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* Paper is a preliminary version and should not be reviewed.
Finite homogeneous geometries
by B. Zil'ber

The notion of a pregeometry (matroid) was introduced at the beginning of the 1930s to study a general notion of dependence. Recently it was found out that the combinatorics of homogeneous pregeometries is closely connected with important problems in stability theory. From the other hand the techniques and ideology of stability theory allow one to get serious results on homogeneous geometries. The aim of the present paper is to give a proof of the following:

Main Theorem. A finite homogeneous geometry of (projective) dimension not less than 7 with more than 2 points on its lines is an affine or projective geometry (possibly truncated).

Strictly speaking we present here only the draft of the proof omitting details. However we hope the draft is quite comprehensible, in fact, the details omitted could be reconstructed using the proof of the infinite version of the theorem in [Z1], [Z2] and a close work [Z3].

The methods of the proof are based on simple ideas of stability theory and develop those of [Z1]-[Z3].

A pregeometry is a set $A$ together with a closure operator $\text{cl}: 2^A \rightarrow 2^A$ satisfying the following conditions for any $X, Y \subseteq A, x, y \in A$:

(i) $X \subseteq \text{cl}(X)$;
(ii) $X \subseteq \text{cl}(Y) \Rightarrow \text{cl}(X) \subseteq \text{cl}(Y)$;
(iii) $x \in \text{cl}(X \cup (Y)) \setminus \text{cl}(X) \Rightarrow y \in \text{cl}(X \cup (Y))$.

If $A$ is allowed to be infinite then usually the following condition is added:

(iv) $\text{cl}(X) = \bigcup \{\text{cl}(X') : X' \subseteq X, X' \text{ is finite}\}$.

Here we consider only finite $A$. 
An automorphism of a pregeometry is any bijection $\alpha: A \to A$ for which
\[ \text{cl}(\alpha(X)) = \alpha(\text{cl}(X)) \]
holds for any $X \subseteq A$. The group of all automorphisms fixing a set $X$ pointwise is denoted $\text{Aut}(A/X)$ and $\text{Aut}(A/\emptyset) = \text{Aut}(A)$.

A pregeometry is said to be **homogeneous** if $x, y \in A \setminus \text{cl}(X)$ implies the existence of an $\alpha \in \text{Aut}(A/X)$ such that $\alpha(x) = y$.

A pregeometry is called a **geometry** if $\text{cl}(\emptyset) = \emptyset$ and $\text{cl}(\{x\}) = \{x\}$ for any $x \in A$.

For any pregeometry $A$ one can construct the geometry $\hat{A}$ by putting
\[ \hat{X} = \{\text{cl}(\{x\}) : x \in X \setminus \text{cl}(\emptyset)\} \]
for any $X \subseteq A$ and defining the closure on $\hat{A}$ to be as follows: $\text{cl}(\hat{X}) = \text{cl}(X)^*$.

Another construction called localization gives a new pregeometry on the set $A$ given a subset $C \subseteq A$. Define the new closure $\text{cl}_C$ to be: $\text{cl}_C(X) = \text{cl}(X \cup C)$ for any $X \subseteq A$. The new pregeometry on $A$ is denoted $A_C$. $\dim X$ denotes the cardinality of a maximal independent (in the sense of cl) subset of $X$, called a base of $X$. The cardinality does not depend on the choice of the base.

$\dim_C X$ is the dimension of $X$ in $A_C$.

Note that $\dim X - 1$ is what is called the projective dimension of $X$.

1. **Sets over a pregeometry**

We shall call a subset $S \subseteq A^X$ **X-definable** for an $X \subseteq A$ if $S$ is invariant under all automorphisms from $\text{Aut}(A/X)$. This definition defines also $X$-definable relations on $S$ as subsets of $A^{nk}$.

An **X-definable set over** $A$ is a set of the form $S/E$, where $S$ is an
X-definable subset of $A^n$ and $E$ is an $X$-definable equivalence relation on $S$.

It is easy to see that $\text{Aut}(A/X)$ acts on any $X$-definable set $U = S/E$. Any $\text{Aut}(A/X)$-invariant subset of $U$ can be in a natural way presented as an $X$-definable set, so we call it $X$-definable too.

If $E$ is trivial then $S/E$ can be identified as $S$, so the $X$-definable subsets of $A^n$ are in this sense $X$-definable sets over $A$.

If $u \in U$ and $U$ is an $X$-definable set then denote by $O(u/X)$ the orbit of $u$ under the action of $\text{Aut}(A/X)$. This is an $X$-definable set (cf. $\text{tp}(u/X)$ in model theory).

We shall call an $X$-definable set $S/E$ ($S \subseteq A^n$) strictly coordinatizable over $X$ if for any $\langle s_1, \ldots, s_n \rangle, \langle s'_1, \ldots, s'_n \rangle \in S$, $\langle s_1, \ldots, s_n \rangle E \langle s'_1, \ldots, s'_n \rangle$ implies

$\text{cl}_X(s_1, \ldots, s_n) = \text{cl}_X(s'_1, \ldots, s'_n)$.

Throughout the paper all $X$-definable sets are considered to be strictly coordinatizable over $X$.

An example: The set $L$ of all lines in a geometry $A$ is a $0$-definable set over $A$. More precisely $L = S/E$, where $S = \{\langle x, y \rangle \in A^2 : x \neq y\}$.

$\langle x, y \rangle E \langle x', y' \rangle$ iff $\text{cl}(x, y) = \text{cl}(x', y')$.

If $U = S/E$ is an $X$-definable set, $u_1, \ldots, u_k \in U$, $u_i = \tilde{s}_i E \tilde{s}_i = \langle s_1, \ldots, s_n \rangle \in S \subseteq A^n$ then we put

$\langle u_1, \ldots, u_k, X \rangle = \text{cl}((s_1, \ldots, s_n, \ldots, s_{k-1}, \ldots, s_k) \cup X)$.

Note that for $a_1, \ldots, a_k \in A$

$\langle a_1, \ldots, a_k \rangle = \text{cl}(a_1, \ldots, a_k)$,

thus we can use the operator $\langle \rangle$ instead of $\text{cl}$.

For $u \in U$ we define
\[ \text{rank}(u/X) = \dim_X(u,X). \]

It follows from the definition that

1.1. \[ \text{rank}(\langle u_1,u_2\rangle/X) = \]
\[ = \text{rank}(u_1/(u_2,X)) + \text{rank}(u_2/X) \]
\[ = \text{rank}(u_2/(u_1,X)) + \text{rank}(u_1/X). \]

Define for sets

\[ \text{rank}(U/X) = \max (\text{rank}(u/X) : u \in U). \]

1.2. From the homogeneity it follows that \( \text{rank}(U/X) = \text{rank}(U/Y) \)
provided \( U \) is \( X \)-definable, \( X \subseteq Y \subseteq A \), \( \text{rank}(U/X) = r, r < \dim_X A, r < \dim_Y A. \)

For any \( Y \subseteq A \), define \( U[Y] = \{ u \in U : (u,X) \subseteq (Y) \}. \)

1.3. Polynomial Theorem. For any \( X \)-definable strictly coordinatizable set \( U \) over \( A \) there is a unique polynomial \( p_U(v) \) of one variable over the rationals such that

(i) for any closed \( Y \subseteq A \), if \( |Y| = n, Y \supseteq X \), then

\[ |U[Y]| = p_U(n). \]

(ii) \( \deg p_U = \text{rank}(U/X). \)

(iii) if \( U' \) is an \( X' \)-definable set over \( A \) such that for some \( \alpha \in \text{Aut}(A) \), \( X' = \alpha(X), U' = \alpha(U) \), then \( p_{U'} = p_U. \)

A proof of the theorem is in fact given in [Z1], Theorem 2.2.

1.4. Let \( U \) be an \( X \)-definable set, \( \text{rank}(U/X) = r \). Define for any \( n \) a binary relation \( E_n \) on \( U \):
If $n + 2r \leq \text{codim } X$, $(X) \neq \emptyset$ and planes in $A$ are not projective, then $E_n$ is an equivalence relation on $U$.

Proof. The only problem is transitivity. Let $u_1 E_n u_2$ and $u_2 E_n u_3$.

By homogeneity to prove $u_1 E_n u_3$ it is sufficient to find $y_1,...,y_n$ independent over $(u_1,u_2,X)$ as well as over $(u_2,u_3,X)$ and over $(u_1,u_2,X)$. If $y_1,...,y_i (i < n)$ have been found already then

$$y_{i+1} \in A \setminus (u_1,u_2,y_1,...,y_i,X) \cup (u_2,u_3,y_1,...,y_i,X) \cup (u_1,u_3,y_1,...,y_i,X).$$

The sum of the three subspaces is less than $A$ since the number of points on a line in $A(x,y_1,...,y_i)$ is greater than 3. □

1.5. Suppose $n + 2r \leq \text{codim } X$, $E_n$ is an equivalence relation on $U$, $n \geq r = \text{rank}(U/X)$.

Under these conditions any class $U_0$ of the equivalence $E_n$ is $(Z)$-definable, provided $X \subseteq Z \subseteq A$, $\dim_XZ \geq r$.

Proof. It suffices to find $u_0 \in U$ such that $(u_0,X) \subseteq (Z)$. Let $u_1 \in U_0$, $\text{rank}(U_0/(X,u_1)) = r_0$. By 1.2 we can find $u_2 \in U$ with $\text{rank}(u_2/(Z)) = r_0$ and $u_3 \in U$ with $\text{rank}(u_3/(Z,u_2)) = r_0$. Since

$$\dim(X,u_2)(X,u_3) = r_0 \leq \dim(X,u_2)Z,$$

there is $\alpha \in \text{Aut}(A/(X,u_2))$ such that $\alpha((u_3)) \subseteq (Z)$, $U_0$ is invariant under $\alpha$. Put $u_0 = \alpha(u_2)$. □

Let $U_0$ be an $X'$-definable set $X \subseteq X' \subseteq A$, $\text{rank}(U_0/X') = r$. $U_0$ is called almost $X$-definable if for any $Z \supseteq X$ with $\dim_XZ \geq r$, $U_0$ is $(Z)$-definable.
\[ \text{rank}(u/X) = \dim_X(u/X). \]

It follows from the definition that

1.1. \[ \text{rank}(u_1, u_2 / X) = \]
\[ = \text{rank}(u_1 / (u_2, X)) + \text{rank}(u_2 / X) \]
\[ = \text{rank}(u_2 / (u_1, X)) + \text{rank}(u_1 / X). \]

Define for sets

\[ \text{rank}(U/X) = \max \{ \text{rank}(u/X) : u \in U \}. \]

1.2. From the homogeneity it follows that \[ \text{rank}(U/X) = \text{rank}(U/Y) \]
provided \( U \) is \( X \)-definable, \( X \subseteq Y \subseteq A \), \[ \text{rank}(U/X) = r, \ r < \dim_X A, \ r < \dim_Y A. \]

For any \( Y \subseteq A \), define \( U[Y] = \{ u \in U : (u, X) \subseteq (Y) \} \).

1.3. Polynomial Theorem. For any \( X \)-definable strictly coordinatizable set \( U \) over \( A \) there is a unique polynomial \( p_U(v) \) of one variable over the rationals such that

(i) for any closed \( Y \subseteq A \), if \( |Y| = n \), \( Y \supseteq X \), then
\[ |U[Y]| = p_U(n). \]

(ii) \( \deg p_U = \text{rank}(U/X) \),

(iii) if \( U' \) is an \( X' \)-definable set over \( A \) such that for some \( \alpha \in \text{Aut}(A) \), \( X' = \alpha(X) \), \( U' = \alpha(U) \), then \( p_{U'} = p_U \).

A proof of the theorem is in fact given in [Z1], Theorem 2.2.

1.4. Let \( U \) be an \( X \)-definable set, \( \text{rank}(U/X) = r \). Define for any \( n \) a binary relation \( E_n \) on \( U \):
$u_1 E_n u_2 \iff$ there are $y_1, ..., y_n \in A$ independent over $(u_1, u_2, x)$ and
\[ \alpha \in \text{Aut}(A/(y_1, ..., y_n, x)) \text{ such that } \alpha(u_1) = u_2. \]

If $n + 2r \leq \text{codim } X, (x) \neq \emptyset$ and planes in $A$ are not projective, then $E_n$ is an equivalence relation on $U$.

Proof. The only problem is transitivity. Let $u_1 E_n u_2$ and $u_2 E_n u_3$.

By homogeneity to prove $u_1 E_n u_3$ it is sufficient to find $y_1, ..., y_n$ independent over $(u_1, u_2, x)$ as well as over $(u_2, u_3, x)$ and over $(u_1, u_2, x)$. If $y_1, ..., y_i (i < n)$ have been found already then

\[ y_{i+1} \in A \setminus (u_1, u_2, y_1, ..., y_i, x) \cup (u_2, u_3, y_1, ..., y_i, x) \cup (u_1, u_3, y_1, ..., y_i, x). \]

The sum of the three subspaces is less than $A$ since the number of points on a line in $A(x, y_1, ..., y_i)$ is greater than 3. \qed

1.5. Suppose $n + 2r \leq \text{codim } X, E_n$ is an equivalence relation on $U$.

$n \geq r = \text{rank}(U/X)$.

Under these conditions any class $U_0$ of the equivalence $E_n$ is

$(Z)$-definable, provided $X \subseteq Z \subseteq A$, $\dim_{X} Z \geq r$.

Proof. It suffices to find $u_0 \in U$ such that $(u_0, x) \subseteq (Z)$. Let $u_1 \in U_0, \text{rank}(U_0/(X, u_1)) = r_0$. By 1.2 we can find $u_2 \in U$ with $\text{rank}(u_2/(Z)) = r_0$ and $u_3 \in U$ with $\text{rank}(u_3/(Z, u_2)) = r_0$. Since

\[ \dim(X, u_2)(X, u_3) = r_0 \leq \dim(X, u_2) Z, \]

there is $\alpha \in \text{Aut}(A/(X, u_2))$ such that $\alpha((u_3)) \subseteq (Z), U_0$ is invariant under $\alpha$. Put $u_0 = \alpha(u_2)$. \qed

Let $U_0$ be an $X$-definable set $X \subseteq X' \subseteq A, \text{rank}(U_0/X') = r$. $U_0$ is called

\textit{almost $X$-definable} if for any $Z \supseteq X$ with $\dim_{X} Z \geq r$, $U_0$ is $(Z)$-definable.
1.6. Under the conditions of 1.5, $U_0$ satisfies the following: for any $Z$ with $\dim_X Z \leq n$ and any $(Z)$-definable set $V$,

$$\text{rank}(U_0 \cap V/(Z)) < r_0 \text{ or } \text{rank}(U_0 \setminus V/(Z)) < r_0.$$ 

This follows from the definition of $F_n$. □

$U_0$ as in 1.6 will be called $n$-irreducible.

2. Parallelism

In what follows in this section $A$ is a finite homogeneous geometry, $L$ the set of all lines in $A$.

Two lines $f_1, f_2$ are called weakly parallel if $f_1 = f_2$ or $\dim(f_1, f_2) = 3$ and $(f_1) \cap (f_2) = \emptyset$. The fact is denoted $f_1 \parallel f_2$.

We say three lines $f_1, f_2, f_3$ satisfy the relation of triple parallelism if

$$f_1 \parallel f_3 \land f_2 \parallel f_3 \land f_1 \neq f_2 \land (f_3) \cap (f_1, f_2) \neq \emptyset.$$ 

This fact is denoted $f_1 \parallel f_2 \parallel f_3$.

2.1. Suppose $f_1 \parallel f_2 \parallel f_3$ holds. Then:

(i) $\dim(f_1, f_2, f_3) = 4$;

(ii) $(f_1, f_2) \cap (f_3) = \emptyset$;

(iii) for any $a \in A \setminus (f_1, f_2)$ there is a unique $t \in L$ such that $a \in (t)$ and $f_1 \parallel f_2 \parallel t$;

(iv) $f_1 \parallel f_2$;

(v) $(f_1, f_2, f_3)$ for any permutation $(i_1, i_2, i_3)$.

The proof is an exercise in elementary properties of homogeneous geometries.
Fix a pair of distinct points \( a, b \in A \) and put
\[
R_{ab} = \{ \langle t_1, t_2 \rangle \in L^2 : a \in (t_1) \land b \in (t_2) \land (\exists \ell \in L) \, t_1 \uparrow t_2 \uparrow t\}.
\]

For \( \tau = \langle t_1, t_2 \rangle \in R_{ab} \) denote
\[
\overline{\tau} = \{ \ell \in L : t_1 \uparrow t_2 \uparrow \ell \}.
\]

2.2. If \( \tau_1, \tau_2 \in R_{ab}, \tau_1 \neq \tau_2 \), then \( \overline{\tau_1 \cap \tau_2} \) contains at most one line.

Proof. Let \( \tau_1 = \langle t_{11}, t_{12} \rangle, \tau_2 = \langle t_{21}, t_{22} \rangle, m_1, m_2 \in \overline{\tau_1 \cap \tau_2}, m_1 \neq m_2. \)

For some \( i, j \in \{1, 2\}, (m_i) \notin \langle t_{ij}, t_{2j} \rangle \). Otherwise \((m_1, m_2) \subseteq (t_{11}, t_{12}) \cap (t_{21}, t_{22}), \) this implies \( (t_{11}, t_{12}) = (t_{21}, t_{22}), \) since \( \dim(m_1, m_2) \geq 3. \)

Moreover \((m_1, m_2) = (t_{11}, t_{12}) = (t_{21}, t_{22}). \) This contradicts with \( t_{11} \neq t_{21}. \)

So, let \((m_1) \notin \langle t_{11}, t_{21} \rangle. \) Together with \( m_1 \in \overline{\tau_1 \cap \tau_2} \) it implies
\[t_{11} \uparrow t_{21} \uparrow m_1, \] provided \( t_{11} \neq t_{21}. \) By 2.1(iv) it contradicts \( a \in (t_{11}) \cap (t_{21}). \) Thus
\[t_{11} = t_{21}. \] Now we have \( t_{11} \uparrow t_{12} \uparrow m_1 \) and \( t_{11} \uparrow t_{22} \uparrow m_1 \) and \( b \in (t_{12}) \cap (t_{22}). \) By 2.1(v) and (iii) we get \( t_{12} = t_{22}, \) thus \( \tau_1 = \tau_2. \) \( \Box \)

2.3. It is easy to see that \( R_{ab} \) is an \( (a, b) \)-definable set with
\[\text{rank}(R_{ab}/(a, b)) = 1. \]
Let \( R^1_{ab}, ..., R^m_{ab} \) be all the \( E_1 \)-classes. \( R^i_{ab} \) are almost \( (a, b) \)-definable and 1-irreducible by 1.6, provided \( \dim A \geq 6, R_{ab} \neq \emptyset. \)

If \( \tau_1 \in R^i_{ab}, \tau_2 \in R^j_{ab}, i, j \in \{1, ..., m\}, \tau_1 \neq \tau_2, \overline{\tau_1 \cap \tau_2} \neq \emptyset \) then for any distinct \( \tau_1 \in R^i_{ab}, \tau_2 \in R^j_{a}, \) it holds that \( \overline{\tau_1 \cap \tau_2} \neq \emptyset \) and \( (\tau_1) \neq (\tau_2). \)

Proof. One can assume \( \tau_1 = \tau'_1. \) Note that \( (\tau_1) \neq (\tau_2), \) since there is \( t \in \overline{\tau_1 \cap \tau_2} \) and by 2.2, \( (t) \subseteq (\tau_1, \tau_2), \) but by the definition of \( \overline{\tau_1 \cap \tau_2} \) \( (t) \notin (\tau_1). \) We show that we may assume \( (\tau_1) \notin (\tau_1, \tau'_2) \) and this will finish the proof by using the definition of \( E_1. \)
So, suppose \((\tau_1) \subseteq (\tau_2, \tau'_2)\), then either \((\tau_2) = (\tau'_2) = (\tau_1)\) or \(\dim((\tau_1) (\tau_2, \tau'_2)) = 1\). The first one is impossible. If the second holds there is \(\alpha \in \text{Aut}(\Lambda/(\tau_1))\) such that \((\alpha(\tau_2)) \not\subseteq (\tau_2, \tau'_2)\). Denote \(\alpha(\tau'_2) = \tau''_2\), then \((\tau_1) \not\subseteq (\tau_2, \tau''_2)\). Take \(\tau''_2\) instead of \(\tau'_2\). □

2.4. Denote

\[
S_{ij} = \{ (\tau_1, \tau_2, \tau) : \tau_1 \in \mathbb{R}^i, \tau_2 \in \mathbb{R}^i, \tau \in \overline{\tau_1 \tau_2}, \tau_1 \not\subseteq \tau_2\},
\]

fix \(t_0 \in L\), such that \((t_0) \cap (a,b) = \emptyset\), and a plane of the form \((a,b,c), c \in A \setminus (a,b)\).

Denote

\[
\lambda = |t_0|, \quad \rho^i = |(\tau \in \mathbb{R}^i_{ab} : t_0 \in \tau)|, \quad \pi = |(a,b,c)|,
\]

\[
\mu^i = |(\tau \in \mathbb{R}^i : (\tau) = (a,b,c))|,
\]

\[
\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j.
\end{cases}
\]

If we put \(z = |Z|\) for any closed set \(Z \subseteq A\) containing \(c, a, b\), then the following hold:

(i) \(|L[Z]| = \frac{z(z-1)}{\lambda(\lambda-1)}\);

(ii) \(|R^i[Z]| = \frac{z-\pi}{\lambda-1} \rho^i + \mu^i\);

if \(S_{ij} \neq \emptyset\)

(iii) \(|S_{ij}[Z]| = |R^i[Z]| \cdot (|R^j[Z]| - \mu^j)\);

and also

(iv) \(|S_{ij}[Z]| = \frac{(z-\lambda)(z-\pi)}{\lambda(\lambda-1)} \cdot \rho^i \cdot (\rho^j - \delta_{ij})\).
Proof. (i) is well-known and easy. (ii) follows from computations of the number of elements in

\[ T^i[Z] = \{ \langle t, t \rangle : c \in (t), t \in L[Z], t \in \bar{t}, \bar{t} \in R^i[Z] \} . \]

For \( (\tau) = (a,b,c) \) there is no \( t \in \bar{t} \) with \( c \in t \) by 2.1(ii). If \( (\tau) \neq (a,b,c) \) then there is a unique \( t \) such that \( \langle z, t \rangle \in T^i \). It follows that

\[ |T^i[Z]| = |R^i[Z]| - \mu^i . \]

From the other hand for any \( t \in L \), provided \( c \in (t) \) and \( (t) \notin (a,b,c) \) there are exactly \( \rho^i \) elements \( \tau \in R^i_{ab} \) such that \( \langle \tau, t \rangle \in T^i \). Using 2.1(iii) one gets

\[ |T^i[Z]| = \frac{z - \pi}{\lambda - 1} \cdot \rho^i \]

where \( (z - \pi)/(\lambda - 1) \) is counted as the number of \( t \in L[Z] \) such that \( c \in (t) \) \( \notin (a,b,c) \).

(iii) follows from 2.3 and 2.2 if one counts \( |S^{ij}[Z]| \) as the number of \( \langle \tau_1, \tau_2 \rangle \in R^i[Z] \times R^j[Z] \) such that \( \bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset \).

(iv) is the result of counting first the number of the lines in

\[ \{ t \in L[Z] : (\exists \tau_1 \in R^i_{ab})(\exists \tau_2 \in R^j_{ab})(\tau_1, \tau_2, t) \in S^{ij} \} = \{ t \in L[Z] : \dim(a,b,t) = 4 \} . \]

This number is equal to \( (z - \lambda)(z - \pi)/\lambda(\lambda - 1) \). Now for each \( t \) from the set there are exactly \( \rho^i(\rho^j - \delta^{ij}) \) pairs of different \( \tau_1, \tau_2 \) such that \( \langle \tau_1, \tau_2, t \rangle \in S^{ij}[Z] \). \( \square \)

2.5. If \( \dim A \geq 6 \), then for any \( \tau_1, \tau_2 \in R_{ab} \)

\[ \bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset \hspace{1em} \text{iff} \hspace{1em} \tau_1 = \tau_2 . \]

Proof. It suffices to show that \( S^{ij} = \emptyset \) for all \( i, j \in \{1,...,m\} \). For this use 2.4 and compare the leading coefficients of the polynomials given by (iii) and (iv)
if $S^i \neq \emptyset$. The coefficients are distinct though the polynomials must coincide by 1.3. □

2.6. If $\dim A \geq 6$ then one of the following hold:
(i) every plane in $A$ is projective;
(ii) every plane in $A$ is affine;
(iii) there are two distinct lines $t_1, t_2$ such that $t_1 \| t_2 \& \neg(\exists t) t_1 \| t_2 \| t$.

Proof. Suppose (i) and (iii) do not hold. Then there are $t_1, t_2 \in L$, $t_1 \neq t_2$, and there is $t \in L$ such that $t_1 \parallel t_2 \parallel t$. Let $a \in (t), a \notin (t_1, t_2), t' \in L, a \in (t')$ and $t_1 \parallel t'$. Then $t' \parallel t_2 \parallel t_1$ and by 2.1, $t' = t$. Thus we have proved that through any $a \in (t_1, t_2)$ there is a unique $t$ such that $t \parallel t_1$. By homogeneity we get the same for any $t_1$ and any $a \in (t_1)$. This is exactly (ii). □

A geometry $(A, cl)$ is called truncated projective (affine) if one can define a new closure $cl^*$ on $A$ such that $(A, cl^*)$ is isomorphic to a projective (affine) geometry over a field and there is $d \leq \dim A$ (dimension of $A$ with respect to $cl^*$) such that $cl(X) = cl^*(X)$ if $\dim^* X \leq d$ and $cl(X) = A$ if $\dim^* X > d$.

2.7. If all planes in $A$ are projective (affine), then $A$ is a truncated projective (affine) geometry.

This is a consequence of the transitivity of Aut($A$) on the set of all non-collinear triples of points from $A$ and Theorem 1 of [CK]. □

2.8. If $\dim A \geq 6$ then one of the following hold:
(i) $A$ is a truncated projective geometry;
(ii) $A$ is a truncated affine geometry;
(iii) the binary relation $I$ on the set of lines is not empty:

$t_1 \parallel t_2 \iff t_1 \parallel t_2 \& \neg(\exists t) t_1 \parallel t_2 \parallel t$.

This is a reformulation of 2.6 taking into account 2.7. □
3. **Quasi-design over A.**

In this section we suppose dim A is finite, homogeneous and the relation I defined in 2.8 is not empty. We denote for \( t \in L \)

\[ It = \{ t' \in L : tI' \}. \]

The results of the section and their proofs are completely analogous to those of [Z1, section 3]. We only improved the proofs and modified them to the finite-dimensional case.

3.3. (i) rank(I(U)/(t)) = 1 for all \( t \in L \);
(ii) if \( t_1 \neq t_2 \) for \( t_1, t_2 \in L \), then \( \text{rank}(It_1It_2/(t_1,t_2)) = 0 \) or \( It_1It_2 = \emptyset \).

The proof is immediate from the definitions. \( \square \)

Studying \( L \) with respect to \( I \) it is convenient to treat elements of \( L \) as points and subsets of the form \( It \) as blocks. As in [Z1] we will call this incidence system a quasi-design.

To the end of the section we fix \( X \subseteq A \) such that codim \( X \geq 3 \) and the partition of \( L \)

\[ L = L_1 \cup ... \cup L_n \]

where \( L_1 \) are orbits with respect to \( \text{Aut}(A/X) \). By homogeneity among \( L_1, ..., L_n \)
there is exactly one set of rank 2. Let

3.2. \( \text{rank}(L_1/X) = 2 \); \( \text{rank}(L_i/X) \leq 1 \) for \( i > 1 \).

3.3. If \( \text{rank}(L_i/X) = 1, t \in L \), \( \text{rank}(L_iIt/(X,t)) = 1 \) then \( t \in L[X] \).

Proof. Under the hypotheses there is \( t' \in L_1 \cap It \) such that \( (t') \notin (t,X) \).

Since \( \text{rank}(t'/X) = 1 \) and \( \text{rank}(t/(t')) = 1 \), one has
2 \geq \operatorname{dim}_X(t',t') > \operatorname{dim}_X(t).

Since \operatorname{codim}(t',X,t') \geq 1, hence supposing \( t \cap X \) we can find \( t'' \in L \) such that 
\( (t'') \nsubseteq (t',X) \) and there is \( \alpha \in \operatorname{Aut}(A/(t',X)) \) such that \( \alpha(t') = t'' \). Then \( t'' \nsubseteq (t') \), 
\( (t') \nsubseteq (t'',t'') \), thus it holds that \( t'' \nsubseteq t' \), which contradicts \( \forall \ell \). \( \square \)

3.4. If \( \operatorname{rank}(L_i/X) = 1 \) and \( \operatorname{rank}(L_i \cap i/(X,t)) < 1 \) for all \( t \in L \) then for any \( q \in L_i \setminus L_i[X] \) there is \( t_1 \in L_i \) such that \( \operatorname{rank}(t_1/(X,q)) = 1 = \operatorname{rank}(t_1/(q)). \)

Proof. Fix \( L_i \). Denote for an \( t \in L \)

\[ S_t = \{ \langle t_1,t_2 \rangle \in I : t_2 \nsubseteq t_1 \in L_i \text{ and } t_1 \neq t \}. \]

It is easy to compute \( \operatorname{rank}(S_t/(X,t)) = 1. \)

Take an arbitrary closed \( Y \subseteq A \) such that \( (t,X) \subseteq Y \). By 1.3, \( |S_t[Y]| \) is the value of a polynomial of degree 1 depending on \( |Y| \). Denote \( 0_{t,j} \), \( j = 1, \ldots, m \), all orbits on \( L \) under \( \operatorname{Aut}(A/(t)) \), except \( \langle t \rangle \). Denote

\[ L^t_{ij} = L_i \cap 0^t_j, \]

and let \( L_i \setminus \langle t \rangle = L^t_{i1} \cup \ldots \cup L^t_{im} \). Then

\begin{equation}
|S_t[Y]| = \sum_{1 \leq j \leq m} |L^t_{ij}(Y)| \cdot \nu^t_j,
\end{equation}

where \( \nu^t_j \) is \( |R(t_1)| \) when \( t_1 \in L^t_{ij}. \)

From the other hand

\begin{equation}
|S_t[Y]| = \sum_{1 \leq k \leq n} |R(t_2 \cap L_i)(Y)| \cdot \chi^t_k,
\end{equation}

where \( \chi^t_k = |R(t_2 \cap L_i)(t)| \) when \( t_2 \in L_k. \)

Count now the ranks of all the subsets involved and the degrees of all the polynomials and consider the leading coefficients of the polynomials (lcp).
Then from (2) we have

\[ \text{lcp } |S_p[Y]| = \text{lcp } |ItnL_1[Y]| \cdot \lambda^t_1. \]

Now we assume \( t \notin (X) \). Then by 3.2 and 3.3

\[ \text{lcp } |It \cap L_1[Y]| = \text{lcp } |It[Y]| \]

and thus

\[ \text{lcp } |S_t[Y]| = \text{lcp } |It[Y]| \cdot \lambda^t_1. \]

Now we consider two possibilities for \( t: t = q \in L_1[Y] \) and \( t = p \in L_1[Y] \).

It is easy to see that \( \lambda q_1 = \lambda p_1 - 1 \), therefore

\[ \text{lcp } |S_q[Y]| < \text{lcp } |S_p[Y]|. \]

Looking to (1) we get

\[ \text{lcp } |S_p[Y]| = \text{lcp } |L_{i1}[Y]| \cdot v^p_1 \]

since any two \( t_1, t'_1 \in L_i \setminus L_i[p] \) are conjugated by \( \text{Aut}(A/(p)) \). And also \( v^p_1 = |It_{1nL_1p}| \) when \( \text{rank}(t_1/(p)) = 2 \). (5), (6) and (1) imply that \( v^p_1 > v^q_j \) for any \( j \) such that \( \text{rank}(L_{ij}/(X,q)) = 1 \). It implies that \( \langle t_1,p \rangle \) and \( \langle t_1,q \rangle \) are not conjugated when \( t_1 \in L_{ij} \), i.e. \( \text{rank}(t_1/q) \neq 2 \). \( \square \)

3.5. If \( z \in (y,X) \) and \( \text{codim } X \geq 3 \), then there are \( x_1, x_2 \in (X) \) such that \( z \in (y,x_1,x_2) \).

Proof. If (i) or (ii) of 2.8 holds then it is evident. Otherwise we use 3.3 and 3.4.

Let \( y \neq z, z \in (X) \), let \( q \) be the line through \( y \) and \( z \), \( L_i \) the orbit of \( q \) under \( \text{Aut}(A/X) \). Then \( \text{rank}(L_i/X) = 1 \).
If there is \( t \in L \) such that \( \text{rank}(L \cap \mathbb{R}) = 1 \) then \( (t) \subseteq (X) \) by 3.3 and let \((x_1,x_2) = (t)\).

If not then use 3.4. There are two possibilities for \( t_1 \) from 3.4:

- \( (t_1)^{(q)} \neq 0 \) or there is \( t \in L \) such that \( q \parallel t \parallel t \). In the first case \((x_1) = (x_2) = (t_1)^{(q)} \subseteq (X) \). In the second case \((t_1) \subseteq (X) \) or it is possible to find \( t \) such that \((t) \subseteq (X) \) and \( q \parallel t \parallel t \). Then \( \text{rank}(q/(x_1,x_2)) = 1 \) for \((x_1,x_2) = (t_1) \) or \((x_1,x_2) = t \) respectively. \( \square \)

### 4. Definable transformations

Under the assumption \( \dim A \geq 7 \) and \( A \) is neither a projective nor an affine geometry, we construct here a definable set \( V \) over \( A \) so that there are "sufficiently many" definable transformations on \( V \).

We begin with a broader notion. An almost \( X \)-definable semitransformation on \( A \) is an almost \( X \)-definable set \( f \subseteq A \times A \) of rank 1 which is 2-irreducible and does not contain \( \langle v,u \rangle \) with \( v \in (X) \) or \( u \in (X) \).

4.1. If \( \text{codim} \ X \leq 5 \), \( \langle u,v \rangle \in A^2 \), \( \text{rank} \langle u,v \rangle / X) = 1 \), \( v,u \in (X) \), then there is an almost \( X \)-definable semitransformation \( f \) on \( A \) with \( \langle u,v \rangle \in f \).

This follows from 1.6.

4.2. If \( f_1 \) is an almost \( X_1 \)-definable semitransformation on \( A \) for \( i = 1,2 \) and \( \dim X_1 u X_2 < \dim X_1 + 2 \), \( \text{rank} \langle f_1 n_{f_2} / X_1 u X_2 \rangle > 0 \) then \( \text{rank} \langle f_1 \setminus f_2 / X_1 u X_2 \rangle = 0 \).

This is a consequence of 2-irreducibility. \( \square \)

4.3. If \( \dim A \geq 7 \), \( \text{codim} \ X \geq 3 \), \( \langle u,v \rangle \) as in 4.1, then there are \( x_1, x_2 \in (X) \) and an almost \( (x_1,x_2) \)-definable semitransformation \( f \) on \( A \) with \( \langle u,v \rangle \in f \).

This follows from 3.5 and 4.1. \( \square \)
Denote $F$ the set of all almost $(x_1, x_2)$-definable semitransformations on $A$ for all $x_1, x_2 \in A$. It is easy to see that if $f \in F$, then $f^{-1} \in F$, where $f^{-1} = \{ \langle v, u \rangle : \langle u, v \rangle \in f \}$.

For almost $X$-definable sets $g_1, g_2$ of rank 1 we denote by $g_1 \sqsubseteq g_2$ the fact that $\text{rank}(g_2 \setminus g_1 / X) = 0$, and $g_1 \sqcap g_2$ denotes $g_1 \sqsubseteq g_2 \sqcap g_2 \sqsubseteq g_1$.

4.4. It follows from 4.2 that $\sqsubseteq$ coincides with $\sqcap$ on $F$ and $\sqcap$ is an equivalence relation on $F$. It follows from 4.3 that for any $f_1, f_2 \in F$ there are $g_1, \ldots, g_k \in F$ such that $g_1 \ldots g_k \sqsubseteq f_1 \circ f_2$, where

$$f_1 \circ f_2 = \{ \langle u, w \rangle : (\exists v) \langle u, v \rangle \in f_1 \land \langle v, w \rangle \in f_2 \}.$$ 

If $f_i$ is almost $(x_{i1}, x_{i2})$-definable for $i = 1, 2$ then $g_i$ are almost $(y_{j1}, y_{j2})$-definable for some $y_{j1}, y_{j2} \in (x_{11}, x_{12}, x_{21}, x_{22})$. The set $(g_1, \ldots, g_k)$ is uniquely determined up to $\sqcap$.

Define $F_1$ to be the subset of $F$ containing all almost $(x_1, x_2)$-definable semitransformations $f$ such that: if $\langle u, v \rangle \in f$, $(u, v) = (q)$, $q \in \mathcal{L}$, $u \notin (x_1, x_2)$,

$$(x_1, x_2) = (t), t \in \mathcal{L},$$

then

$$\max(|s|, \text{rank}(\mathcal{g})) \text{ exists} ;$$

let $\langle v, u \rangle$.

Since $f$ iff $\langle v, u \rangle$.

The observation above makes it possible to treat the quotient set $F_1/\sqsubseteq = \hat{F}_1$ as $\emptyset$-definable. An element of $\hat{F}_1$ corresponding to $f \in F_1$ will be denoted $\hat{f}$. 


(f) = (x_1, x_2), \text{ rank}(f/\mathcal{X}) = \dim_X(x_1, x_2) \text{ if } f \text{ is almost } (x_1, x_2)\text{-definable.}

4.7. (i) If \( f \in F_1 \), then \( f^{-1} \in F_1 \);

(ii) if \( f_1 \in F, f_2 \in F_1 \), \( \text{rank}(f_2/(f_1)) = 2 \), \( f \subset f_1 \circ f_2 \), then \( f \in F_1 \).

(i) is evident. (ii) is again a consequence of 2.1 and elementary geometric considerations. \( \Box \)

It is natural to use the following notation for \( v \in A, f \in F \):

\[ f(v) = \{ u : \langle v, u \rangle \in f \}. \]

4.8. If \( g, f \in F_1 \), \( \text{rank}(g/(f)) = 2 \), \( v \in A \setminus (f, g) \), \( u_1, u_2 \in f(v) \) and \( u_1 \neq u_2 \), then \( g(u_1) \cap g(u_2) = \emptyset \).

Proof. Assume the contrary, \( w \in g(u_1) \cap g(u_2) \). Then \( u_1, u_2 \in (f, v) \cap (g, w) \), hence \( u_2 \in (f, u_1) \cap (g, u_1) \). It follows that \( \text{rank}(g/(u_1, u_2)) \leq 1 \), which contradicts

\[ \text{rank}(g/(u_1, u_2)) \geq \text{rank}(g/(f, u_1, u_2)) = \text{rank}(g/(f, v)) = 2. \] \( \square \)

4.9. Let \( f, h \in F_1 \), \( \text{rank}(f/(h)) = 2 \) and \( k = |f(v)| = \max(|s(v)| : s \in F_1, v \in A \setminus (f, g)) \). Taking \( g \subset f^{-1} \circ h \) we get \( h \subset f \circ g \), \( g \in F_1 \) (by 4.7) and \( \text{rank}(g/(f)) = 2 \). Under these assumptions for any \( v \in A \setminus (f, g) \), \( u \in f(v) \) there exists a unique \( w \in g(u) \cap h(v) \).

Proof. Let \( f(v) = (u_1, \ldots, u_k) \), \( u_i \neq u_j \) if \( i \neq j \). Denote \( m_i = |g(u_i) \cap f(v)| \), let \( \langle v, u \rangle \in f \), \( \langle w, u \rangle \in g \), \( \langle v, w \rangle \in h \),

\[ f' = \{ \langle v', u' \rangle : \exists w')(w' \in g(u') \cap h(v') \}. \]

Since \( f' \subseteq f \) and \( \langle v, u \rangle \in f' \), \( \text{rank}(f'/(f, h)) = 1 \), hence \( f' \circ f \). It follows that \( \langle v, u_i \rangle \in f \) iff \( \langle v, u_i \rangle \in f' \), therefore \( g(u_i) \cap h(v) \neq \emptyset \) and \( m_i > 0 \) for \( i = 1, \ldots, k \).

From the other hand \( \sum_{i \leq k} m_i < k \), since \( \bigcup_{i \leq k} g(u_i) \cap f(v) = h(v) \). Thus
4.10. Fix \( f \in F_1 \) as a set. For any \( \langle v, u \rangle \in f, \langle t, w \rangle \in f \) such that 
\[ \text{rank}(\langle v, w \rangle/f) = 2 \]
there exist \( x_1, x_2 \in A \) and an \( (f, x_1, x_2) \)-definable mapping 
\[ \gamma : f \to f \]
such that 
\[ \text{rank}(\langle v, u \rangle/(f, x_1, x_2)) = 1 \] and 
\[ \gamma(\langle v, u \rangle) = \langle t, w \rangle. \]

**Proof.** For given \( \langle v, u \rangle \in f \) take \( g, h \in F_1 \) as in 4.9 so that 
\( w \in g(u) \cap h(v) \). Such a choice is possible by homogeneity. Note that 4.9 gives an 
\( (f, h) \)-definable mapping \( \alpha : f \to h \) by the law \( \alpha : \langle v, u \rangle \to \langle v, w \rangle \). Let \( i \) be the 
inversion \( i : \langle v, w \rangle \to -\langle v, w \rangle \). Let \( \beta \) be again an \( (f, h) \)-definable mapping \( h^{-1} \to f^{-1} \) 
such that \( \langle w, v \rangle \to \langle w, t \rangle \). Then \( \gamma = \alpha \circ \beta \circ i \) is the required mapping.

\[ (x_1, x_2) = (h) \cdot 0. \]

4.11. \( \gamma \) in 4.10 is a bijection of \( f \setminus (f, x_1, x_2)^2 \) onto itself.

This is easily seen from the construction. \( \square \)

An \( (f, x_1, x_2) \)-definable bijection of \( f \setminus (f, x_1, x_2)^2 \) onto itself will be 
called a transformation of \( f \). One constructed as in 4.10 will be called generic.

4.12. For any \( v_1, v_2, t_1, t_2 \in A \) such that \( \text{rank}(\langle v_1, v_2, t_1, t_2 \rangle/f) = 4 \), any 
\( u_1, u_2 \in A \) such that \( \langle v_1, u_1 \rangle \in f, \langle v_2, u_2 \rangle \in f \), there exists a transformation \( \gamma \) and 
\( w_1, w_2 \in A \) with \( \langle t_1, w_1 \rangle \in f, \langle t_2, w_2 \rangle \in f \), \( \gamma(\langle v_1, u_1 \rangle) = \langle t_1, w_1 \rangle, \gamma(\langle v_2, u_2 \rangle) = \langle t_2, w_2 \rangle \).

**Proof.** Let \( \gamma, h \) be as in the proof of 4.10, \( \text{rank}(\langle v_1, v_2 \rangle/(f, h)) = 2 \). Let 
\[ \gamma(\langle v_1, u_1 \rangle) = \langle t_1, w_1 \rangle, \gamma(\langle v_2, u_2 \rangle) = \langle t_2, w_2 \rangle . \]
It is easy to see that \( (v_1, v_2, t_1, t_2, f) = (v_1, v_2, w_1, w_2, f) = (h_1, h_2, f) \) \( (f) \) and therefore \( v_1, v_2, t_1, t_2 \) are independent over \( (f) \). Take \( \alpha \in \text{Aut}(A/(f, v_1, v_2)) \) such that \( \alpha(t_1') = t_1, \alpha(t_2') = t_2, \) and put \( w_1 = \alpha(w_1'), w_2 = \alpha(w_2'), \gamma' = \alpha(\gamma) . \) \( \square \)

4.13. Let \( \gamma_1 \) be a \( (f, x_1, x_2) \)-definable transformation, \( \gamma_2 \) a generic 
\( (f, h_2) \)-definable transformation and \( \text{rank}(h_2/(f, x_1, x_2)) = 2 \). Then there is a 
unique generic \( \gamma_3 \) which is \( (f, h) \)-definable for \( (h) \subseteq (f, h_2, x_1, x_2) \) such that for
any \( \langle v, u \rangle \in f \setminus (f, h_1, x_1, x_2)^2 \).

\[
\gamma_3(\langle v, u \rangle) = \gamma_2(\gamma_1(\langle v, u \rangle)).
\]

**Proof.** Let \( \langle v, u \rangle \in f \setminus (f, h_2, x_1, x_2)^2 \), \( \gamma_1(\langle v, u \rangle) = \langle s, r, \gamma_2(\langle s, r \rangle) = \langle t, w \rangle \).

Then by 3.5, \( r \in h_1(v) \) for \( h_1 \in F \), \( h_1 \) almost \( (f, x_1, x_2) \)-definable, \( s \in f^{-1}(r) \) and \( w \in h_2(s) \), i.e. \( w \in h_1^{-1}h_2(v) \). By 4.7 there is \( h \in F_1 \) such that \( (h) \subseteq (h_1, f_1, h_2) \) and \( w \in h(v) \), rank \( (h/f) = 2 \). This is sufficient to construct \( \gamma_3 \) as in 4.10 with \( \gamma_3(\langle v, u \rangle) = \langle t, w \rangle \).

By 4.3, \( \gamma_3 \) is unique. \( \square \)

**4.14.** If \( \beta_1 \) is a \( (f, x_1, x_2) \)-definable transformation, \( i = 1, 2 \), and

\[
\dim(f, x_{11}, x_{12}, x_{21}, x_{22}) \leq 5 \text{ then there is a unique } (f, y_1, y_2) \text{-definable}
\]

transformation with \( y_1, y_2 \in \langle f, x_{11}, x_{12}, x_{21}, x_{22} \rangle \) such that \( \beta_3(\langle v, u \rangle) = \beta_2(\beta_1(\langle v, u \rangle)) \)

for any \( \langle v, u \rangle \in f \setminus (f, x_{11}, x_{12}, x_{21}, x_{22})^2 \).

**Proof.** As in the proof of 4.13 there are \( h_1, h_2 \in F \), such that \( r \in h_1(v), w \in h_2(s) \), \( w \in h_1^{-1}h_2(v) \) and \( h_1, h_2 \) are almost

\( (f, x_{11}, x_{12}, x_{21}, x_{22}) \)-definable. Hence \( w \in h(v) \), \( h \in F_1 \), \( h \) is almost \( (y_1, y_2) \)-definable,

\( y_1, y_2 \in \langle f, x_{11}, x_{12}, x_{21}, x_{22} \rangle \).

There are three possibilities for \( h \):

1. \( h \in F_1 \), rank \( (h/f) = 2 \). In this case \( \beta_3 \) can be constructed as in 4.10.

2. \( h \in F_1 \), rank \( (h/f) = 1 \). Then \( \dim(f, h) = k \leq 3 \) and let \( \beta_3 \) be an almost \( (f, h) \)-definable \( (5-k) \)-irreducible subset of

\[
\{ \langle v', u', t', w' \rangle : w' \in h(v') \text{ and } u' \in f(v') \text{ and } w' \in f(t') \}
\]

by 1.4. Then \( \beta_3 \subseteq \beta_1 \beta_2 \).

3. \( h \notin F_1 \). Then \( w \in (v, f, y) \) for some \( y \in (f, y_1, y_2) \) and we get \( \beta_3 \).
repeating the previous point. □

4.15. Any \((f,x_1,x_2)\)-definable transformation \(\beta\) satisfies one of the following:

(i) \(\beta\) is generic;

(ii) there is \(y \in (f,x_1,x_2)\) and \(\beta'\) such that \(\beta'\) is almost \((f,y)\)-definable and \(\beta' \sqsupset \beta\); if \(\beta'' \sqsupset \beta\) and \(\beta''\) is almost \((f,y')\)-definable then \((f,y) = (f,y')\);

(iii) there is \(\beta'\) which is almost \((f)\)-definable and \(\beta' \sqsupset \beta\).

Proof. Let \(\langle v,u \rangle \in f \setminus (f,x_1,x_2)^2\), \(\beta(\langle v,u \rangle) = \langle t,w \rangle\). Since \(w \in (v,f,x_1,x_2)\), there is \(h \in F\) which is \((y_1,y_2)\)-definable, \(y_1, y_2 \in (f,x_1,x_2)\). There are three possibilities:

(1) \(h \in F_1\), \(\text{rank}(h/(f)) = 2\). This case like case 1 of 4.14 gives (i).

(2) \(h \in F_1\), \(\text{rank}(h/(f)) = 1\). Again act like in 4.14 and get \(\beta' \sqsupset \beta\) which is almost \((f,h)\)-definable, \((f,h) = (f,y)\) and we get (ii) if \(y \in (f)\) or (iii) if \(y \in (f)\).

(3) \(h \notin F_1\). The same as (2). □

For any transformation \(\beta\) of 4.15, \((f,\beta)\) is defined as \((f,x_1,x_2)\) in the case (i), \((f,y)\) in (ii) and \((f)\) in (iii).

4.16. The set of all transformations forms a group \(\Gamma\). The set \(\Gamma\) and multiplication in \(\Gamma\) are \((f)\)-definable, as well as the partial action of \(\Gamma\) on \(f\):
if \(\gamma \in \Gamma, \forall \in f \setminus (f,\gamma)\) then \(\gamma(\forall)\) is defined.

In general \(\Gamma\) is not strongly coordinatizable over \((f)\) but:

(i) the subset \(\Gamma_0 = \{\gamma \in \Gamma : \gamma\) is generic\) is strongly coordinatizable over \((f)\);

(ii) \(\Gamma\) is strongly coordinatizable over any \(a_1, a_2 \in A\) which are independent over \((f)\);

(iii) \(\text{rank}(\Gamma/(a_1,a_2,f)) = 2, \text{rank}(\Gamma_0/(f)) = 2, \text{rank}(\Gamma \setminus \Gamma_0/(a_1,a_2,f)) < 2\).
5. The structure of $\Gamma$.

If $\Gamma$ has a proper $(f)$-definable subgroup of rank 2, take a minimal such one instead of $\Gamma$. So we may assume $\Gamma$ has no proper $(f)$-definable subgroup of rank 2.

5.1. The center $C$ of $\Gamma$ is an $(f)$-definable subgroup of rank 0.

Proof. For $\gamma \in f \setminus (f)^2$ there is $\omega \in f \setminus (f,\gamma)^2$ and a subset

$$\Gamma_{(f,\gamma,\omega)} = \{ \gamma \in \Gamma_0 : \gamma(\gamma) = \omega \}$$

with $\text{rank}(\Gamma_{(f,\gamma,\omega)}/(f,\gamma,\omega)) = 1$. Suppose $\text{rank}(C/(f,\gamma,\omega)) > 0$. Then for any $\gamma_1, \gamma_2 \in \Gamma_{(f,\gamma,\omega)}$, $u \in f \setminus (f,\gamma,\omega,\gamma_1,\gamma_2)^2$ one can find $\alpha \in C$ such that

$$\alpha(\gamma) = u, \quad \gamma \in (f,\gamma_1,\alpha) \cup (f,\gamma_2,\alpha).$$

Then $\gamma_1(u) = \gamma_1(\alpha(\gamma)) = \alpha(\gamma_1(\gamma)) = \alpha(\gamma_2(\gamma)) = \gamma_2(\alpha(\gamma)) = \gamma_2(u)$. It follows that $\gamma_1 = \gamma_2$, contradiction. □

5.2. $\Gamma$ is 2-irreducible, provided $\text{dim} A \geq 8$.

Proof. Let $E_2$ be the equivalence relation on $\Gamma_0$ defined in 1.4. $U_0$ an $E_2$-class of rank 2. It is easy to see that if $\gamma \in \Gamma$ then $\gamma.U_0 \sqsubset U_1$ for some $E_2$-class $U_1$. It follows that the $(f)$-definable group

$$(\gamma \in \Gamma : \gamma U_i \sqsubset U_i \text{ for any } E_2 \text{-class } U_i \text{ of rank 2})$$

is of rank 2, thus it coincides with $\Gamma$. Moreover, if $\gamma \in U_0$ then $U_0.\gamma^{-1} \sqsubset \Gamma$, thus $\Gamma$ is 2-irreducible. □

5.3. $\Gamma = \Gamma/C$ is a centerless $(f)$-definable group.

Proof. If $\gamma$ is a central element of $\Gamma$ and $\gamma$ the corresponding element
of $\Gamma$, then $\gamma^\Gamma = \gamma.C$ is finite, therefore $C_\Gamma(\gamma)$ is of rank 2. Thus it coincides with $\Gamma$, $\gamma \in C, \overline{\gamma} = \bar{\gamma}$. □

5.4. The same arguments show that $\gamma^\Gamma$ can not be of rank 0 for $\gamma \neq \bar{\gamma}$. □

5.5. Suppose $\Delta$ is an $X$-definable group over $A$, rank($\Delta/X$) = 1, codim $X \geq 3$. Then there is a unique 1-irreducible $X$-definable normal subgroup $\Delta^0$ of $\Delta$ with rank($\Delta^0/X$) = 1.

The proof is analogous to 5.2. □

The subgroup $\Delta^0$ will be called the connected component of $\Delta$. If $\Delta = \Delta^0$, $\Delta$ is called connected.

5.6. If $\Delta$ is as in 5.5 and connected then $\Delta$ is abelian.

Proof. For $\delta \in \Delta \setminus C(\Delta)$ consider the conjugacy class $\delta^\Delta = \phi \subseteq \Delta$. $\phi$ or $\Delta \setminus \phi$ is of rank 0 over $(X, \delta)$, only the second is possible, since $C_\Delta(\delta)$ is of rank 0. Take now the polynomials $p_\phi$ and $p_\Delta$ given by 1.3. From $\phi \cap \Delta$ it follows that the leading coefficients of the polynomials coincide. On the other hand $p_\Delta = |C_\Delta(\delta)|.p_\phi$, thus $|C_\Delta(\delta)| = 1$, contradiction. □

We assume now that $\Gamma$ is a centerless 2-irreducible $0$-definable group over a pregeometry $A'$, rank($\Gamma/0$) = 2, dim $A' \geq 6$.

5.7. There is no normal subgroup $\Delta$ of $\Gamma$ which is $(x_1, x_2)$-definable for some $x_1, x_2 \in A'$ and rank($\Delta/(x_1, x_2)$) = 1.

Proof. Repeating the known construction [C] we can define a $(x_1, x_2, x_3)$-definable field structure on $\Delta$, provided $\Delta$ is connected, which we may assume by 5.5. But such a field can not exist since the mapping definable in the field $v \mapsto v^2 - v$ maps $\Delta$ on a subset $\phi \subseteq \Delta$ and contradicts 1.3 as in 5.6. □

5.8. Let $P$ be a maximal $p$-subgroup of $\Gamma$ for some prime $p$, $Y$ a closed subset of $A'$, dim $Y \geq 3$, $P[Y]$ a maximal $p$-subgroup of $\Gamma[Y]$. Then one and only one of the following holds:
(i) \(|P[Y]|\) does not depend on \(|Y|\);

(ii) \(P\) is an almost \((\gamma)\)-definable subgroup for some \(\gamma \in P\), \(\text{rank}(P/(\gamma)) = 1\), \(|P[Y]|\) is a polynomial of \(|Y|\) of degree 1, its connected component \(P^0\) is \((\gamma)\)-definable.

Proof. Choose \(\gamma \in P[Y] \cap C_{\Gamma}(P) \setminus \{e\}\), denote \(\Delta = C_{\Gamma}(\gamma)\). Then \(P \subseteq \Delta\), \(\text{rank}(\Delta/(\gamma)) \leq 1\). If \(\text{rank}(\Delta/(\gamma)) = 0\) then \(\Delta[Y]\) does not depend on \(Y\) by 1.3, the same is true for \(P[Y]\).

If \(\text{rank}(\Delta/(\gamma)) = 1\) and \(\Delta^0nP \neq \{e\}\) then \(\Delta^0 \subseteq P\) and all \(\Delta^0\)-cosets in \(P\) are almost \((\gamma)\)-definable, so is \(P\). This gives (ii). If \(\Delta^0nP = \{e\}\) then \(P\) intersects with any \(\Delta^0\)-coset at most in one point. The cosets are almost \((\gamma)\)-definable, therefore the number of cosets in \(\Delta[Y]\) which intersect with \(P\) does not depend on \(|Y|\).

5.9. There is at most one prime \(p\) for which 5.8(ii) holds.

Proof. From 5.8(ii) and 5.7 it follows that the set of all \(p\)-elements is of rank 2. Now recall 5.2. \(\square\)

5.10. The polynomial \(p_{\Gamma}(y)\) counting \(\Gamma[Y]\) by 1.3 is of degree 2. On the other hand the Sylow Theorem together with 5.8 and 5.9 gives

\[|\Gamma[Y]| = p_1^{m_1} \cdots p_n^{m_n} \cdot pp(y)\]

where \(p_1,\ldots,p_n\) are all the prime divisors of \(\Gamma[Y]\) for which 5.8(i) holds and \(pp(y)\) is the polynomial of degree 1 counting \(P[Y]\) satisfying 5.8(ii). This is the final contradiction. Thus \(\Gamma\) does not exist. \(\square\)

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* Paper is a preliminary version and should not be reviewed.