Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety

B. Zilber,
University of Oxford

July 31, 2003
latest misprint corrections 24.10.05

1 Introduction

I believe that it is a common feeling among experts that nowadays model theory establishes itself more and more as a universal language of mathematics. "Universal" might be not quite a right word here as very few people outside logic speak this language, but surely its system of notions and ideas developed on a very high level of abstraction is proving to have a power to see many fields of mathematics in a new and unifying way. In many cases this new angle of view yields new results but sometimes even a new interpretation itself might be a good cause for research. The present paper is pursuing rather the latter goal.

We study the $L_{\omega_1,\omega}$-theory of universal covers of semi-abelian varieties over algebraically closed fields of characteristic 0, in fact over the complex numbers $\mathbb{C}$. Slightly simplifying and extending the definition, by a semi-abelian variety over $\mathbb{C}$ we mean an algebraic group $\mathbb{A}(\mathbb{C})$ (we write the group multiplicatively) such that its universal cover is $\mathbb{C}^d$, $d = \dim \mathbb{A}$. This assumes that there is an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{i} \mathbb{C}^d \xrightarrow{\exp} A(\mathbb{C}) \longrightarrow 1, \tag{1}$$

where exp is an analytic homomorphism from the additive group $(\mathbb{C}^d, +)$ and $\Lambda = \ker \exp$ is a discrete Zariski dense subgroup of $\mathbb{C}^d$ isomorphic to $\mathbb{Z}^N$, for some $N = N_A$, $d \leq N \leq 2d$. It follows immediately that the torsion in $\mathbb{A}$ can be described uniquely by N_A :

Fact 1 *Given a semi-abelian variety $\mathbb{A}$ and an algebraically closed field F containing the field of definition of $\mathbb{A}$, for any n the group*

$$\mathbb{A}_n = \{a \in \mathbb{A}(F) : a^n = 1\}$$

is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^N$.

We are going to discuss the following Uniqueness Problem for covers of semi-Abelian varieties.

Let $\mathbb{A}$ be a semi-abelian variety defined over some k_0, a finitely generated extension of $\mathbb{Q}$, let V be an abelian divisible torsion-free group and ex_V an abstract group homomorphism such that

$$0 \longrightarrow \mathbb{Z}^N \xrightarrow{i_V} V \xrightarrow{\mathrm{ex}_V} \mathbb{A}(\mathbb{C}) \longrightarrow 1 \qquad (2)$$

is an exact sequence.

Uniqueness Problem *Does there exist an isomorphism between the sequences (1) and (2), that is a pair of bijections (ρ, π) such that $\rho : \mathbb{C}^d \longrightarrow V$ is a group isomorphism and $\pi : \mathbb{A}(\mathbb{C}) \longrightarrow \mathbb{A}(\mathbb{C})$ is a bijection induced by a field isomorphism fixing k_0 (a Galois automorphism over k_0), and the diagram commutes?*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \xrightarrow{i} & \mathbb{C}^d & \xrightarrow{\exp} & \mathbb{A}(\mathbb{C}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow \pi & & \\ 0 & \longrightarrow & \mathbb{Z}^N & \xrightarrow{i_V} & V & \xrightarrow{\mathrm{ex}_V} & \mathbb{A}(\mathbb{C}) & \longrightarrow & 1 \end{array}$$

Notice that the positive answer to the question would signal that (2) is a reasonable 'algebraic' substitute for the classical complex universal cover. This, in turn, could be extended to suggest an algebraic substitute for universal covers for semi-abelian varieties over fields of positive characteristic, replacing $\mathbb{Z}^N$ by a suitable finite rank subgroup, e.g. for $\mathbb{A}$ equal to a one-dimensional algebraic torus, the kernel of ex_V in characteristic p has to be the additive group

$$\mathbb{Z}[\frac{1}{p}] = \left\{ \frac{m}{p^k} : m, k \in \mathbb{Z}, k \geq 0 \right\}.$$

We studied the Uniqueness Problem in [Z0] for $\mathbb{A}(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of the complexes, that is of the complex one-dimensional torus, and managed to answer the question positively with the help of some field arithmetic results as well as some quite advanced model theory. Notice that if we require π to be identity the answer is negative even for this simple case.

The uniqueness problem remains open even for the cases where we believe the answer is positive, e.g. elliptic curves without complex multiplication, but it is rather clear that these can be solved in positive provided the obvious generalisations of the arithmetic results of [Z0] can be proved. In this paper we show the converse, that is, in order for the answer to the problem to be positive generalisations of the arithmetic results used in [Z0] must hold. In other words, the geometrically motivated Uniqueness Problem in a rather non-trivial way is equivalent to some profoundly arithmetical questions. The link between arithmetic and model theory is provided by deep results of J.Keisler [K] and S.Shelah [Sh] after an observation that the uniqueness problem can be reformulated as a problem on categoricity in uncountable cardinals of an appropriate $L_{\omega_1,\omega}$-sentence. In section 5 we give a list of arithmetic properties which are necessary and sufficient for the sentence to be categorical in all uncountable cardinals.

The criterion, as remarked above, holds for some classes but it does not cover the general case. In particular, Theorem 1 of this paper states that a necessary condition for the existence of the isomorphism is that the action of the Galois group $\mathrm{Gal}(\tilde{k}_0 : k_0)$ on the Tate module $T_l(\mathbb{A})$ is represented by a subgroup of $\mathrm{GL}_N(\mathbb{Z}_l)$ of finite index. This is true for elliptic curves without complex multiplication by a result of Serre, but is false e.g. if the elliptic curve has a complex multiplication.

A more appropriate version of the Uniqueness Problem assumes the introduction of a more sophisticated structure on V, which yet should not be too complicated. An expanded version may bear a structure of a module of complex multiplications on V as well as, say, a bilinear form on Λ. This choice is restricted by the model-theoretic criterion on keeping the structure analysable (preferably stable) and on the other hand we want the analysis to cover a wider class of arithmetic examples. We would like to address these matters in a further research.

The results of this paper were conjectured by the author in a vague form after the main result of [Z0] was obtained. The author is grateful to

E.Hrushovski for a suggestive discussion of the topic. Thanks are also due to O.Lessmann for his educating lectures on Keisler-Shelah theory of excellency and many helpful discussions. My special thanks to the anonymous referee who suggested a number of important improvements to the paper.

Misha Gavrilovich achieved some progress in the solution of the Uniqueness Problem for elliptic curves without complex multiplication, and discussions with him not only were useful but also substantially influenced the final form of results of section 5.

2 The first order theory of group covers

We consider a natural language of two sorted structures $\mathbb{V}$ to describe the universal covers. The first sort, denoted usually V, corresponding to $\mathbb{C}^d$, is going to be a group structure in the language $(+, q\cdot)_{q\in\mathbb{Q}}$, which treats V as a rational vector space.

The second sort describes the algebraic group $\mathbb{A}$ as a group on the set $A = \mathbb{A}(F)$ of F-points of $\mathbb{A}$, for some algebraically closed field F of characteristic zero (which is just $\mathbb{C}$ in the initial setting). Such a group can be represented as a constructible (Boolean combination of Zariski closed) subset $\mathbb{A}(F) \subseteq \mathbf{P}^n(F)$ of the projective space over F, with an algebraic group operation. Let $k_0 = \mathbb{Q}(c)$ be a field which contains the field of definition of $\mathbb{A}$, c a finite tuple from F. We consider all Zariski closed k_0-definable relations $W \subseteq \mathbb{A}^n$ on A as part of the language, that is each of the relations is named in the language. Notice that the group operation corresponds to one of the relations. So, from now on when we refer to $\mathbb{A}$ as a substructure of $\mathbb{V}$ we have all the Zariski closed relations over k_0 on $\mathbb{A}$ in mind. We now refer to a well-known

Fact 2 *(Folklore and [Z1], [Z2]) In $\mathbb{A}(F)$ an algebraically closed field $(F(\mathbb{A}), +, \cdot)$ is definable. Moreover, if we choose c in the definition of k_0 big enough,*

$$\mathbb{A}(F) \subseteq \mathrm{dcl}\ F(\mathbb{A}) \text{ and } F(\mathbb{A}) \subseteq \mathrm{dcl}\ (\mathbb{A}(F)),$$

or equivalently: for any point $a \in A$ there is a finite tuple $[a]$ in F such that any automorphism of the structure that induces identity on $[a]$ acts as identity on a and vice versa.

Corollary 1 *We can identify the initial field F with $F(\mathbb{A})$ and $\mathbb{A}(F)$ with the A.*

Remark Technically, the identifications can be realised via a finite collection of meromorphic functions $f_1, \ldots, f_n$ on $\mathbb{A}$ such that

$$f_1(a) = f_1(a') \ \& \ldots \ \& f_n(a) = f_n(a') \text{ iff } a = a', \text{ for generic } a, a' \in A.$$

Then for such an a one can let $[a] = \langle f_1(a), \ldots, f_n(a) \rangle$.

To complete the description of $\mathbb{V}$ we indicate that one more operation $\mathrm{ex} : V \to A$ acts between the two sorts.

The first-order axioms for group covers of a fixed semi-abelian variety $\mathbb{A}$ say:

A1. $(V, +, q\cdot)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space;

A2. The complete first order theory of $\mathbb{A}(F)$ in the relational language having a name for each algebraic variety $W \subseteq \mathbb{A}^n$ defined over $k_0 = \mathbb{Q}(c)$;

A3. ex is a group homomorphism from $(V, +)$ onto the group $(\mathbb{A}(F), \cdot)$.

We let T_A be the first order theory axiomatised by A1 - A3.
It follows from the uncountable categoricity of the theory of algebraically closed fields of fixed characteristic and Fact 2

Fact 3 *Given T_A, an uncountable cardinal κ and a model of T_A with* card $F = \kappa$, *the isomorphism type of the structure on $\mathbb{A}(F)$ described by axioms A2 is determined uniquely.*

In other words, if there is another model with card $F' = \kappa$, *then there is an isomorphism $\pi : \mathbb{A}(F) \to \mathbb{A}(F')$ of the substructures inducing a field isomorphism $F \to F'$ over k_0.*

Moreover, the theory of $\mathbb{A}(F)$ has elimination of quantifiers in the language of Zariski closed relations.

In what follows we usually denote $\mathbb{V} = (V, A)$, with $A = \mathbb{A}(F)$, models of T_A.

Given a subgroup $S \subseteq V$ we write $S \otimes \mathbb{Q}$ for the divisible hull of the subgroup. Also we denote $\Lambda(V)$ the kernel of ex in V (which is definable by the quantifier-free formula $\mathrm{ex}(x) = 1$) and often we omit mentioning V when no ambiguity can arise.

Lemma 2.1 *T_A implies that*

$$V \cong V_0 \dotplus \Lambda(V) \otimes \mathbb{Q}, \tag{3}$$

with V_0 a linear subspace, and

$$\Lambda(V)/n\Lambda(V) \cong (\mathbb{Z}/n\mathbb{Z})^N, \quad N = N_A. \tag{4}$$

Proof The first follows from the general theory of linear spaces, since $\Lambda(V) \otimes \mathbb{Q}$ is a subspace. It follows also from the axioms that

$$\mathbb{A}(F) \cong V_0 \times (\Lambda(V) \otimes \mathbb{Q})/\Lambda(V).$$

The second component of the decomposition is isomorphic to the torsion subgroup of $\mathbb{A}(F)$, which is described in Fact 1, and the description is first order. Hence (4) follows. □

We say that the kernel in $\mathbb{V}$ is **standard** if

$$\Lambda(V) \cong \mathbb{Z}^N.$$

3 Types and elimination of quantifiers

We write the group operation in A multiplicatively.

Let $W \subseteq A^n$ be an algebraic variety defined and irreducible over some field $K \supseteq k_0$. With any such W and K we associate a sequence $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ of algebraic varieties which are definable and irreducible over K and satisfy the following:

$W^1 = W$, and for any $l, m \in \mathbb{N}$ the mapping

$$[m] : \langle y_1, \dots y_n \rangle \mapsto \langle y_1^m, \dots y_n^m \rangle$$

maps $W^{\frac{1}{lm}}$ onto $W^{\frac{1}{l}}$.

Such a sequence is said to be **a sequence associated with W over K.**

Also with any $\langle w_1, \ldots w_n\rangle \in W$ as above we associate a sequence

$$\{\langle w_1, \ldots w_n\rangle^{\frac{1}{l}} : l \in \mathbb{N}\}$$

such that for any $l, m \in \mathbb{N}$ the mapping

$$[m] : \langle y_1, \ldots y_n\rangle \mapsto \langle y_1^m, \ldots y_n^m\rangle$$

maps $\langle w_1, \ldots w_n\rangle^{\frac{1}{lm}}$ onto $\langle w_1, \ldots w_n\rangle^{\frac{1}{l}}$. Such a sequence is said to be **associated** with $\bar{w} = \langle w_1, \ldots w_n\rangle$.

A sequence associated with $\bar{w}$ is not uniquely determined; for $\bar{w}^{\frac{1}{l}}$ there are l^{Nn} possible values. Obviously, one can get all the values multiplying a value $\bar{w}^{\frac{1}{l}}$ by all the $\bar{\xi} = \langle \xi_1, \ldots, \xi_n\rangle$, with ξ_i's torsion points of order l, which we sometimes denote $\bar{1}^{\frac{1}{l}}$. We say that other possible choices of the sequence associated with the same $\bar{w}$ are **conjugated** to the given one. The same is applied to sequences associated with a variety W.

Lemma 3.1 *Let $\bar{w} \in W$ and $\{\bar{w}^{\frac{1}{l}} : l \in \mathbb{N}\}$, $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ be sequences associated with $\bar{w}$ and W correspondingly. Then there is a sequence $\{\bar{1}^{\frac{1}{l}} : l \in \mathbb{N}\}$ of torsion points associated with $\bar{1} = \langle 1, \ldots, 1\rangle \in A^n$ such that*

$$\bar{1}^{\frac{1}{l}} \cdot \bar{w}^{\frac{1}{l}} \in W^{\frac{1}{l}}$$

for all $l \in \mathbb{N}$. Moreover, if for every l there is $z_l \in F$ such that

$$\langle w_1^{\frac{1}{l}}, \ldots, w_{n-1}^{\frac{1}{l}}, z_l\rangle \in W^{\frac{1}{l}},$$

then we may assume

$$\bar{1}^{\frac{1}{l}} = \langle 1, \ldots, 1, 1^{\frac{1}{l}}\rangle, \text{ for some associated sequence } \{1^{\frac{1}{l}} : l \in \mathbb{N}\}.$$

Proof Immediate from the definitions.□

Lemma 3.2 *Assume that* Λ *in* $\mathbb{V}$ *is algebraically compact (which is the case if* $\mathbb{V}$ *is* ω*-saturated),* W *a nonempty algebraic subvariety of* F^n *and* $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ *a sequence associated with* W *over* k*. Then there is* $\bar{x} \in V^n$ *such that*

$$\mathrm{ex}(\frac{1}{l} \cdot \bar{x}) \in W^{\frac{1}{l}}$$

for all $l \in \mathbb{N}$*. In fact, given any* $\bar{v} = \langle v_1, \ldots, v_n \rangle$ *such that* $\mathrm{ex}(\bar{v}) \in W$*, we can get the required* $\bar{x}$ *in the form* $\bar{x} = \langle v_1+\tau_1, \ldots, v_n+\tau_n \rangle$ *for some* $\tau_1, \ldots, \tau_n \in \Lambda$*. Moreover, if for every* l *there is* $z_l \in A$ *such that*

$$\langle \mathrm{ex}(\frac{v_1}{l}), \ldots, \mathrm{ex}(\frac{v_{n-1}}{l}), z_l \rangle \in W^{\frac{1}{l}}$$

then we may assume $\tau_1 = \cdots = \tau_{n-1} = 0$*.*

Proof By 3.1 we need to choose $\bar{\tau}$ such that

$$\mathrm{ex}(\frac{\bar{\tau}}{l}) = \bar{1}^{\frac{1}{l}} \text{ for all } l \in \mathbb{N}.$$

This defines a consistent type in Λ in terms of group operation, and we are done by algebraic compactness. □

Lemma 3.3 *Given a finitely generated extension* k *of* k_0 *and* $\bar{v} \in V^n$*, linearly independent, the quantifier-free type of* $\bar{v}$ *over* k *is determined by the following three sets of formulas:*

$$\left\{ \mathrm{ex}(\frac{1}{l} \cdot \bar{x}) \in W^{\frac{1}{l}} : l \in \mathbb{N} \right\}; \tag{5}$$

$$\{\mathrm{ex}(\bar{x}) \notin V : V \subset W, \ k\text{-variety}, \ \dim V < \dim W\}; \tag{6}$$

$$\{m_1 \cdot x_1 + \cdots + m_n \cdot x_n \neq 0 : \ \langle m_1, \ldots, m_n \rangle \in \mathbb{Z}^n \setminus \{\bar{0}\}\}, \tag{7}$$

for W *the minimal* k*-variety containing* $\mathrm{ex}(\bar{v})$ *and a sequence* $W^{\frac{1}{l}}$ *associated with the variety.*

Proof Check all atomic formulas in the type of $\bar{v}$:

Any atomic formula containing a term with ex is equivalent to a Zariski closed relation between $\mathrm{ex}(\frac{x_i}{l})$, $i =, 1 \ldots, n$, with a common l. This is included in (5). The negation of such an atomic formula follows from (6). And (7) lists all the negations of atomic formulas, which do not contain terms with ex. Positive atomic formulas with no ex terms can not hold by the assumptions.□

Remark If $\dim W = 0$ the part given by (6) is void.

Lemma 3.4 *Let*

$$\mathbb{V} = (V, A) \text{ and } \mathbb{V}' = (V', A')$$

be ω-saturated models of T_A and

$$\rho : (V \cup A) \to (V' \cup A')$$

a partial L-isomorphism, with finitely generated domain D. Then given any $z \in V \cup A$, ρ extends to the substructure generated by $D \cup \{z\}$.

Proof By definition the V-part of D is a linear subspace generated by some linearly independent $v_1, \ldots, v_{n-1} \in V$.

First consider the case $z \in A$. We may assume that $z \notin \mathrm{ex}(V \cap D)$, for otherwise z is in D already. Then the quantifier-free type $\mathrm{qftp}(z/D)$ of z over D is determined by the quantifier-free type $\mathrm{qftp}_A(z/D \cap A)$ of the structure $\mathbb{A}$ of z over $D \cap A$, since the only terms over $D \cap H$ that may appear in the atomic formulas concerning z are of the form $\mathrm{ex}(q \cdot v)$, and these can be replaced by their values in $D \cap A$. In this case we can extend ρ by choosing a realisation of the type $\rho(\mathrm{qftp}_A(z/D \cap A))$, which is consistent because of the quantifier elimination for $\mathbb{A}$.

Now consider the case when $z \in V \setminus D$. Let C be a finite subset of A which, along with $\{v_1, \ldots, v_{n-1}\}$, generates D. We can replace the field k_0 by its extension $k_0(C)$ and thus w.l.o.g. assume that $D \cap A$ is generated by $\mathrm{ex}(v_1, \ldots, v_{n-1})$ alone.

Let, for $l \in \mathbb{N}$, $W^{\frac{1}{l}}$ be the minimal algebraic variety over k_0 which contains $\langle \mathrm{ex}(\frac{v_1}{l}), \ldots, \mathrm{ex}(\frac{v_{n-1}}{l}), \mathrm{ex}(\frac{z}{l}) \rangle$. Obviously, $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ is a sequence associated with W. By assumptions on ρ and the elimination of quantifiers in $\mathbb{A}$, for every l there is $y_l \in A'$ such that $\langle \mathrm{ex}(\frac{\rho v_1}{l}), \ldots, \mathrm{ex}(\frac{\rho v_{n-1}}{l}), y_l \rangle \in W^{\frac{1}{l}}$. By Lemma 3.2 there is $v'_n \in V$ such that $\langle \mathrm{ex}(\frac{\rho v_1}{l}), \ldots, \mathrm{ex}(\frac{\rho v_{n-1}}{l}), \mathrm{ex}(\frac{v'_n}{l}) \rangle \in W^{\frac{1}{l}}$

for all $l \in \mathbb{N}$. Letting $\rho(z) = v'_n$ and extending to the subspace generated by $D \cap V \cup \{z\}$ by linearity, we have by Lemma 3.3 the required partial isomorphism. □

Corollary 2 *The first-order theory T_A is submodel complete, allows elimination of quantifiers and is complete and superstable.*

Corollary 3 *The structure induced in $\mathbb{V}$ on the sort $\mathbb{A}$ is the structure induced by Zariski closed k_0-definable relations only.*

Elimination of quantifiers also yields

Corollary 4 *Given a model $\mathbb{V} = (V, A)$ of T_A, the decomposition (3) of Lemma 2.1 and elements $\tau_1, \ldots, \tau_N \in \Lambda(V)$ such that*

$$n_1\tau_1 + \cdots + n_N\tau_N \in m\Lambda \text{ iff } g.c.d.(n_1, \ldots, n_N) \in m\mathbb{Z} \tag{8}$$

for any $n_1, \ldots, n_N, m \in \mathbb{Z}$, $m > 1$, let

$$V' = V_0 + \mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N.$$

Then the substructure $\mathbb{V}' = (V', A)$ of $\mathbb{V}$ is a model of T_A with standard kernel.

Proof Indeed, $\mathrm{ex}(V') = A(F)$, since $\mathrm{ex}(\mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N)$ contains all the m-torsion points of $\mathbb{A}(F)$, for all m, by Fact 1, and thus

$$\mathrm{ex}(\mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N) = \mathrm{ex}(\Lambda(V) \otimes \mathbb{Q}).$$

This proves that $\mathbb{V}'$ is a model of T_A.

Since $\Lambda(V') \otimes \mathbb{Q} \cap V_0 = 0$ and $\mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N \subseteq \Lambda(V') \otimes \mathbb{Q}$, we have $\mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N = \Lambda(V') \otimes \mathbb{Q}$ and thus $\mathbb{Z}\tau_1 + \cdots + \mathbb{Z}\tau_N = \Lambda(V')$. □

We call an N-tuple $\langle \tau_1, \ldots, \tau_N \rangle$ in $\Lambda(V)$ with the property (8) a **pseudo-generating tuple** of $\Lambda(V)$.

Lemma 3.5 *Let $\mathbb{V} = (V, A)$ be an ω-saturated model of T_A, K a subfield of F, $V(K) = \mathrm{Ln}(\mathbb{A}(K)) = \{v \in V : \mathrm{ex}(v) \in \mathbb{A}(K)\}$ and let $v = \langle v_1, \ldots, v_n \rangle$ be an n-tuple in V linearly independent over $V(K) \otimes \mathbb{Q}$.*

Then there is a model $(V', A') = \mathbb{V}' \prec \mathbb{V}$ with standard kernel such that $\mathbb{A}(K) \subseteq A'$ and $\mathbb{V}'$ realises the type $\mathrm{tp}(v/\mathbb{A}(K))$.

Proof Consider the decomposition of the vector space V into the direct sum

$$V = V(K) \otimes \mathbb{Q} \dot{+} V_1$$

for V_1 a linear subspace containing v. We can further decompose

$$V(K) \otimes \mathbb{Q} = \Lambda(V) \otimes \mathbb{Q} \dot{+} V_2,$$

for a linear subspace V_2. Thus we have

$$V = (V_1 + V_2) \dot{+} \Lambda(V) \otimes \mathbb{Q}.$$

Choose pseudogenerators $\tau_1, \ldots, \tau_N$ in $\Lambda(V)$ and let

$$V' = (V_1 + V_2) + \mathbb{Q}\tau_1 + \cdots + \mathbb{Q}\tau_N.$$

By Corollary 4 we have that $\mathbb{V}' = (V', A)$ is a model of T_A. The rest follows from elimination of quantifiers and the fact that v is in V'.$\Box$

Given a sequence $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with W in n variables over a field K and a type p in n variables we say that the sequence **stabilises modulo type** p if there is an $l \in \mathbb{N}$ such that for all m there is only one K-definable variety V with $V^m = W^{\frac{1}{l}}$, such that $p(x_1, \ldots, x_n)$ and $\langle \mathrm{ex}(\frac{x_1}{ml}), \ldots, \mathrm{ex}(\frac{x_n}{ml}) \rangle \in V$ is consistent.

An obvious equivalent condition is that

$$p(x_1, \ldots, x_n) \cup \{\langle \mathrm{ex}(\frac{x_1}{ml}), \ldots, \mathrm{ex}(\frac{x_n}{ml}) \rangle \in W^{\frac{1}{lm}}\} \models \{\langle \mathrm{ex}(\frac{x_1}{k}), \ldots, \mathrm{ex}(\frac{x_n}{k}) \rangle \in W^{\frac{1}{k}} : k \in \mathbb{N}\}.$$

If p is trivial we omit mentioning the type.

Lemma 3.6 *Let $U \subseteq \mathbb{A}^N$ be the variety given by equations $x_1 = \cdots = x_N = 1$. Let $K \supseteq k_0$ be a field and, given a pseudogenerating N-tuple τ, the sequence $U_\tau^{\frac{1}{n}}$, $n \in \mathbb{N}$, associated with U over K. Then the number of distinct sequences $U_\tau^{\frac{1}{n}}$ over K, for all pseudogenerating τ, is either finite or $2^{\aleph_0}$.*

Proof Notice first that if one defines $\alpha_i = \mathrm{ex}(\frac{\tau_i}{l})$ then $U_\tau^{\frac{1}{n}}$ describes the locus of $\langle \alpha_1, \ldots, \alpha_N \rangle$ (and so determines the complete type of the tuple over $\mathbb{A}(k_0)$) and $\{\alpha_1, \ldots, \alpha_N\}$ is a basis over $\mathbb{Z}/n\mathbb{Z}$ of the free finite module

$$\mathbb{A}_n = \{a \in \mathbb{A}(F) : a^n = 1\}$$

(see also Fact 1). Moreover, it follows from definition that an N-tuple τ' with $\mathrm{ex}(\frac{\tau'}{l}) \in U^{\frac{1}{l}}$, for all l, pseudogenerates kernel if and only if $U^{\frac{1}{n}}$ determines the type of a generating N-tuple $\langle \beta_1, \ldots, \beta_N \rangle$ of $\mathbb{A}_n$, for every n.

Since the α's and β's above are independent generators of the same group, there is an invertible $(\mathbb{Z}/n\mathbb{Z})$-linear (in multiplicative form) map $\sigma : \mathbb{A}_n^N \to \mathbb{A}_n^N$ such that

$$\sigma : \langle \alpha_1, \ldots, \alpha_N \rangle \mapsto \langle \beta_1, \ldots, \beta_N \rangle.$$

σ is a group automorphism $\mathbb{A}_n^N \to \mathbb{A}_n^N$ 0-definable in $\mathbb{A}$, and $\sigma(U_\tau^{\frac{1}{n}})$ meets $U_{\tau'}^{\frac{1}{n}}$. Since both are atoms, we have $\sigma(U_\tau^{\frac{1}{n}}) = U_{\tau'}^{\frac{1}{n}}$.

Letting $n = lm$ we see that the number $Dg_{l,m}(U_\tau)$ of choices for varieties $U_{\tau'}^{\frac{1}{n}}$, some pseudogenerating $\bar{\tau}'$, such that $(U_{\tau'}^{\frac{1}{n}})^m = U_\tau^{\frac{1}{l}}$ depends only on K. In case the sequence $\{U_\tau^{\frac{1}{l}} : l \in \mathbb{N}\}$ stabilises we obviously have that the number of all such sequences is finite.

In the alternative case consider the sequence $L = \{l_1, \ldots, l_i, \ldots,\}$ constructed by induction as $l_1 = 1$ and $l_{i+1} = l_i \cdot m$ for m minimal such that $Dg_{l_i,m}(U_\tau) > 1$. Now, given an $l_i \in L$, for any choice of $U_{\tau'}^{\frac{1}{l_i}}$ there are at least 2 choices of $U_{\tau'}^{\frac{1}{l_{i+1}}}$ such that

$$(U_{\tau'}^{\frac{1}{l_{i+1}}})^{\frac{l_{i+1}}{l_i}} = U_{\tau'}^{\frac{1}{l_i}}.$$

Hence there are $2^{\aleph_0}$ sequences associated with U over K.□

Proposition 1 *Let $\mathbb{V}$ be a model of T_A and $\langle \tau, v\rangle = \langle \tau_1, \dots, \tau_N, v_1, \dots, v_n\rangle$ with $v_1, \dots, v_n \in V$ linearly independent over $\Lambda(V) \otimes \mathbb{Q}$ and $\langle \tau_1, \dots, \tau_N\rangle$ pseudo-generators of the kernel.*

(i) Suppose that the sequence $\{W^{\frac{1}{l}}_{\tau,v} : l \in \mathbb{N}\}$ associated with $\{\mathrm{ex}(\frac{\tau,v}{l}) : l \in \mathbb{N}\}$ over k_0 does not stabilise modulo the type 'x is a pseudogenerating N-tuple' (see (8)). Then there are $2^{\aleph_0}$ distinct complete types over k_0 realisable in uncountable models of T_A with standard kernel.

(ii) Suppose that $K \supseteq k_0$ is a field such that $\mathbb{A}_{tors} \subseteq \mathbb{A}(K)$, v is linearly independent over $V(K) \otimes \mathbb{Q}$ and the sequence $\{W^{\frac{1}{l}}_v : l \in \mathbb{N}\}$ associated with $\{\mathrm{ex}(\frac{v}{l}) : l \in \mathbb{N}\}$ over K does not stabilise. Then there are $2^{\aleph_0}$ distinct complete types over K realisable in uncountable models of T_A with standard kernel.

Proof We may assume that $\mathbb{V}$ is ω-saturated.

(i) Consider first the N-type

$$p_\tau(x) = \{\mathrm{ex}(\frac{x}{l}) \in U^{\frac{1}{l}}_\tau : l \in \mathbb{N}\},$$

with $\mathrm{ex}(\frac{\tau}{l}) \in U^{\frac{1}{l}}_\tau$ for all $l \in \mathbb{N}$. In case p_τ does not stabilise modulo the type 'x is a pseudogenerating N-tuple' by Lemma 3.6 we have $2^{\aleph_0}$ distinct types of the form p_τ for pseudogenerating τ. By Corollary 4 each such type is realised in a model of T_A with standard kernel.

In the opposite case $p_\tau(x)$ can be replaced by a formula $\psi(x)$ and the type 'x is a pseudogenerating tuple'.

Let, for each $l \in \mathbb{N}$,

$$Z^{\frac{1}{l}}_{\tau,v} = \{z \in \mathbb{A}^n : \langle \mathrm{ex}(\frac{\tau}{l}), z\rangle \in W^{\frac{1}{l}}_{\tau,v}\}$$

These varieties can be also represented as the fibres over $\mathrm{ex}(\frac{\tau}{l})$ of the projection of $W^{\frac{1}{l}}_{\tau,v}$ into the N-space on the first N co-ordinates.

Claim. For any l there is m and $a_{l,m} \in \mathbb{A}^n_m$ such that

$$Z^{\frac{1}{lm}}_{\tau,v} \cap a \cdot Z^{\frac{1}{lm}}_{\tau,v} = \emptyset.$$

Proof. Since $\{W_{\tau,v}^{\frac{1}{l}} : l \in \mathbb{N}\}$ does not stabilise, for any given l there is an m and some k_0-irreducible $W^{\frac{1}{lm}}$ such that

$$W_{\tau,v}^{\frac{1}{lm}} \cap W^{\frac{1}{lm}} = \emptyset \tag{9}$$

and

$$(W_{\tau,v}^{\frac{1}{lm}})^m = W_{\tau,v}^{\frac{1}{l}} = (W^{\frac{1}{lm}})^m. \tag{10}$$

We can assume that l is big enough in order for both $\mathrm{pr}(W_{\tau,v}^{\frac{1}{lm}})$ and $\mathrm{pr}(W^{\frac{1}{lm}})$ to be equal to the same $U_\tau^{\frac{1}{l}}$, the member of the stabilised sequence above. It follows from (9) that the fibres over same point of the projection of the sets do not intersect. It follows from (10) that the fibres are conjugated by a multiple a of order m. This proves the claim.

Now consider a sequence $L = \{l_1, \ldots, l_i, \ldots,\}$ constructed by induction as $l_1 = 1$ and $l_{i+1} = l_i \cdot m$ for m minimal given by the claim.

Now, given a sequence $\mu : \mathbb{N} \to \mathbb{A}^n_{tors}$ with the property

$$\mu(1) = 1 \text{ and } \mu(i+1)^{l_{i+1}} = \mu(i)$$

we have that $\mu(m) \in \mathbb{A}^n_{l_1 \ldots l_m}$ for all m and we can construct an n-type over $k_0(\mathbb{A}_{tors})$

$$q_{\mu,\tau}(y) = \{\mathrm{ex}(\frac{y}{l_1 \ldots l_m}) \in \mu(m) \cdot V_{\tau,v}^{\overline{\frac{1}{l_1 \ldots l_m}}} : m \in \mathbb{N}\}.$$

By the claim there are $2^{\aleph_0}$ mutually inconsistent such types.

Notice that, since the N-tuple $\mathrm{ex}(\frac{\tau}{l})$ generates the group $\mathbb{A}_l$ for all l, $\mu(m) = \mathcal{M}(m,\tau)$ is a term of τ. Now we consider the $N+n$-types in variables $\{x_1, \ldots, x_N, y_1, \ldots, y_n\}$

$$Q_\mu(x,y) = \{\mathrm{ex}(\frac{y}{l_1 \ldots l_m}) \in \mathcal{M}(m,x) \cdot V_{x,v}^{\overline{\frac{1}{l_1 \ldots l_m}}} : m \in \mathbb{N}\}$$

obtained by replacing all occurrences of τ in $q_{\mu,\tau}$ by x. Let

$$Q^*_\mu(x,y) = Q_\mu(x,y) \& \psi(x) \& \{\ x \text{ is a pseudogenerating tuple}\}. \tag{11}$$

We claim that $Q^*_{\mu_1}(x,y)$ and $Q^*_{\mu_2}(x,y)$ are consistent if and only if $Q^*_{\mu_1}(\tau,y)$ and $Q^*_{\mu_2}(\tau,y)$ are. This follows from the fact that $\psi(x)$& 'x is a pseudogenerating tuple' is a complete type, by Lemma 3.3, since it is equivalent to p_τ. Hence there are $2^{\aleph_0}$ mutually inconsistent types of the form (11).

For each such type there is a realisation of the form $\langle \tau, v' \rangle$ in $\mathbb{V}$. It follows that $v'_1, \ldots, v'_n$ are linearly independent over $\Lambda(V) \otimes \mathbb{Q}$. Hence the linear subspace $L = \mathbb{Q}v'_1 + \ldots \mathbb{Q}v'_n$ does not intersect $\Lambda(V) \otimes \mathbb{Q}$. Hence we can choose linear subspace $V'_0 \supseteq L$ of V such that

$$V'_0 \dot{+} \Lambda(V) = V.$$

By Corollary 4, for

$$V' = V'_0 + \mathbb{Q}\tau'_1 + \cdots + \mathbb{Q}\tau'_N$$

$\mathbb{V}' = (V', A)$ is a model of T_A with standard kernel. By elimination of quantifiers the types of $\langle \tau, v' \rangle$ in $\mathbb{V}$ and $\mathbb{V}'$ coincide. This finishes the proof of (i).

(ii) Given the K-irreducible variety $W = W_v \subseteq \mathbb{A}^n$, any sequence $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with W over K has the property that for every $l, m \in \mathbb{N}$ the number $Dg_{l,m}(W, K)$ of K-varieties $X \subseteq \mathbb{A}^n$ such that $X^m = W^{\frac{1}{l}}$ depends on l and m but not on the way the associated sequence has been chosen. This follows from the fact that if $Z^{\frac{1}{l}}$ is another choice for the lth member of the sequence then, for some $a \in \mathbb{A}^n_{lm}$,

$$W^{\frac{1}{l}} = a^m \cdot Z^{\frac{1}{l}}$$

and the map $x \mapsto ax$ sets a K-definable bijection

$$\{x \in \mathbb{A}^n : x^m \in W^{\frac{1}{l}}\} \to \{x \in \mathbb{A}^n : x^m \in Z^{\frac{1}{l}}\}.$$

This property allows us to construct under the assumption that $\{W_v^{\frac{1}{l}}\}$ does not stabilise a binary tree of $2^{\aleph_0}$ mutually inconsistent sequences associated with W. For each such sequence $\{W^{\frac{1}{l}}\}$ we can, using Lemma 3.5, construct a model of T_A with standard kernel in which the sequence is realised. $\square$

Remark (i) The assumption that $v_1, \ldots, v_n$ are linearly independent over $\Lambda \otimes \mathbb{Q}$ is equivalent to the fact that the elements $\mathrm{ex}(v_1), \ldots, \mathrm{ex}(v_n)$ are multiplicatively independent in the group A, that is no non-trivial group word on the elements is equal to 1.

(ii) The assumption that $v_1, \ldots, v_n$ are linearly independent over $V(K) \otimes \mathbb{Q}$ is equivalent to the fact that the elements $\mathrm{ex}(v_1), \ldots, \mathrm{ex}(v_n)$ are multiplicatively independent in the group A over $\mathbb{A}(K)$, that is no non-trivial group word on the elements is in $\mathbb{A}(K)$.

4 The $L_{\omega_1,\omega}$-theory of group covers

We start with

Remark There is an $L_{\omega_1,\omega}$-formula stating that $\Lambda \cong \mathbb{Z}^N$:

$$\exists \tau_1, \ldots, \tau_N \in \Lambda \bigwedge_{(z_1,\ldots z_N)\in\mathbb{Z}^N\setminus\{\bar{0}\}} z_1\tau_1 + \cdots + z_N\tau_N \neq 0 \wedge$$

$$\wedge \forall u \in \Lambda \bigvee_{(z_1,\ldots z_N)\in\mathbb{Z}^N} z_1\tau_1 + \cdots + z_N\tau_N = u.$$

Now we observe that the Uniqueness Problem as formulated in the introduction (without complex multiplication) is equivalent to the following model theoretic question:

Given a semi-abelian variety $\mathbb{A}$, *is the* $L_{\omega_1,\omega}$*-sentence* $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ *categorical in power* $2^{\aleph_0}$?

Naturally, there are no model theoretical reasons to distinguish $2^{\aleph_0}$, except maybe for the case $2^{\aleph_0} = \aleph_1$, so we consider rather two versions of the problem:

Given a semi-abelian variety $\mathbb{A}$, *is the* $L_{\omega_1,\omega}$*-sentence* $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ *categorical in*

(i) power $\aleph_1$?

(ii) all uncountable powers?

In these forms the problem can be treated in the frames of Keisler-Shelah theory of $L_{\omega_1,\omega}$-categoricity.

The first step in this theory is to reduce the study of a categorical sentence to the case when all the models of the sentence are first-order atomic. This is done in general by extending the language by appropriate $L_{\omega_1,\omega}$-definable predicates.

We assume below that $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ is $\aleph_1$-categorical.

Notation Let

(i) for each $n > 0$, $Ind^n(x_1, \ldots, x_n)$ denote an n-type stating that $x_1, \ldots x_n$ are linearly independent in the $\mathbb{Q}$-space V;

(ii) $PG^l(x_1, \ldots, x_l)$, $l \in N$, be the l-type:

$$\langle x_1, \ldots, x_l \rangle \in \Lambda^l \ \& \bigwedge_{\text{g.c.d.}(m_1,\ldots,m_l)=1,\ m>1} m_1x_1 + \cdots + m_lx_l \notin m\Lambda;$$

(iii) for each n and a k_0-irreducible variety W in n variables $Gen^W(\bar{y})$ be the n-type on A stating that $\bar{y}$ is k_0-generic point in W.

Remark A new predicates of type (i) is equivalent to the set of formulas (7) and of (iii) to the set (6).

Remark It is immediate by definitions that an l-tuple $\langle x_1, \ldots, x_l \rangle$ can be extended to a pseudo-generating N-tuple iff $PG^l(x_1, \ldots, x_l)$ holds.

In the following definition we use vector and matrix notations: $x = (x_1, \ldots, x_n)$, and for $r = (r_1, \ldots, r_n) \in \mathbb{Q}^n$, we denote $rx = r_1x_1 + \cdots + r_nx_n$.

Call a quantifier-free L-type $p(x_1, \ldots, x_n)$ **almost principal** if p is equivalent to a union of the following subtypes, for some rational vectors $q_1, \ldots, q_m \in \mathbb{Q}^n$, $(m \le n)$, a rational $n \times m$-matrix Q, a non negative integer $l \le m$, a positive integer M and a k_0-irreducible variety W in m variables:

(i) $Ind^m(q_1x, \ldots, q_mx)$ & $x = \langle q_1x, \ldots, q_mx \rangle Q$;
(ii) $PG^l(q_1x, \ldots, q_lx)$;
(iii) $\langle \mathrm{ex}(q_1x), \ldots, \mathrm{ex}(q_mx) \rangle \in W^{\frac{1}{M}}$ & $Gen^W(\mathrm{ex}(q_1x), \ldots, \mathrm{ex}(q_mx)$

Let S_A be the set of all complete n-types in V-variables, for all n, realisable in models of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$.

Lemma 4.1 *Any type in S_A is almost principal or $|S_A| = 2^{\aleph_0}$.*

Proof Assume that there is a non-almost principal type realised in a model $\mathbb{V}$ of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ by some $\langle v_1, \ldots, v_n\rangle \in V^n$.

We claim that then, for $\tau_1, \ldots, \tau_N$ generating $\Lambda(V)$, the type of $\langle \tau_1, \ldots, \tau_N, v_1, \ldots, v_n\rangle$ is not almost principal as well. Indeed, we may assume w.l.o.g. that $\tau_1 = v_1, \ldots, \tau_l = v_l$ and $\{v_{l+1}, \ldots, v_n\}$ is linearly independent over $\Lambda \otimes \mathbb{Q}$. Let W be the minimal k_0-variety containing $\langle \mathrm{ex} v_1, \ldots, \mathrm{ex} v_n\rangle$. Our assumptions imply that, for no positive integer M, the type

$$Ind^n(v_1, \ldots, v_n) \ \& \ PG^l(v_1, \ldots, v_l) \ \& \ Gen^W(\mathrm{ex}(v_1), \ldots, \mathrm{ex}(v_n)) \ \&$$

$$\& \langle \mathrm{ex}(v_1), \ldots, \mathrm{ex}(v_n)\rangle \in W^{\frac{1}{M}}$$

implies

$$\{\langle \mathrm{ex}(\frac{v_1}{m}), \ldots, \mathrm{ex}(\frac{v_n}{m})\rangle \in W^{\frac{1}{Mm}} : m \in \mathbb{N}\}.$$

This is also true if we replace $PG^l(v_1, \ldots, v_l)$ with $PG^N(v_1, \ldots, v_l, u_{l+1}, \ldots, u_N)$, for new variables $u_{l+1}, \ldots, u_N$, since an l-tuple can be extended (in a saturated model of T_A) to a pseudogenerating N-tuple iff the l-tuple satisfies PG^l.

So, we now are under assumptions of Proposition 1 and hence $|S_A| = 2^{\aleph_0}$.□

Keisler's Theorem (Theorem 5.6 of [K]) *If an $L_{\omega_1,\omega}$-sentence Σ is $\aleph_1$-categorical then the set of complete n-types realisable in models of Σ is at most countable.*

Corollary 5 *All types in S_A are almost principal.*

Extend the language L to a new language L^* by adding predicate symbols to be interpreted as $Ind^n(x_1, \ldots, x_n)$, $PG^l(x_1, \ldots, x_l)$ and $Gen^W(\bar{y})$ on A for all n, l, W $(l \leq N)$.

Remark All the new predicates are $L_{\omega_1,\omega}$-definable in T_A.

Recall that an L-structure is said to be **L-atomic** if the type of any finite tuple of elements in the structure is principal, i.e. is determined by a finite set of L-formulas.

The following is stating that all the uncountable models of $T_A+\{\Lambda\cong\mathbb{Z}^N\}$ are ω-homogeneous in the language L^*.

Lemma 4.2 *Let*

$$\mathbb{V}=(V,A) \text{ and } \mathbb{V}'=(V',A')$$

be models of $T_A+\{\Lambda\cong\mathbb{Z}^N\}$ such that the underlying fields $F(A)$ and $F(A')$ are of infinite transcendence degree. Suppose

$$\rho:(V\cup A)\to(V'\cup A')$$

is a partial L-isomorphism with finitely generated domain D. Then, given any $z\in V\cup A$, ρ extends to the substructure generated by $D\cup\{z\}$.

In particular, if $\mathbb{V}$ and $\mathbb{V}'$ are countable, ρ extends to an isomorphism between the structures.

Proof Similarly to the proof of Lemma 2.1.

Let $v_1,\dots,v_{n-1}\in V$ generate the linear subspace $V\cap D$ and be independent.

First consider the case $z\in A\setminus \mathrm{ex}(V\cap D)$. The quantifier-free type $\mathrm{qftp}(z/D)$ of z over D is determined by the quantifier-free type $\mathrm{qftp}_A(z/D\cap A)$. The latter by QE for $\mathbb{A}$ is equivalent to a collection of formulas stating that z is generic in W over $A\cap D$, for some irreducible over $A\cap D$ variety W. Then $\rho(\mathrm{qftp}_A(z/D\cap A))$ states the corresponding genericity condition about $z'\in\rho(W)$. Such a z' must exist in A' by the assumptions of the lemma. Hence we are done in this case.

Now consider the case when $z\in V\setminus D$. Let C be a finite subset of A which along with $\{v_1,\dots,v_{n-1}\}$ generates D. We can replace the field k_0 by its extension $k_0(C)$ and thus w.l.o.g. assume that $D\cap A$ is generated by $\mathrm{ex}(v_1,\dots,v_{n-1})$ alone.

Let, for $l\in\mathbb{N}$, $W^{\frac{1}{l}}$ be the minimal algebraic variety over k_0 which contains $\langle \mathrm{ex}(\frac{v_1}{l}),\dots,\mathrm{ex}(\frac{v_{n-1}}{l}),\mathrm{ex}(\frac{z}{l})\rangle$. By Corollary 5 the type

$$\{\langle \mathrm{ex}(\frac{v_1}{l}),\dots,\mathrm{ex}(\frac{v_{n-1}}{l}),\mathrm{ex}(\frac{z}{l})\rangle\in W^{\frac{1}{l}}:l\in\mathbb{N}\}$$

is equivalent to its finite subset, in fact just to one of the formulas. By assumptions on ρ and the elimination of quantifiers in $\mathbb{A}$, given l, there is

$y_l \in A'$ such that $\langle \mathrm{ex}(\frac{\rho v_1}{l}), \ldots, \mathrm{ex}(\frac{\rho v_{n-1}}{l}), y_l \rangle \in W^{\frac{1}{l}}$. Then, letting $\rho(z) = z'$ for a $z' \in V'$ such that $\mathrm{ex}(\frac{z'_1}{l}) = y_l$, we get, extending to $V \cap D + \mathbb{Q}z$ by linearity, the required partial isomorphism. □

Proposition 2 *Any model of the $L_{\omega_1,\omega}$-sentence*

$$T_A + \{\Lambda \cong \mathbb{Z}^N\} + \{\mathrm{tr.d.}F(A) \geq \aleph_0\} \tag{12}$$

is L^*-atomic.

Proof By Lemma 4.2 any complete L^*-type p (in fact any complete $L_{\omega_1,\omega}$-type) in a model of the sentence is determined by its quantifier-free L-subtype, which is almost principal by Corollary 5. This immediately implies that p is equivalent to a finite L^*-type.□

Keisler's theory can say more about our Σ. To get stronger consequences notice first

Proposition 3 $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ *has the amalgamation property, that is for any three models $\mathbb{V}_0$, $\mathbb{V}_1$ and $\mathbb{V}_2$ of the sentence with embeddings $\mathbb{V}_0 \preccurlyeq_{\pi_i} \mathbb{V}_i$, $i = 1, 2$ there is a model $\mathbb{V}$ and embeddings $\mathbb{V}_i \preccurlyeq_{\phi_i} \mathbb{V}$, $i = 1, 2$ agreeing with the π_i's.*

Proof Notice that by quantifier elimination the embeddings are just usual embeddings. Let $\mathbb{V}_0 \subseteq \mathbb{V}_i$, $i = 1, 2$ be embeddings of models and $V_0 \subseteq V_i$, $\mathbb{A}(F_0) \subseteq \mathbb{A}(F_i)$ the corresponding embeddings of the underlying sets. By QE and the definable correspondence between $\mathbb{A}(F)$ we have $F_0 \subseteq F_i$, $i = 1, 2$.

Now consider an algebraically closed field F which is a free amalgam of algebraically closed subfields F_1 and F_2 over F_0, that is, up to isomorphism, we can think of F as $\mathrm{acl}(F_1F_2)$ with $F_1, F_2 \subseteq F$, $F_0 = F_1 \cap F_2$ and transcendence bases B_i of F_i over F_0, $i = 1, 2$, independent over F_0. Let also $V_1 + V_2$ be a free amalgam of the two vector spaces over V_0. Let $\mathrm{ex}_i : V_i \to \mathbb{A}(F_i)$, $i = 0, 1, 2$, denote $\mathrm{ex}_{|V_i}$. We then have a natural homomorphism

$$\mathrm{ex}' : V_1 + V_2 \to \mathbb{A}(F_1) \cdot \mathbb{A}(F_2) \subseteq \mathbb{A}(F),$$

defined by $\mathrm{ex}(v_1 + v_2) = \mathrm{ex}_1(v_1) \cdot \mathrm{ex}_2(v_2)$, for $v_1 \in V_1, v_2 \in V_2$.

By the theory of Abelian groups we have a direct decomposition

$$\mathbb{A}(F) = (\mathbb{A}(F_1) \cdot \mathbb{A}(F_2)) \times B.$$

Let

$$V = (V_1 + V_2) \times B$$

and $\mathrm{ex} : V \to \mathbb{A}(F)$ be defined as $\langle v, b\rangle \mapsto \langle \mathrm{ex}'(v), b\rangle$. We then have that $\ker \mathrm{ex} = \ker \mathrm{ex}_0 \cong \mathbb{Z}^N$ and all the axioms of T_A satisfied for $\mathbb{V} = (V, \mathrm{ex}, \mathbb{A}(F))$ by construction. Thus $\mathbb{V}$ is a model of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ extending the amalgam $\mathbb{V}_0 \subseteq \mathbb{V}_i$, $i = 1.2\Box$

Now, in the proof of Keisler's theorem the unique model of cardinality $\aleph_1$ is an Ehrenfeucht-Mostowski model, that is it realises countably many complete types over any countable set. Thus if we use the amalgamation property for countable models of Σ then a stronger version of Keisler's Theorem holds
*In the presence of the amalgamation property Σ is ω-**stable,** that is the set of complete n-types over a countable model realisable in models of Σ is at most countable.*

Corollary 6 *$T_A + \{\Lambda \cong \mathbb{Z}^N\}$ is ω-stable.*

Under assumption of ω-stability and the amalgamation property for countable models Shelah develops the theory of splitting, ranks and independence.

Recall that a complete type p over B **splits** over $C \subseteq B$ if there are b_1, b_2 in B realising the same type over C and a formula $\phi(x, y)$ such that $\phi(x, b_1) \in p$ and $\neg\phi(x, b_2) \in p$.

Non-splitting in ω-stable classes is an **independence relation** $\downarrow$. More precisely for subsets A, B, C with $C \subseteq A$ and $C \subseteq B$

$$A \downarrow_C B \text{ iff } \mathrm{tp}(a/B) \text{ doesn't split over } C \text{ for all finite } a \text{ in } A.$$

In what follows we are interested in the case when A, B and C are models.

It is helpful to notice that

Lemma 4.3 *In the theory of algebraically closed fields*

$F_1 \underset{F_0}{\downarrow} F_2$ *iff the fields* F_1 *and* F_2 *are* **linearly disjoint** *over* $F_0 = F_1 \cap F_2$.

Proof Immediate from definitions. See [L], Chapter VIII, section 4 for the definition and properties of linear disjointness.□

By Fact 2 the elements and types of sort A in $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ are in a direct correspondence to types in the theory of algebraically closed fields. It follows

Lemma 4.4 *Let* $\mathbb{V}_0 \subseteq \mathbb{V}_i$, $i = 1, 2$ *be models of* $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ *intersecting at* V_0 *and* $A_i = \mathbb{A}(F_i)$ *the sort* A *subset for* $\mathbb{V}_i$, *and* F_i *the corresponding algebraically closed field,* $i = 0, 1, 2$. *Then*

$A_1 \underset{A_0}{\downarrow} A_2$ *iff the fields* F_1 *and* F_2 *are linearly disjoint over* $F_0 = F_1 \cap F_2$.

By quantifier elimination we know that to check non-splitting of a general type of $\mathrm{tp}(\bar{x}/V_2)$, $\bar{x}$ in V_1, over V_0 it is enough to check that corresponding algebraic varieties over A_0 in (5) of Lemma 3.3 are irreducible over A_2, which follows if F_1 and F_2 are linearly disjoint over F_0. Thus we get

Corollary 7

$\mathbb{V}_1 \underset{\mathbb{V}_0}{\downarrow} \mathbb{V}_2$ *iff the fields* F_1 *and* F_2 *are linearly disjoint over* $F_0 = F_1 \cap F_2$.

Finally observe by definitions

Remark Algebraically closed fields F_1 and F_2 are linearly disjoint over $F_0 = F_1 \cap F_2$ if and only if there is an algebraically independent set B such that

$$B = B_1 \cup B_2, \quad B_1 \cap B_2 = B_0$$

and B_i is a transcendence basis for F_i for $i = 0, 1$ and 2.

Now we can introduce the notion of excellency of a sentence Σ ([Sh]). It is based on the notion of an **independent n-system of countable models.**

By this we mean a collection of countable models $\{M_s : s \subset n\}$ of Σ (here $n = \{0, \ldots, n-1\}$ and $\subset$ means the proper subset relation) with the property that

$$M_s \prec M_t \quad \text{iff} \quad s \subset t$$

and

$$M_s \mathop{\smile\!\!\!\!\!\!|}_{M_{s\cap t}} M_t, \quad \text{for any } s, t \subset n.$$

Example and definition Let F be an algebraically closed field containing k_0, $B \subseteq F$ a subset algebraically independent over k_0 and

$$B = B_1 \dot{\cup} \ldots \dot{\cup} B_n$$

its partition into non-empty subsets. Define, for $s \subset n$:

$$F_s = \mathrm{acl}(k_0(\bigcup\{B_i : i \in s\})). \tag{13}$$

By the observation about linear disjointness $\{F_s : \ s \subset n\}$ is an n-independent system of subfields. By the above remark characterising linear disjointness of algebraically closed fields *any independent n-system of algebraically closed fields containing k_0 has this form.*

Proposition 4 *An independent n-system of countable models of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ has the form*

$$\{\mathbb{V}_s = (V_s, A_s) : \ s \subset n\}, \ \textit{for } A_s = \mathbb{A}(F_s)$$

where $\{F_s : \ s \subset n\}$ is a system of the form (13).

Proof Immediate by Corollary 7 and following remarks.□

Definition A sentence Σ all of whose models are atomic is said to be **excellent** if the class of models of Σ has the amalgamation property for countable models and for any independent n-system of countable models $\{M_s \prec M : s \subset n\}$ there is a model M_n of Σ, such that $M_s \prec M_n$ for all $s \subset n$ and M_n prime over $\{M_s \prec M : s \subset n\}$.

Remark In particular, it follows that $\{M_s \prec M : s \subset n\}$ is **good**, that is any type over the set realisable in a model of Σ is principal.

Definition Given a countable algebraically independent over k_0 subset $B \subseteq F$ and its partition $B = B_1 \dot\cup \ldots \dot\cup B_n$ define **acl-B-generated extension of** k_0 to be the field

$$k_0^B = k_0(\bigcup_{s \subset n} F_s), \text{ an extension of } k_0 \text{ by the algebraically closed fields.}$$

Shelah's Theorem [Sh] *Suppose that all models of an $L_{\omega_1,\omega}$-sentence Σ are atomic, Σ is categorical in $\aleph_n$, all n, and assume also that $2^{\aleph_{n+1}} > 2^{\aleph_n}$, for all $n \in \mathbb{N}$ (or just GCH). Then the class of models of Σ is excellent and ω-stable.*

Assuming below GCH and that $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ is categorical in all uncountable powers we have

Corollary 8 *For any algebraically independent countable B with a partition,*

$$|S_A(\mathbb{V}(k_0^B))| = \aleph_0.$$

Proposition 5 *Given $\mathbb{V}$, a model of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$, and an algebraically independent $B \subseteq F(\mathbb{V})$ with a partition, $\mathbb{V}$ is L^*-atomic over $V(k_0^B)$.*

Proof We can reduce any type over $V(k_0^B)$ to the type of an n-tuple $v = \langle v_1, \ldots, v_m \rangle \in V^n$ linearly independent over $V(k_0^B)$. By Proposition 1(ii) and Corollary 8 any such type is L^*-principal. $\square$

5 Arithmetic consequences

Given an algebraically closed field $F \supseteq k_0$ let, for $a \in \mathbb{A}(F)$,

$$a^{\mathbb{Q}} = \{x \in \mathbb{A}(F) : x^n = a^m \text{ some } m, n \in \mathbb{Z}, n \neq 0\}.$$

In particular, $1^{\mathbb{Q}} = \mathbb{A}_{tors}$ is the set of all torsion points of $\mathbb{A}(F)$.

Given a subset $X \subseteq \mathbb{A}$ we denote

$$k_0(X) = k_0(\bigcup\{[x] : x \in X\}) = \text{dcl}\ X\ \cap F(\mathbb{A}).$$

In particular we consider for a given finite collection of points $a_1, \dots, a_n \in A(F)$, $n \geq 0$,

$$k_a = k_0(\mathbb{A}_{tors}, a_1^{\mathbb{Q}}, \dots, a_n^{\mathbb{Q}})$$

the subfield of F generated over k_0 by all the coordinates of the elements of $a_i^{\mathbb{Q}}$, including $a_0 = 1$. In other words k_a is the field obtained by adjoining to k_0 the coordinates of all the points in the division hull of the group generated by $a_1, \dots, a_n$. In particular, we need $\mathbb{A}_{tors}$ only in case $n = 0$.

We are going to consider the group $\mathbb{A}(k_a)$ of k_a-points of $\mathbb{A}$, and the group $\mathbb{A}(k_0)$.

We denote $\tilde{K} = \mathrm{acl}(K)$ for a field K.

Theorem 1 (Torsion points) *Assuming $\aleph_1$-categoricity of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ we have for any finitely generated extension k of k_0 :*

(i) the torsion subgroup $\mathbb{A}_{tors}(k)$ of $\mathbb{A}(k)$ is finite;

(ii) there is a number d such that for any l the Galois group $\mathrm{Gal}(\tilde{k} : k)$ has at most d orbits on the set

$$\{\langle a_1, \dots, a_N\rangle \in \mathbb{A}_l^N : \ a_1, \dots, a_N \textit{ generate } \mathbb{A}_l\};$$

(iii) for all but finitely many prime p the group $\mathrm{Gal}(\tilde{k} : k)$ acts on the Tate module $T_p(\mathbb{A})$ as $\mathrm{GL}_N(\mathbb{Z}_p)$, and for remaining finite number of p the group acts as a subgroup of $\mathrm{GL}_N(\mathbb{Z}_p)$ of finite index.

Proof In fact we are going to use only the fact that all models of infinite transcendence degree of the sentence are L^*-atomic (Proposition 2). This property is preserved under expansion of the language with finitely many constant, thus we can assume that $k = k_0$.

(i) follows from (ii). Indeed, let $a \in \mathbb{A}(k_0)$ be a primitive solution of the equation $x^l = 1$. Then a can be included in some $\bar{a}$, a generating N-tuple of $\mathbb{A}_l$. For any m co-prime with l we have that $\bar{a}^m$ (component-wise) is generating $\mathbb{A}_l$ as well. But a and a^m are not conjugated by a Galois automorphism over k_0, unless $m \equiv 1 \mod l$. Hence, by (ii), $\varphi(l) \leq d$, so l is bounded.

(ii) Follows from Proposition 2 or, more precisely, from the fact that the type of any N-tuple τ pseudogenerating kernel stabilises. This implies that there are at most d complete types of pseudogenerating N-tuples, some finite d, and by ω-saturatedness of the structure on $\mathbb{A}$ we get d orbits under the automorphism group, which acts on $\mathbb{A}(\tilde{k}_0) \supseteq \mathbb{A}_{tors}$ as $\mathrm{Gal}(\tilde{k}_0 : k_0)$.

(iii) Essentially the same argument as for (ii) but in a different language.

Consider a model $\mathbb{V}$ of T_A with $\Lambda(V) \cong \widehat{\mathbb{Z}}^N$, where $\widehat{\mathbb{Z}}$ denotes the compactification of $\mathbb{Z}$ in profinite topology. It is known that equivalently we can represent

$$\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p, \tag{14}$$

as a direct product of additive groups of p-adic numbers, considered as topological groups. We then have correspondingly

$$\Lambda(V) \cong \prod_{p \text{ prime}} \mathbb{Z}_p^N \tag{15}$$

and each p-component of the direct product is a module over the ring $\mathbb{Z}_p$ which can be identified as the Tate module $T(\mathbb{A})$ of $\mathbb{A}$.

Identifying $\Lambda(V)$ with $\widehat{\mathbb{Z}}^N$, notice that a pseudogenerating tuple τ of $\widehat{\mathbb{Z}}_p^N$ generates a dense subgroup of the group. Hence an $\alpha \in \mathrm{Aut}(\widehat{\mathbb{Z}}^N)$ is uniquely determined by the pseudogenerating tuple $\tau' = \alpha(\tau)$. This α obviously preserves the profinite topology and is $\mathbb{Z}$-linear so component-wise α is $\mathbb{Z}_p$-linear, hence

$$\mathrm{Aut}(\widehat{\mathbb{Z}}^N) \cong \prod_{p \text{ prime}} \mathrm{GL}_N(Z_p).$$

Any Galois automorphism α on $\mathbb{A}_{tors}$ over k_0, by elimination of quantifiers and Lemma 3.2, can be lifted to an automorphism of the additive group $\Lambda(V)$. Since there are only finitely many types of generating tuples, the Galois automorphisms induce a subgroup $\mathrm{Aut}_{\mathrm{Gal}}(\widehat{\mathbb{Z}}^N)$ of $\mathrm{Aut}(\widehat{\mathbb{Z}}^N)$ of finite index. Component-wise we have

$$\mathrm{Aut}_{\mathrm{Gal}}(\widehat{\mathbb{Z}}^N) \cong \prod_{p \text{ prime}} G_p, \qquad G_p \subseteq \mathrm{GL}_N(Z_p)$$

with $G_p = \mathrm{GL}_N(Z_p)$ for almost all p and of finite index in the remaining finite number of p.$\square$

Comments (i) of Theorem 1 and (i) of the next theorem for a number field k is an immediate consequence of the Mordell-Weil Theorem (and Dirichlet's theory in the case $\mathbb{A}$ is a one-dimensional torus). (iii) of Theorem 1 is well-known for the one-dimensional torus (the multiplicative group of the field) and is a difficult theorem of Serre for elliptic curves without complex multiplication. No other cases are known to the author.

The known cases of the next theorem are classically proved via the theory of Abelian Galois extensions and the method of infinite descent. For elliptic curves without multiplication (iii) is given by Bashmakov's Theorem.

Theorem 2 (Kummer's Theory and heights) *Assuming* $\aleph_1$*-categoricity of* $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ *we have:*

(i) $\mathbb{A}(k_0) \cong A_0 \times \mathbb{A}_{tors}(k_0)$ *for some free abelian group* A_0*;*

and for $a_1, \dots, a_n \in \mathbb{A}$ *multiplicatively independent:*

(ii)

$$\mathbb{A}(k_a) \cong A_a \times \mathbb{A}_{tors} \cdot a_1^{\mathbb{Q}} \cdot \dots \cdot a_n^{\mathbb{Q}}$$

for some A_a *free abelian;*

(iii) given $b_1, \dots, b_k \in \mathbb{A}(k_0)$ *multiplicatively independent there is an* $l \in \mathbb{N}$ *such that for any* $m \in \mathbb{N}$

$$\mathrm{Gal}(k_0(\mathbb{A}_{ml}, b_1^{\frac{1}{ml}}, \dots, b_k^{\frac{1}{ml}}) : k_0(\mathbb{A}_{ml}, b_1^{\frac{1}{l}}, \dots, b_k^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nk}.$$

(iv) given $b_1, \dots, b_k \in \mathbb{A}(k_a)$ *such that* $a_1, \dots, a_n, b_1, \dots, b_k$ *are multiplicatively independent, there is an* $l \in \mathbb{N}$ *such that for any* $m \in \mathbb{N}$

$$\mathrm{Gal}(k_a(b_1^{\frac{1}{ml}}, \dots, b_k^{\frac{1}{ml}}) : k_a(b_1^{\frac{1}{l}}, \dots, b_k^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nk}.$$

(v) Let $F_0 \subseteq F = F(\mathbb{A})$ *be a countable algebraically closed subfield and* $b_1, \dots, b_k \in \mathbb{A}(F)$ *multiplicatively independent over* $\mathbb{A}(F_0)$*. Then there is an* $l \in \mathbb{N}$ *such that for any* $m \in \mathbb{N}$

$$\mathrm{Gal}(F_0(b_1^{\frac{1}{ml}}, \dots, b_k^{\frac{1}{ml}}) : F_0(b_1^{\frac{1}{l}}, \dots, b_k^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nk}.$$

Proof We first prove (v). This follows directly from ω-stability and Proposition 1.

Now we consider (iv). We start with the remark that ω-stability and atomicity is preserved when the language is extended by naming a finite number of elements of a model. Thus if we name $\tau_1, \ldots, \tau_N, h_1, \ldots, h_n \in V$, such that $\tau_1, \ldots, \tau_N$ generate the kernel of a model and $\mathrm{ex}(h_i) = a_i$, we still have atomicity of the model. Notice that such an expansion of a model $\mathbb{V}$ names all elements of the subfield $k_a = k_0(\mathbb{A}_{tors}, a_1^{\mathbb{Q}}, \ldots, a_n^{\mathbb{Q}})$ of $F(\mathbb{V})$. Hence we have that any sequence $\{W_v^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with a v in V over k_a stabilises, by Proposition 1. Consider the sequence $\{W_v^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with $v_1, \ldots, v_k$, where $\mathrm{ex}(v_i) = b_i$. We then have an l such that for any m all the k-tuples of roots of order m of $\mathrm{ex}(\frac{v}{l})$ are conjugated by a Galois automorphism over $k_a(\mathrm{ex}(\frac{v}{l}))$, that is over $k_a(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})$. Then the group in (iv) is transitive on k-tuples of roots of order m of $\mathrm{ex}(\frac{v}{l})$. It follows that the group action is of the form

$$\mathrm{ex}(\frac{v}{ml}) \mapsto \alpha \cdot \mathrm{ex}(\frac{v}{ml}), \text{ for } \alpha \in \mathbb{A}_m^k.$$

Hence the group in (iv) is isomorphic to $\mathbb{A}_m^k$, and so to $(\mathbb{Z}/m\mathbb{Z})^{Nk}$.

The proof of (iii) is very similar, with the use of (i) instead of (ii) of Proposition 1. We thus get an l such that for any m all the $(N+k)$-tuples of roots of order m of $\mathrm{ex}(\frac{\tau, v}{l})$ are conjugated by a Galois automorphism over $k_0(\mathrm{ex}(\frac{\tau, v}{l}))$. In particular any two values of $(b_1^{\frac{1}{lm}}, \ldots, b_k^{\frac{1}{lm}})$ are conjugated over $k_0(\mathbb{A}_{lm}, b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})$. Hence (iii) follows.

For (i) and (ii) we need to prove that the groups

$$A_0 = \mathbb{A}(k_0)/\mathbb{A}_{tors}(k_0) \text{ and } A_a \cong \mathbb{A}(k_a)/\mathbb{A}_{tors} \cdot a_1^{\mathbb{Q}} \cdot \cdots \cdot a_n^{\mathbb{Q}}$$

are free.

Then (i) and (ii) follow by the general theory of Abelian groups, see [F, Th 14.4].

Obviously, A_a and A_0 are torsion-free. By Pontryagin Theorem [F, Th 19.1] we only need to prove

Claim. For any finitely generated subgroup $U \subseteq A_a$ (correspondingly $U \subseteq A_0$) the pure hull

$$\tilde{U} = \{u \in A_a : u^m \in U \text{ for some } m \in \mathbb{N}\}$$

of the subgroup in A_a (in A_0) is finitely generated.

Notice that U itself is free since it is finitely generated torsion-free [F, Th. 15.5].

Proof of Claim.

We prove it for the group $\mathbb{A}(k_a)$ using (iv) proved above and notice that the proof for $\mathbb{A}(k_0)$ is very similar but uses (iii) in place of (iv).

Let $\{u_1, \ldots, u_k\}$ be independent generators of U, and $\{b_1, \ldots, b_k\}$ elements in $\mathbb{A}(k_a)$ which correspond to $\{u_1, \ldots, u_k\}$ under the natural projection $\mathbb{A}(k_a) \to A_a$. Thus $\{a_1, \ldots, a_n, b_1, \ldots, b_k\}$ are multiplicatively independent.

We claim that for some l

$$\tilde{U} \subseteq gp(u_1^{\frac{1}{l}}, \ldots, u_k^{\frac{1}{l}}). \tag{16}$$

(Here and below $gp(S)$ for a subset $S \subseteq A$ stands for a subgroup of A generated by elements of S.)

By (iv) we can choose an $l \in \mathbb{N}$ such that for any $m \in \mathbb{N}$

$$\mathrm{Gal}(k_a(b_1^{\frac{1}{ml}}, \ldots, b_k^{\frac{1}{ml}}) : k_a(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nk}.$$

If (16) does not hold for this l then there is $g \in \mathbb{A}(k_a)$ such that

$$g^m \in gp(\mathbb{A}_{tors}, a_1^{\mathbb{Q}}, \ldots, a_n^{\mathbb{Q}}, b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})$$

but g is not in the subgroup.

Then, replacing g by $g \cdot c$, some $c \in gp(\mathbb{A}_{tors}, a_1^{\mathbb{Q}}, \ldots, a_n^{\mathbb{Q}})$, we can have

$$g^m \in gp(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}}), \quad g \notin gp(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}}),$$

and

$$g \in gp(b_1^{\frac{1}{lm}}, \ldots, b_k^{\frac{1}{lm}}).$$

The latter can be written as

$$b_1^{\frac{s_1}{lm}} \cdot \cdots \cdot b_k^{\frac{s_k}{lm}} = g, \tag{17}$$

for some integers $s_1, \ldots, s_k \in \{0, \ldots, m-1\}$. We may assume $s_1 \neq 0$. Then for $\alpha \in \mathbb{A}_m$, a torsion point of order m, by (iv)

$$(b_1^{\frac{1}{lm}}, b_2^{\frac{1}{lm}}, \ldots, b_k^{\frac{1}{lm}}) \mapsto (\alpha b_1^{\frac{1}{lm}}, b_2^{\frac{1}{lm}}, \ldots, b_k^{\frac{1}{lm}})$$

generates a Galois automorphism $\hat{\alpha}$ over $k_a(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})$. By (17) $\hat{\alpha}(g) = \alpha^{s_1} g \neq g$, contradicting $g \in \mathbb{A}(k_a)$. Claim proved and the Proposition follows. □

The previous statements can be generalised to the fields of the form k_0^B, acl-B-generated extensions of k_0, introduced in section 4. Given finite subset $a = \{a_1, \ldots, a_r\} \subseteq F$ and k_0-algebraically independent B with an n-partition denote also

$$k_a^B = k_0^B(a_1^{\mathbb{Q}}, \ldots, a_r^{\mathbb{Q}}).$$

Theorem 3 *Assuming GCH and the categoricity of $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ in all uncountable cardinals, we have*

(i)

$$\mathbb{A}(k_a^B) \cong A_a^B \times a_1^{\mathbb{Q}} \cdot \cdots \cdot a_r^{\mathbb{Q}} \cdot \prod_{s \subset n} \mathbb{A}(F_s),$$

for some free abelian A_a^B;

(ii) if $b_1, \ldots, b_k \in A$ are multiplicatively independent over $\prod_{s \subset n} \mathbb{A}(F_s) \cdot a_1^{\mathbb{Q}} \cdot \ldots \cdot a_r^{\mathbb{Q}}$, there is an $l \in \mathbb{N}$ such that for any $m \in \mathbb{N}$

$$\mathrm{Gal}(k_a^B(b_1^{\frac{1}{ml}}, \ldots, b_k^{\frac{1}{ml}}) : k_a^B(b_1^{\frac{1}{l}}, \ldots, b_k^{\frac{1}{l}})) \cong (\mathbb{Z}/m\mathbb{Z})^{Nk}.$$

Proof Obviously (ii) in the statement of the theorem is very similar to (iv) of Theorem 2 with k_a replaced by k_a^B. On the other hand by Corollary 8 and Proposition 5 we have ω-stability and L^*-atomicity of models of the sentence over k_0^B, and this is the only fact we use to prove (iv) of Theorem 2. Thus (ii) follows by repeating the same argument. □

Remark This theorem has been proved without any assumptions for the one-dimensional torus $\mathbb{A}$ (the multiplicative group of the field) in [Z0] in a stronger form: we don't need B to be independent. Nevertheless the proof in [Z0] uses heavily the theory of linearly disjoint extensions of a field.

Now we want to show that the statements of Theorems 1 -3 imply excellency and ω-stability. Moreover,

Theorem 4 *Assume that the statements (ii) of Theorem 1, (iii) and (v) of Theorem 2 and (ii) of Theorem 3 hold. Then $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ is almost quasi-minimal excellent in the sense of [Z3] and is categorical in all uncountable cardinalities.*

Proof First we remark that (ii) of Theorem 1 implies, taking into account quantifier elimination and the description of types (Lemma 3.3), that any type over $\emptyset$ of a tuple of elements in the standard kernel is principal (in the language L^*). (iii) of Theorem 2 in combination with the previous statement implies that any type over $\emptyset$ is principal.

Our proof of categoricity is based on the definition of almost quasi-minimality and the categoricity theorem 5 in section 3 of [Z3].

Let $B \subseteq \mathbb{A}$ be an irreducible algebraic curve in $\mathbb{A}$ defined over a finite extension k of k_0 and thus defined over some choice of constants in A. Notice that by the previous remark the type of constants is principal, hence if we prove categoricity of the class of models with the new constants we get also the categoricity of the original class. (In fact, Fact 2 implies that one can choose B defined over k_0).

Let $U = \text{Ln}B = \{v \in V : \text{ex}(v) \in B\}$. By elimination of quantifiers B is strongly minimal and U is quasi-minimal. Moreover, since the algebraic closure acl (in model theoretic sense) of B contains $F = F(A)$ and the algebraic closure of F contains $\mathbb{A}(F)$, we have $A \subseteq \text{acl}(B)$. Obviously $V = \text{Ln}(A)$ and so $V \subseteq \text{cl}(U)$, where by definition

$$\text{cl}(X) = \text{Ln}(\text{acl}(\text{ex}X)).$$

It is now easy to see that cl satisfies the Assumption 1 of [Z3].

The observation above stating that any type is principal proves ω-homogeneity over $\emptyset$. ω-homogeneity over models follows immediately from (v) of Theorem 2, by the same argument. This proves Assumption 2 of [Z3].

Finally, comparing definitions one sees that (ii) of Theorem 3 with $r = 0$ states in terms of [Z0] that the type of a tuple $\langle v_1, \ldots, v_k \rangle$ in V such that $\text{ex}(v_i) = b_i$ over a special set $V(k_0^B)$ is principal, thus definable over a finite subset. This proves Assumption 3 and the theorem.$\square$

Corollary 9 *The conditions in Theorems 1 - 3 are equivalent to the statement that $T_A + \{\Lambda \cong \mathbb{Z}^N\}$ is categorical in all uncountable cardinals.*

References

[F] L.Fuchs, **Infinite Abelian Groups**, vol.1, Academic Press, New York and London, 1970

[K] H.J.Keisler, *On the quantifier 'there exist uncountably many x'*, Ann.Math.Logic v.1(1970), 1-91

[L] S.Lang, **Algebra**, Addison-Wesley Publishing Co., Reading, Mass., 1965

[LB] H.Lange, Ch.Birkenhake, **Complex Abelian Varieties**, Springer-Verlag, Berlin Heidelberg, 1992

[M] D.Mumford, **Abelian Varieties**, Oxford University Press, 1970

[Sh] S.Shelah, *Classification theory for non-elementary classes.I: the number of uncountable models of $\psi \in L_{\omega,\omega}(Q)$.*, Parts A,B, Isr.J.Math., v.46, (1983), 212-240, 241-273.

[Z0] B.Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero.* Submitted to JLMS

[Z1] ———, *Some Remarks on Model Theory of Algebraically Closed Fields* In: Proc. 7th Easter Conference in Model Theory, ed.B.I.Dahn et al., 204-214. Berlin: Humboldt-Univ.1989

[Z2] ———, *Algebraic Geometry via Model Theory* In: Contemp.Math. 131(1992) (part 3), 523-537

[Z3] ———, *A categoricity theorem for quasi-minimal excellent classes,* Submitted

Available on www.maths.ox.ac.uk/˜zilber