itself being multi-sorted.

One needs to keep in mind that the symbols $\mathbf{u}(u, v)$ and $\mathbf{v}(v, u)$ have different meaning in different (U^a, V^b)-systems.

We will use a uniform notation $M^{a,b}_{u,v}$ for an irreducible (U^a, V^b)-module and for eigenvectors:

$$\mathbf{u}^{a,b}(u, v), \mathbf{u}^{a,b}(uq^{ab}, v) \ldots, \mathbf{u}^{a,b}(uq^{kab}, v), \ldots$$

is a basis of U^a-eigenvectors in the module, and

$$\mathbf{v}^{a,b}(v, u), \mathbf{v}^{a,b}(vq^{ab}, u) \ldots, \mathbf{v}^{a,b}(vq^{kab}, u) \ldots$$

is a basis of V^b-eigenvectors in the same module.

Correspondingly, (2), (4) become

$$\begin{array}{l} U^a : \mathbf{u}^{a,b}(uq^{kab}, v) \mapsto uq^{kab}\mathbf{u}^{a,b}(uq^{kab}, v) \\ V^b : \mathbf{u}^{a,b}(uq^{kab}, v) \mapsto v\mathbf{u}^{a,b}(uq^{(k-1)ab}, v) \end{array} \tag{9}$$

$$\begin{array}{c} U^a : \mathbf{v}^{a,b}(vq^{kab}, u) \mapsto u\mathbf{v}^{a,b}(vq^{(k+1)ab}, u) \\ V^b : \mathbf{v}^{a,b}(vq^{kab}, u) \mapsto vq^{kab}\mathbf{v}^{a,b}(vq^{kab}, u) \end{array} \tag{10}$$

and the covering maps of subsection 3.1 between levels are

$$p_{n,m} : \mathbf{u}^{a,b}(u, v) \mapsto \mathbf{u}^{na,mb}(u^n, v^m),$$

$$p_{n,m} : \mathbf{v}^{a,b}(v, u) \mapsto \mathbf{v}^{a,b}(v^m, u^n).$$

Model-theoreticall we treat all the (U^a, V^b)-systems together as sorts of a structure and the covering maps $p_{n,m}$ as definable maps in the structure.

The whole picture becomes even more uniform if we introduce a **universal cover** (covering structure) (Q, P) consisting of two sorts Q and P, with Q of elements (vectors) written as $|x, p\rangle$, for $x, p \in \mathbb{C}$, and P of elements $|p, x\rangle$, for $x, p \in \mathbb{C}$. Call this structure **a** (Q, P)**-system**.

We consider (Q, P) along with the covering map

$$\begin{array}{c} \mathbf{exp}^{a,b} : |x, p\rangle \mapsto \mathbf{u}^{a,b}(e^{2\pi iax}, e^{ibp}) \\ \mathbf{exp}^{a,b} : |p, x\rangle \mapsto \mathbf{v}^{a,b}(e^{ibp}, e^{2\pi iax}). \end{array} \tag{11}$$

This can be presented as the following (noncommutative!) diagram

$$\begin{array}{ccc} |x, p\rangle & \xrightarrow{\mathbf{exp}^{a,b}} & \mathbf{u}^{a,b}(e^{2\pi iax}, e^{ibp}) \\ \Big\downarrow \mathrm{Q} & & \Big\downarrow \frac{U^a - 1}{2\pi ia} \\ & \mathbf{exp}^{a,b} & \frac{e^{2\pi iax} - 1}{2\pi ia}\mathbf{u}^{a,b}(e^{2\pi iax}, e^{ibp}) \\ x \cdot |x, p\rangle & \longrightarrow & x \cdot \mathbf{u}^{a,b}(e^{2\pi iax}, e^{ibp}) \end{array}$$

Note that the bottom right corner of the diagram has two different entries, which converge as a tends to 0.

Similar diagram holds for P, $\mathbf{v}^{a,b}(e^{ibp}, e^{2\pi iax})$ and V^b correspondingly.

Recall that a complex number z can be uniquely identified by the sequence $\{\exp \frac{z}{n} : \ n \in \mathbb{N}\}$, that is the map

$$z \mapsto \{\exp \frac{z}{n} : \ n \in \mathbb{N}\}$$

is injective. So, elements $|x, p\rangle$ of the (Q, P)-system can be uniquely identified with sequences $\{\mathbf{u}^{a,b}(q^{\frac{x}{n\hbar}}, q^{\frac{p}{2\pi n\hbar}}) : \ n \in \mathbb{N}\}$ and $|p, x\rangle$ with $\{\mathbf{v}^{a,b}(q^{\frac{p}{2\pi n\hbar}}, q^{\frac{x}{n\hbar}}) : n \in \mathbb{N}\}$. In this sense the (Q, P)-system can be considered a limit structure for the (U^a, V^b)-systems.

3.5 Problem Eventually one would like to have a **complete universal cover** $\mathcal{H}$, an extension of (Q, P) and an action of operators P and Q on $\mathcal{H}$ which satisfies the Heisenberg relation (1) and agrees with U^a and V^b via $\mathbf{p}_{a,b}$. The (heuristic) Dirac calculus assumes that such an $\mathcal{H}$ exists in the form of a Hilbert space of generalised function $\mathbb{R} \to \mathbb{C}$ with P given by operator $f \mapsto -i\hbar\frac{d}{dx}f$ and Q by $f \mapsto x \cdot f$, but this approach encounters a lot of mathematical difficulties and has not succeded so far.

3.6 Principal module

Given a (U^a, V^b)-system, normally only one among (U^a, V^b)-modules has a meaning compatible with physics interpretations.

The module $M^{a,b}_{1,1}$ with U^a and V^b-eigenvalues

$$\{q^{abk} : k = 0, 1, 2, \ldots, \frac{N}{ab} - 1\}$$

will be called **the principal module**.

According to our construction of étale coverings there is a whole system of principal modules, one for each pair of rational numbers a and b. Note that an application of the covering map $p_{n,m}$ to eigenvectors of a corresponding principal module, by definition produces eigenvectors of another principal module, so we have a well-defined system of étale coverings of principal modules which corresponds to a **universal principal object** corresponding to the system of Q-eigenvectors $|x, 0\rangle$ and P-eigenvectors $|p, 0\rangle$ with $x \in \mathbb{Q}$, $p \in 2\pi\mathbb{Q}$.

We assume the notation for the particular values of parameters

$$|x\rangle := |x, 0\rangle \quad |p\rangle := |p, 0\rangle,$$

which establishes an exact link with Dirac's bra-ket notation used by physicist.

In this case we also may invert the correspondence in (11) using the principal branch of logarithm, so

$$\begin{array}{lll} |x\rangle & \leftrightarrow & \mathbf{u}^{a,b}(e^{2\pi i a x},1) = \mathbf{u}(q^{\frac{ax}{\hbar}},1) \\ |p\rangle & \leftrightarrow & \mathbf{v}^{a,b}(e^{ibp},1) = \mathbf{v}(q^{\frac{bp}{2\pi\hbar}},1) \\ x = \frac{n}{N}, & & n = 0,1\dots,\frac{N}{ab}-1; \quad p = \frac{2\pi k}{N}, \quad k = 0,1\dots,\frac{N}{ab}-1. \end{array} \tag{12}$$

3.7 Structural approximation and rescaling.

We use in later calculations below the fact that an (U,V)-system corresponding to a generic $q = e^{ih}$ can be approximated by (U,V)-systems for $q = \epsilon$ a root of unity of order N, see [5] for the corresponding notion of approximation.

An essential fact in this context is that we approximate the generic q and the infinite cyclic group generated by it by $\epsilon = \exp\frac{2\pi i}{N}$ and the corresponding finite cyclic groups, along an ultrafilter D. The fact established in [5], section 4, states that it is necessary and sufficient then that

the sequence of roots of unity, $\exp 2\pi i\frac{M_N}{N}$, *(*$N, M_N \in \mathbb{N}$*) satisfies the following:*

(i)

$$\lim_D \frac{M_N}{N} = \frac{h}{2\pi} =: \hbar$$

(ii) for every given positive $m \in \mathbb{Z}$, *almost all* N *modulo the ultrafilter are divisible by* m.

In particular, the condition (ii) allows us to assume the divisibility for N as in 3.1 and so the formula (8) holds. Correspondingly, we also make the following assumption about the constant $\hbar$.

Physics assumption. In some natural physics units

(PA1) $\hbar = \frac{1}{N}$;

(PA2) the integer N is divisible by any positive integer $m << N$.

4 "Time evolution" for the free particle

4.1 Motivation. According to quantum mechanics the time evolution operator for the free particle is $K^t := e^{-it\frac{\mathrm{P}^2}{2\hbar}}$ and the expression on the right makes sense in the (Banach) operator algebra[1]. Then the Heisenberg relation (1) gives

$$[\mathrm{Q}, K^t] = t\mathrm{P}K^t.$$

Hence $\mathrm{Q}K^t - K^t\mathrm{Q} = t\mathrm{P}K^t$ and

$$K^t 2\pi i \mathrm{Q} K^{-t} = 2\pi i(\mathrm{Q} - t\mathrm{P}).$$

Note that by the Baker-Campbell-Hausdorff formula applicable in Banach algebras

$$U \cdot V^{-2\pi t} = \exp 2\pi i\mathrm{Q} \cdot \exp -2\pi it\mathrm{P} = \exp(2\pi i\mathrm{Q} - 2\pi it\mathrm{P} + \frac{1}{2}[2\pi i\mathrm{Q}, -2\pi it\mathrm{P}]) =$$

$$= \exp(2\pi i\mathrm{Q} - 2\pi it\mathrm{P} + \pi ith) = q^{\pi t} K^t U K^{-t},$$

so one gets

$$K^t U K^{-t} = q^{-\pi t} U V^{-2\pi t} \tag{13}$$

Here and below we have chosen once and for all a certain value for $q^{\pi t}$.

Another property that follows from the form of the time evolution operator, is that it must commute with P and so with V^a for all rational a :

$$K^t V^a K^{-t} = V^a \tag{14}$$

Note that (13) and (14) can at best determine K^t up to a scalar coefficient.

In the particular case of the free particle there is an alternative way of determining K^t on the principal module (not on an arbitrary one!)

Recall that according to the Dirac notation $\mathrm{P} : |p\rangle \mapsto p|p\rangle$, so

$$e^{-it\frac{\mathrm{P}^2}{2\hbar}} : \ |p\rangle \mapsto e^{-\frac{itp^2}{2\hbar}}|p\rangle$$

and since $p = 2\pi tk\hbar = \frac{2\pi t}{N}k$, $k = 0, 1 \ldots, \frac{N}{2\pi t} - 1$,

$$e^{-\frac{itp^2}{2\hbar}} = q^{-\pi tk^2}.$$

Finally, by the correspondence (12) we have from above

$$e^{-it\frac{\mathrm{P}^2}{2\hbar}} : \ \mathbf{v}^{1,2\pi t}(q^{2\pi tk}, 1) \mapsto q^{-\pi tk^2}\mathbf{v}^{1,2\pi t}(q^{2\pi tk}, 1).$$

This can be rewritten as the determination of eigenvector and eigenvalues

$$K^t : \ \mathbf{v}^{1,2\pi t}(q^{2\pi tk}, 1) \mapsto q^{-\pi tk^2}\mathbf{v}^{1,2\pi t}(q^{2\pi tk}, 1). \tag{15}$$

[1]This also would assume that operators P and Q are bounded which is causes technical problems

4.2 Definition of the time evolution operator in an (U, V)-system. Let $t = \frac{m}{2\pi n}$, $m, n \in \mathbb{Z}$, $n \neq 0$, so $2\pi t$ is a rational number. Let M be a $(U, V^{2\pi t})$-module and $v^{2\pi t}$ a $V^{2\pi t}$-eigenvalue in this module. Define K_v^t to be an operator acting on this module and satisfying the relations

$$\begin{array}{l}\text{(i) } K_v^t U K_v^{-t} = q^{-\pi t} v^{2\pi t} U V^{-2\pi t} \\ \text{(ii) } K_v^t V^{2\pi t} K_v^{-t} = V^{2\pi t}\end{array} \tag{16}$$

In particularly, K_v^t must be invertible.

In addition to this we also assume that

$$K_1^t \mathbf{v}^{1,2\pi tk}(q^{2\pi pt}, 1) = q^{-\pi tk^2} \mathbf{v}^{1,2\pi t}(q^{2\pi ptk}, 1) \tag{17}$$

The relation (ii) is obvious from the motivating paragraph, and the relation (i) differs from (13) by the multiplier $v^{2\pi t}$ on the right. This is necessary if we do not assume that $v^{2\pi t} = 1$ (compare the determinant on the both sides of the equality). In (13) the latter in fact was a part of our assumptions as the only physically relevant module is the principal module (see 3.6).

(iii) is the same as (15).

Denote $S_{t,v} := q^{-\pi t} v^{2\pi t} U V^{-2\pi t}$, the right-hand side of (16(i)).

4.3 We now work in an $(U, V^{2\pi t})$-system and assume that q is a primitive root of unity of order N satifying the assumption (PA2) of 3.7 so that in particular $\frac{N}{2\pi t}$ is an integer.

We assume that K_v^t acts in an $(U, V^{2\pi t})$-module $M_{u,v^{2\pi t}}$ and aim to describe this action and the parameters of the module.

Set

$$\mathbf{s}(uq^{2\pi tm}, v^{2\pi t}) := K_v^t \mathbf{u}^{1,2\pi t}(uq^{2\pi tm}, v^{2\pi t}).$$

Then

$$S_{t,v}\, \mathbf{s}(u, v) = u\, \mathbf{s}(u, v^{2\pi t}), \quad V^{2\pi t}\, \mathbf{s}(u, v^{2\pi t}) = v^{2\pi t}\, \mathbf{s}(q^{-2\pi t} u, v^{2\pi t}). \tag{18}$$

Proof. Using (16),

$$S_{t,v}\mathbf{s}(u, v^{2\pi t}) = K_v^t U K^{-t} K_v^t \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = K_v^t U \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = u K_v^t \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}),$$

so $S_{t,v}\mathbf{s}(u, v^{2\pi t}) = u\mathbf{s}(u, v^{2\pi t})$.

$$V^{2\pi t} K_v^t \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = K_v^t V^{2\pi t} \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = K_v^t v^{2\pi t} \mathbf{u}^{1,2\pi t}(q^{-2\pi t} u, v^{2\pi t}),$$

so $V^{2\pi t}\mathbf{s}(u, v^{2\pi t}) = v^{2\pi t}\mathbf{s}(q^{-2\pi t} u, v^{2\pi t})$, as required.

4.4 Inner product. Recall that in an irreducible $(U, V^{2\pi t})$-module the (generalised) inner product is defined, which treats U and $V^{2\pi t}$ as (generalised) unitary operators. In particular, we will have

$$\begin{aligned}&\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = U\mathbf{s}(u, v^{2\pi t}) * U\mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = \\ &= V^{2\pi t}\mathbf{s}(u, v^{2\pi t}) * V^{2\pi t}\mathbf{u}^{1,2\pi t}(u, v^{2\pi t})\end{aligned} \tag{19}$$

Then, by construction, $S_{t,v}$ is unitary as well.

We have by (18) for arbitrary integer k :

$$\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(uq^{2\pi kt}, v^{2\pi t}) = u^{-1}S_{t,v}\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(uq^{kt}, v^{2\pi t}) =$$

$$= u^{-1}\mathbf{s}(u, v^{2\pi t}) * S_{t,v}^{-1}\mathbf{u}^{1,2\pi t}(uq^{kt}, v^{2\pi t}) = u^{-1}\mathbf{s}(u, v^{2\pi t}) * q^{\pi t}v^{-2\pi t}V^t U^{-1}\mathbf{u}^{1,2\pi t}(uq^{2\pi kt}, v^{2\pi t}) =$$

$$= u^{-1}\mathbf{s}(u, v^{2\pi t}) * q^{\pi t}u^{-1}q^{-2\pi kt}\mathbf{u}^{1,2\pi t}(uq^{(k-1)t}, v^{2\pi t}) =$$

$$= q^{2\pi t(k-\frac{1}{2})} \cdot \mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(uq^{2\pi(k-1)t}, v^{2\pi t}),$$

in accordance with (5).

It follows by induction on k,

$$\begin{aligned}&\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(uq^{2\pi kt}, v^{2\pi t}) = c\, q^{\pi tk^2}, \\ &\text{where } c = c(u, v^{2\pi t}) = q^{-\pi t}\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}).\end{aligned} \tag{20}$$

On the other hand, by (18)

$$\mathbf{s}(u, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = V^{2\pi t}\mathbf{s}(u, v^{2\pi t}) * V^{2\pi t}\mathbf{u}^{1,2\pi t}(u, v^{2\pi t}) = \mathbf{s}(uq^{-2\pi t}, v^{2\pi t}) * \mathbf{u}^{1,2\pi t}(uq^{-2\pi t}, v^{2\pi t})$$

which proves that in (20) $c(u, v^{2\pi t}) = c(uq^{2\pi kt}, v^{2\pi t})$ that is depends on the class $\hat{u} = \{uq^{2\pi tk} : k \in \mathbb{Z}\}$ but not on u.

Also note that $c(u, v^{2\pi t})$ does not depend on $v^{2\pi t}$ but on the class $\hat{v}^{2\pi t} = \{v^{2\pi t}q^{2\pi tk} : k \in \mathbb{Z}\}$. Indeed, by (2) $\mathbf{u}^{1,2\pi t}(u, v^{2\pi t}q^{2\pi t}) = q^{2\pi tm}\mathbf{u}^{1,2\pi t}(u, v^{2\pi t})$ for some $m \in \mathbb{Z}$. By definition then $\mathbf{s}(u, v^{2\pi t}q^{2\pi t}) = q^{2\pi tm}\mathbf{s}^{1,2\pi t}(u, v^{2\pi t})$. Then by properties of the inner product

$$\mathbf{s}(u, v^{2\pi t}q^{2\pi t}) * \mathbf{u}(u, v^{2\pi t}q^{2\pi t}) = \mathbf{s}(u, v^{2\pi t}) * \mathbf{u}(u, v^{2\pi t}).$$

So, summing up, $c = c(\hat{u}, \hat{v}^{2\pi t})$ is an invariant of the cosets by the cyclic group generated by $q^{2\pi t}$, equivalently, invariant of the module.

4.5 Corollary

$$K_v^t \mathbf{u}(uq^{2\pi mt}, v^{2\pi t}) = c(\hat{u},\, \hat{v}^{2\pi t}) \sum_{k=0}^{\frac{N}{2\pi t}-1} q^{\pi tk^2} \mathbf{u}^{1,2\pi t}(uq^{2\pi(m+k)t}, v^{2\pi t}). \qquad (21)$$

This follows from (20) and the second formula in (18). One can also check independently and directly that the sum on the right is an $S_{t,v}$-eigenvector with the eigenvalue u.

In particular, *given the values of* $v^{2\pi t}$ *and* $c(\hat{u},\, \hat{v}^{2\pi t})$ *the operator* K_v^t *is determined on the module* M *by the formula (21) uniquely.*

The choice of $v^{2\pi t}$ amounts to a choice of the canonical basis of U-eigenvectors.

4.6 Determining the coefficient.

Note that so far we have only used (16). We are now using (17). This formula allows us to calculate the trace of K_1^t :

$$\mathrm{Tr}\ K_1^t = \sum_{k=0}^{\frac{N}{2\pi t}-1} q^{-\pi tk^2}.$$

We rewrite the latter using the notation $N_t := \frac{N}{2\pi t}$, which is an even integer by (PA2), and use the formula for (generalised) quadratic Gauss sums (see [4]),

$$\mathrm{Tr}\, K_1^t = \sum_{k=0}^{N_t-1} e^{-\frac{\pi i}{N_t}k^2} = \sqrt{N_t} \cdot e^{-\frac{\pi i}{4}}.$$

Alternative calculation using formula (21) gives

$$\mathrm{Tr}\, K_1^t = c(\hat{1},\, \hat{1}) \sum_{k=0}^{N_t-1} 1 = c(\hat{1},\, \hat{1}) \cdot N_t.$$

Hence

$$c(\hat{1},\, \hat{1}) = \frac{1}{\sqrt{N_t}} e^{\frac{\pi i}{4}}.$$

This formula can be rewritten using (PA1) which gives

$$N_t = \frac{N}{2\pi t} = \frac{1}{2\pi t\hbar},$$

so

$$c(\hat{1},\,\hat{1}) = \sqrt{2\pi t\hbar}\,e^{\frac{\pi i}{4}}. \tag{22}$$

We now postulate our choice of the coefficient for an aribtrary module

$$c(\hat{u},\hat{v}^{2\pi t}) = \sqrt{2\pi t\hbar}\,e^{\frac{\pi i}{4}} = \sqrt{\frac{2\pi t}{N}}\,e^{\frac{\pi i}{4}}. \tag{23}$$

4.7 The (U,V,K)-systems.

To simplify the notation we assume for the time being that $2\pi t = 1$.

Given a (U,V)-system we define its K**-expansion**, and call it a (U,V,K)-system as follows.

We expand the language by adding a symbol K of a binary operation $K(v,\mathbf{w})$, where $(v,\mathbf{w})$ is in the domain of the operation if and only if $v \in \mathbb{F}^*$ and $\mathbf{w}$ is an element of an (U,V)-module $M_{u,v}$. The value of $K(v,\mathbf{w})$ will be defined by the formula (21) and the value of the coefficient by (22), which is the same constant for all (U,V)-modules.

Proposition. $K(v,\mathbf{w})$ is definable as a Zariski-closed relation in the language of (U,V)-systems.

Proof. The formula for $K(v,\mathbf{w}_1) = \mathbf{w}_2$ says that

- there is an $u \in \mathbb{F}^*$ and a canonical basis of U-eigenvectors satisfying (2) with regards to u and v;
- there are $a_0,\ldots,a_{N-1} \in \mathbb{F}$ and $b_0,\ldots,b_{N-1} \in \mathbb{F}$ so that

$$\mathbf{w}_1 = \sum_{k=0}^{N-1} a_k \mathbf{u}(uq^k,v) \ \&\ \mathbf{w}_2 = \sum_{m=0}^{N-1} b_m \mathbf{u}(uq^m,v) \ \&$$

$$b_m = \sum_{k=0}^{N-1} a_k q^{\frac{(k-m)^2}{2}} \text{ where } c = \sqrt{\frac{2\pi t}{N}}\,e^{\frac{\pi i}{4}}.$$

This gives us a definition of $K(v,\mathbf{w}_1) = \mathbf{w}_2$ as a positive $\exists$-core formula in the terminology of [1], so Zariski-closed relation in the sense of the Zariski structure.□

Remark 1. Note that $K(v,\cdot)$ could be equally defined as the operator K_v acting on V-eigenvectors of an arbitrary module $M_{u,v}$ as

$$K_v \mathbf{v}(vq^k, u) = q^{-\frac{k^2}{2}} \mathbf{v}(vq^k, u).$$

The fact that the eigenvalues have exactly this form easily follows from (17) and the fact that the matrix of the operator $K^t(v,\cdot)$ in the canonical U-bases remains the same in all modules, as well as the transition matrix from a U-base to a V-base.

Remark 2. We could not claim even ω-stability of the theory of (U,V,K)-systems if we chose to expand (U,V)-systems by a unique operator K^t (not a family of operators) in every module. Indeed, the formula

$$K^t U K^{-t} = q^{-\frac{1}{2}} v U V^{-1}$$

defines, using K^t, the value of v in every module. The invariant v^N of the module is obviously definable too, so we get a definable map $v^N \mapsto v$ for all $x = v^N \in \mathbb{F}^*$ in $\mathbf{M}'$ which contradicts ω-stability of the theory.

Remark 3. One may interprete the above proposition as a statement that the algebra (U,V) determines the operator-valued function $K(v,\cdot)$. This is a non-trivial claim since using conventional operator calculus we would have to find a way from operators $e^{i\mathrm{P}}$ and $e^{2\pi i\mathrm{Q}}$ (that is V and U) to the time evolution operator $e^{-it\frac{\mathrm{P}^2}{2\hbar}}$. This is a non-trivial analytic procedure but in our setting we do not assume any analytic or topological properties of the algebra.

Remark 4. The assumption (22) may look too strong and artificial. In principle we calculate K_v^t on each (U,V)-module separately and $c(\hat{u},\hat{v})$ is not determined, even if one fixes its value in some of the modules. But if we assume that the function $c(\hat{u},\hat{v})$ is defined in a Zariski structure on the (U,V)-system, then the choices for the function must become very limited, it has to be a Zariski function $(\mathbb{F}^*/\Gamma)^2 \to \mathbb{F}$, where Γ is the cyclic subgroup of $\mathbb{F}^*$ generated by q. If in addition we require that in the limit, when Γ becomes an infinite cyclic subgroup, the (U,V)-system is still (analytic) Zariski (see [5]) then constant $c(\hat{u},\hat{v})$ is the only option.

Finally we want to make explicit the dependence of $K(v,\mathbf{w})$ on the parameter t. This comes through the introduction of the dependence of the

(U,V) system on t, namely we consider, as has been done in 4.2, a $(U,V^{2\pi t})$-system and define $K(v,\mathbf{w})$ in it exactly as above, replacing N with $\frac{N}{2\pi t}$. To make this dependence explicit we write $K^t(v,\mathbf{w})$ for the operator in an $(U,V^{2\pi t})$-system.

4.8 Composition of time evolution operators

Suppose $K^t_v,\mathbf{w})$ is the time evoution operator and m is a positive integer. Then we can define K^{tm} in two ways:

first, *ab initio* as in 4.2, denote it $K^{tm}_{v^m}$;

second, as a composition $K^t_v(K^t_v,\ldots))\ldots)$ (m times). Denote it $(K^t_v)^m$.

Note that $K^{tm}_{v^m}$ is an operator on a $(U,V^{2\pi tm})$-module while $(K^t_v)^m$ acts on a $(U,V^{2\pi t})$ one.

We want to find a canonical correspondence between the two definitions.

Proposition. Let $M_{u,v^{2\pi t}}$ be an irreducible $(U,V^{2\pi t})$-module and $M_{u,v^{2\pi tm}}$ an irreducible $(U,V^{2\pi tm})$-submodule of $M_{u,v^{2\pi t}}$.

For any $\mathbf{w}\in M_{u,v^{2\pi tm}}$

$$(K^t_{v^{2\pi t}})^m(\mathbf{w}) = K^{tm}_{v^{2\pi tm}}(\mathbf{w}),$$

that is the two operators coinside on each submodule.

Also, $(K^t_{v^{2\pi t}})^{-1}$ coincides with $K^{-t}_{v^{-2\pi t}}$.

Proof. Consider a canonical V-basis

$$\{\mathbf{v}^{1,2\pi t}(v^{2\pi t}q^{2\pi tk},u):\ k=0,1,\ldots,\frac{N}{2\pi t}-1\}$$

of $M_{u,v^{2\pi t}}$. The action of $(K^t_{v^{2\pi t}})^m$ on the basis gives (see Remark 1 of 4.7)

$$(K^t_{v^{2\pi t}})^m\mathbf{v}(v^{2\pi t}q^{2\pi tk},u) = q^{-\pi tmk^2}\mathbf{v}(v^{2\pi t}q^{2\pi tk},u).$$

According to the Lemma in 3.1 an irreducible $(U,V^{2\pi tm})$-submodule in the module has a $V^{2\pi tm}$-eigenvectors basis of the form

$$\{\check{\mathbf{v}}(v^{2\pi tm}q^{2\pi tnm},u) : n=0,1,\ldots,\frac{N}{2\pi tm}-1\}$$

where

$$\check{\mathbf{v}}(v^{2\pi tm}q^{2\pi tnm},u) := \sqrt{\frac{1}{m}}\sum_{l=0}^{m-1}\mathbf{v}(v^{2\pi t}q^{2\pi t(n+\frac{N}{2m\pi t}l)},u),$$

Claim.

$$(K^t_{v^{2\pi t}})^m \check{\mathbf{v}}(v^{2\pi tm}q^{2\pi tnm}, u) = q^{-\pi tmn^2}\check{\mathbf{v}}(v^{2\pi t}q^{2\pi nm}, u).$$

This follows by definitions once one takes into account that $\frac{N}{\pi t}$ is an integer divisible by $2m$.

Finally we recall that by definition

$$K^{tm}_{v^{2\pi tm}}\check{\mathbf{v}}(v^{2\pi tm}q^{2\pi tnm}, u) = q^{-\pi tmn^2}\check{\mathbf{v}}(v^{2\pi t}q^{2\pi nm}, u).$$

Same argument works for $m = -1$. □

Now observe also that

Corollary. For any integer m, $K^{tm}_{v^{2\pi tm}}$ can be considered as an operator on an $(U, V^{2\pi t})$ module and coincides with $(K^t_{v^{2\pi t}})^m$ on this module.

$$K^{tm}_{v^{2\pi tm}} = (K^t_{v^{2\pi t}})^m$$

4.9 Feynman propagator for the free particle.

The **kernel of the Feynman propagator** is defined as

$$\langle x_1|K^t|x_0\rangle,$$

which should be read as the inner product of $K^t|x_0\rangle$ and $|x_1\rangle$ (linear on the right) renormalised by the Dirac delta function. The latter means that

$$\langle x|x\rangle = \delta(x) = \frac{1}{\Delta x}$$

for every vector $|x\rangle$ of length 1, where Δx is the element of length in a discrete approximation to the delta-function (see [2], 12.1.2-3).

We use the interpretation 3.4 and 3.6 of elements $\mathbf{u}(e^{2\pi ix}, 1)$ of the principal (U^a, V^b)-module as Dirac's vectors $|x\rangle$, calculate the inner product

$$K^t\mathbf{u}(e^{2\pi ix_0}, 1) * \mathbf{u}(e^{2\pi ix_1}, 1) \qquad (24)$$

and normalise it by the Dirac delta function.

Note that $e^{2\pi ix} = q^{\frac{2\pi x}{h}}$. So we rewrite (24), assuming that $\frac{2\pi x_i}{h}$ is rational, as

$$\mathbf{s}(q^{\frac{2\pi x_0}{h}}, 1) * \mathbf{u}(q^{\frac{2\pi x_1}{h}}, 1)$$

and by (20) and (23), substituting $x_0 = mth$, $u = q^{2\pi mt}$, $x_1 = (m+k)th$, $uq^{2\pi kt} = q^{2\pi(m+k)t}$, we get

$$\mathbf{s}(q^{\frac{2\pi x_0}{h}}, 1) * \mathbf{u}(q^{\frac{2\pi x_1}{h}}, 1) = c \cdot q^{\pi tk^2} = e^{\frac{\pi}{4}i} \cdot e^{\pi i \frac{(x_1 - x_0)^2}{th}} = \sqrt{\frac{2\pi t}{iN}} e^{i\frac{(x_1-x_0)^2}{2t\hbar}}.$$

Also, the discrete approximation which we explicitly use in our model corresponds to the element of length

$$\Delta x = \frac{2\pi t}{N}$$

(we split the interval $[0, 1)$ into $\frac{N}{2\pi t}$ equal length subintervals). Recall that we assumed in 3.7 that $N = \hbar^{-1}$.

This finally determines

$$\langle x_1 | K^t | x_0 \rangle = \sqrt{\frac{N}{2\pi it}} e^{i\frac{(x_1-x_0)^2}{2t\hbar}} = \sqrt{\frac{1}{2\pi i\hbar t}} e^{i\frac{(x_1-x_0)^2}{2t\hbar}}$$

The final expression is the well known formula for the kernel of the Feynman propagator for the free particle, see e.g. [2], formula (7.76).

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