On transcendental number theory, classical analytic functions and Diophantine geometry

B. Zilber

University of Oxford

http://www.maths.ox.ac.uk/~zilber/

Background

An efficient way to classify mathematical structures is through answering the following questions:

To what extent can a structure $\mathbf{M}$ *be described by a formal language* L?

What do we need to describe $\mathbf{M}$ *uniquely up to isomorphism?*

Definition A structure $\mathbf{M}$ in a language L is said to have theory **categorical in cardinality** λ if there is exactly one, up to isomorphism, structure of cardinality λ satisfying the L-description [the theory $\mathrm{Th}(\mathbf{M})$] of $\mathbf{M}$.

Definition A structure **M** in a language L is said to have theory **categorical in cardinality** λ if there is exactly one, up to isomorphism, structure of cardinality λ satisfying the L-description [the theory Th(**M**)] of **M**.

Uncountable structures with categorical theories = **logically perfect structures**.

Basic examples of 'perfect' structures:

(1) **Trivial** structures (the language allows the equality only)

(2) **Linear** structures: Abelian divisible torsion-free groups;
Vector spaces over a given division ring
Commutative one-dimensional algebraic groups (with or without "complex multiplication";

(3) **Algebraically closed fields** $(+, \cdot, =)$

One can construct more complicated structures over the basic ones preserving the property of categoricity, e.g.

Algebraic groups

$$\mathrm{GL}(n, \mathbb{C}), \mathrm{PGL}(n, \mathbb{C}), ...$$

One can construct more complicated structures over the basic ones preserving the property of categoricity, e.g.

Algebraic groups

$$\mathrm{GL}(n,\mathbb{C}), \mathrm{PGL}(n,\mathbb{C}), ...$$

More generally, complex **algebraic varieties** $V \subseteq \mathbb{C}^n$ **equipped with algebraic relations** (given by polynomial equations

$$p(\bar{x}_1, \dots, \bar{x}_m) = 0$$

in $n \times m$ variables).

$\mathbb{C}$ can be replaced by any algebraically closed field.

Dimension notions and pregeometries on logically perfect structures

for finite $X \subset \mathbf{M}$:

(1) Trivial pregeometry: **the number of points in** X,
the number of connected components in the subgraph containing X,

(2) Linear structures:
the linear dimension $\mathrm{lin.d}\, X$ of $\langle X \rangle$

(3) Algebraically closed fields:
the transcendence degree $\mathrm{tr.d}\,(X)$ over the prime subfield.

Dimension notions and pregeometries on logically perfect structures

for finite $X \subset \mathbf{M}$:

(1) Trivial pregeometry: **the number of points in** X,
the number of connected components in the subgraph containing X,

(2) Linear structures:
the linear dimension $\mathrm{lin.d}\, X$ of $\langle X \rangle$

(3) Algebraically closed fields:
the transcendence degree $\mathrm{tr.d}\,(X)$ over the prime subfield.

Dual notion: the **dimension of an algebraic variety** V over F

$$\dim V = \max\{\ \mathrm{tr.d}_F(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in V\}.$$

Three basic geometries of stability theory:

(1) **Trivial geometry**

(2) **Linear geometry**

(3) **Algebraic geometry**.

Three basic geometries of stability theory:

(1) **Trivial geometry**

(2) **Linear geometry**

(3) **Algebraic geometry**.

Is any 'logically perfect' structure reducible to basic geometries (1) - (3)?

Three basic geometries of stability theory:

(1) **Trivial geometry**

(2) **Linear geometry**

(3) **Algebraic geometry**.

Is any 'logically perfect' structure reducible to basic geometries (1) - (3)?

YES, for some key classes (1993-2007).

NO in general (E.Hrushovski, 1989)

The analysis of 'NO'.

The analysis of 'NO'.

Hrushovski's construction of new structures

Given a class of structures **M** with a dimension notions d_1, and d_2 we want to consider a *new function* f on **M**.

The analysis of 'NO'.

Hrushovski's construction of new structures

Given a class of structures $\mathbf{M}$ with a dimension notions d_1, and d_2 we want to consider a *new function* f on $\mathbf{M}$.
On $(\mathbf{M}, \mathrm{f})$ introduce a **predimension**

$$\delta(X) = \mathrm{d}_1(X \cup \mathrm{f}(X)) - \mathrm{d}_2(X).$$

Consider structures $(\mathbf{M}, \mathrm{f})$ which satisfy the **Hrushovski inequality**:

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

The analysis of 'NO'.

Hrushovski's construction of new structures

Given a class of structures $\mathbf{M}$ with a dimension notions d_1, and d_2 we want to consider a *new function* f on $\mathbf{M}$.
On $(\mathbf{M}, \mathrm{f})$ introduce a **predimension**

$$\delta(X) = \mathrm{d}_1(X \cup \mathrm{f}(X)) - \mathrm{d}_2(X).$$

Consider structures $(\mathbf{M}, \mathrm{f})$ which satisfy the **Hrushovski inequality**:

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

Amalgamate all such structures to get a *universal and homogeneous* structure in the class.
The resulting structure $(\tilde{\mathbf{M}}, \mathrm{f})$ *will be homogeneous and have a good dimension theory.*

Are Hrushovski structures mathematical pathologies?

Observation (1996): If $\mathbf{M}$ is a field and we want f = ex to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.d}\,(X \cup \mathrm{ex}(X)) - \mathrm{lin.d}\,(X) \geq 0.$$

Are Hrushovski structures mathematical pathologies?

Observation (1996): If **M** is a field and we want f = ex to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.d}\,(X \cup \mathrm{ex}(X)) - \mathrm{lin.d}\,(X) \geq 0.$$

The Hrushovski inequality, in the case of the complex numbers and ex = exp, is equivalent to

$$\mathrm{tr.d}\,(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent (**the Schanuel conjecture**).

Pseudo-exponentiation

Consider the class of fields of characteristic 0 with a function ex: $\mathbf{F}_{\mathrm{ex}} = (F, +, \cdot, \mathrm{ex})$ satisfying

EXP1: $\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$
EXP2: $\ker \mathrm{ex} = \omega\mathbb{Z}$

Consider the subclass satisfying the Schanuel condition

$$\mathrm{SCH}: \quad \mathrm{tr.d}\,(X \cup \mathrm{ex}(X)) - \mathrm{lin.d}\,(X) \geq 0.$$

Amalgamation process produces an *algebraically-exponentially closed* **field with pseudo-exponentiation**, $\mathbf{F}_{\mathrm{ex}}(\lambda)$.

$\mathbf{F}_{\mathrm{ex}}(\lambda)$ satisfies:

Algebraic-exponential closedness (**Existential closedness**):

EC: Every system of algebraic-exponential equations which does not contradict SCH must have a solution.

Countable closure property:

CC: *Analytic* subsets of $\mathbf{F}^n$ of dimension 0 are countable.

Theorem (2001) *Given an uncountable cardinal* λ, *there is a unique, up to isomorphism, algebraically closed field with pseudo-exponentiation* $\mathbf{F}_{\mathrm{ex}}$ *of cardinality* λ *satisfying* $\quad \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$

Theorem (2001) *Given an uncountable cardinal* λ, *there is a unique, up to isomorphism, algebraically closed field with pseudo-exponentiation* $\mathbf{F}_{\rm ex}$ *of cardinality* λ *satisfying* EXP + SCH + EC + CC

Conjecture *The field of complex numbers* $\mathbb{C}_{\rm exp}$ *is isomorphic to the unique field with exponentiation* $\mathbf{F}_{\rm ex}$ *of cardinality* $2^{\aleph_0}$.

Equivalently, $\mathbb{C}_{\rm exp}$ *satisfies* SCH + EC.

Theorem (2001) *Given an uncountable cardinal* λ, *there is a unique, up to isomorphism, algebraically closed field with pseudo-exponentiation* $\mathbf{F}_{\mathrm{ex}}$ *of cardinality* λ *satisfying* EXP + SCH + EC + CC

Conjecture *The field of complex numbers* $\mathbb{C}_{\mathrm{exp}}$ *is isomorphic to the unique field with exponentiation* $\mathbf{F}_{\mathrm{ex}}$ *of cardinality* $2^{\aleph_0}$.

Equivalently, $\mathbb{C}_{\mathrm{exp}}$ *satisfies* SCH + EC.

Model-theoretic geometry suggest a geometry of exponentiation.

Weaker forms of Schanuel's conjecture

$$\mathrm{SCH}' : \quad \mathrm{tr.d}\,(X \cup \exp(X)) - \mathrm{mlt.rk}\ \exp X \geq 0$$

$$\mathrm{mlt.rk}\, Y \text{ the multiplicative group rank of } \langle Y \rangle$$

$$\mathrm{lin.d}\, X - 1 \leq \mathrm{mlt.rk}\ \exp X \leq \mathrm{lin.d}\, X.$$

Weaker forms of Schanuel's conjecture

$$\text{tr.d}\, X + \text{tr.d}\, \exp X - \text{mlt.rk}\, \exp X \geq 0$$

Weaker forms of Schanuel's conjecture

$$\mathrm{tr.d}\, X + \mathrm{tr.d}\, \exp X - \mathrm{mlt.rk}\, \exp X \geq 0$$

$$\mathrm{lin.d}_{\,K} X + \mathrm{tr.d}\, \exp X + c(K) - \mathrm{mlt.rk}\, \exp X \geq 0$$

$$(K \subset \mathbb{C}, \qquad 0 \leq c(K) \leq \mathrm{tr.d}\, K \;\; \text{finite})$$

Weaker forms of Schanuel's conjecture

$$\operatorname{tr.d} X + \operatorname{tr.d} \exp X - \operatorname{mlt.rk} \exp X \geq 0$$

$$\operatorname{lin.d}_K X + \operatorname{tr.d} \exp X + c(K) - \operatorname{mlt.rk} \exp X \geq 0$$
$$(K \subset \mathbb{C}, \qquad 0 \leq c(K) \leq \operatorname{tr.d} K \ \ \text{finite})$$

$$\operatorname{tr.d} \exp X + c(L) - \operatorname{mlt.rk} \exp X \geq 0$$
$$\text{for all } X \subset_{\text{fin}} L, \ L = Ke_1 + \ldots + Ke_m$$

Weaker forms of Schanuel's conjecture

$$\mathrm{tr.d}\, X + \mathrm{tr.d}\, \exp X - \mathrm{mlt.rk}\, \exp X \geq 0$$

$$\mathrm{lin.d}\,_K X + \mathrm{tr.d}\, \exp X + c(K) - \mathrm{mlt.rk}\, \exp X \geq 0$$
$$(K \subset \mathbb{C}, \qquad 0 \leq c(K) \leq \mathrm{tr.d}\, K \ \text{ finite})$$

$$\mathrm{tr.d}\, \exp X + c(L) - \mathrm{mlt.rk}\, \exp X \geq 0$$
$$\text{for all } X \subset_{\mathrm{fin}} L, \ L = Ke_1 + \ldots + Ke_m$$

$$\mathrm{tr.d}\, Y + c(L) - \mathrm{mlt.rk}\, Y \geq 0$$
$$\text{for all } Y \subset_{\mathrm{fin}} G$$

Weaker forms of Schanuel's conjecture

$$\mathrm{tr.d}\, X + \mathrm{tr.d}\ \exp X - \mathrm{mlt.rk}\ \exp X \geq 0$$

$$\mathrm{lin.d}\,_K X + \mathrm{tr.d}\ \exp X + c(K) - \mathrm{mlt.rk}\ \exp X \geq 0$$
$$(K \subset \mathbb{C}, \qquad 0 \leq c(K) \leq \mathrm{tr.d}\, K \ \text{ finite})$$

$$\mathrm{tr.d}\ \exp X + c(L) - \mathrm{mlt.rk}\ \exp X \geq 0$$
$$\text{for all } X \subset_{\text{fin}} L,\ L = Ke_1 + \ldots + Ke_m$$

$$\mathrm{tr.d}\, Y + c(L) - \mathrm{mlt.rk}\, Y \geq 0$$
$$\text{for all } Y \subset_{\text{fin}} G$$

$$2\mathrm{tr.d}\, Y + c(\alpha) - \mathrm{mlt.rk}\, Y \geq 0$$
$$\alpha \notin \mathbb{R} \cup i\mathbb{R}, \ \text{ for all } Y \subset_{\text{fin}} \exp \alpha\mathbb{R}, \ \ 0 \leq c(\alpha) \leq 2$$

Theorem(2004) The Schanuel conjecture SCH^K :

$$\mathrm{lin.d}\,_K X + \mathrm{tr.d}\, \exp X + c(K) - \mathrm{mlt.rk}\, \exp X \geq 0$$

is first order-axiomatisable. The first order theory $\mathbf{F}^K$ of raising to powers $k \in K$ is superstable.
Given a finite $X \subseteq 2\pi i K$, the subgroup $\langle \mathrm{ex}(X) \rangle \subseteq \mathbf{F}^\times$ is definable in $\mathbf{F}^K$.

The proof requires Mordell-Lang for the multiplicative groups of fields.

Theorem(2004) The Schanuel conjecture SCH^K :

$$\mathrm{lin.d}\,_K X + \mathrm{tr.d}\,\exp X + c(K) - \mathrm{mlt.rk}\,\exp X \geq 0$$

is first order-axiomatisable. The first order theory $\mathbf{F}^K$ of raising to powers $k \in K$ is superstable.
Given a finite $X \subseteq 2\pi i K$, the subgroup $\langle \mathrm{ex}(X) \rangle \subseteq \mathbf{F}^\times$ is definable in $\mathbf{F}^K$.

The proof requires Mordell-Lang for the multiplicative groups of fields.

Corollary *Let* Γ *be the subgroup of* $\mathbb{C}^*$ *generated by* $a_1, \ldots, a_n \in \mathbb{C}$ *and* K *be the subfield containing* $\frac{\ln a_1}{2\pi i}, \ldots, \frac{\ln a_n}{2\pi i}$.
Assume Schanuel's conjecture SCH^K. *Then, for every* $W \subseteq \mathbb{C}^m$ *definable in* $\mathbb{C}^K$,
$\Gamma^m \cap W$ *equals a finite union of cosets of subgroups* $\Gamma^m \cap T$, *some tori* T.

Wilkie's Theorem SCH^K *holds for* $K \subseteq \mathbb{R}$ *generated by* **generic** *tuples of real numbers.*

Nonstandard numbers

$$\mathbb{C} \prec {}^*\mathbb{C}, \quad \mathbb{Z} \prec {}^*\mathbb{Z}, \quad \mathbb{Q} \prec {}^*\mathbb{Q}, \ldots$$

Correspondingly, it makes sense in ${}^*\mathbf{F}$ to 'raise' to nonstandard integer powers and have the predimension for $X \subseteq {}^*\mathbb{C}$,

$$\delta(X) = \mathrm{lin.d}_{\,{}^*\mathbb{Q}} X + \mathrm{tr.d}\, \exp X - \mathrm{mlt.rk}\, \exp X.$$

The **relative predimension** with respect to $\mathbb{C}$:

$$\delta(X/\mathbb{C}) = \min\{\delta(X \cup A) - \delta(A) : \; A \subseteq_{\mathrm{fin}} \mathbb{C}, \\ A \text{ large enough}\}.$$

Theorem (with M.Bays, 2006) TFAE:
(i) (CIT) Given $W \subseteq \mathbb{C}^n$, an irreducible algebraic variety over $\mathbb{Q}$, there is finite collection $\tau(W)$ of tori in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$ and an atypical irreducible component $A \subseteq W \cap T$
(that is $\dim A > \dim W + \dim T - n$)
there is $\mathbf{T} \in \tau(W)$ such that $A \subseteq W \cap \mathbf{T}$.

(ii) for all $X \subseteq_{\text{fin}} {}^*\mathbb{C}$, $\quad \delta(X/\mathbb{C}) \geq 0$;

Theorem (with M.Bays, 2006) TFAE:
(i) (CIT) Given $W \subseteq \mathbb{C}^n$, an irreducible algebraic variety over $\mathbb{Q}$, there is finite collection $\tau(W)$ of tori in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$ and an atypical irreducible component $A \subseteq W \cap T$
(that is $\dim A > \dim W + \dim T - n$)
there is $\mathbf{T} \in \tau(W)$ such that $A \subseteq W \cap \mathbf{T}$.

(ii) for all $X \subseteq_{\text{fin}} {}^*\mathbb{C}, \quad \delta(X/\mathbb{C}) \geq 0$;

(iii) (Bombieri - Masser - Zanier's Conjecture) Given $W \subseteq \mathbb{C}^n$, an irreducible algebraic variety over $\mathbb{C}$, there is finite collection $\tau(W)$ of tori in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$ and an atypical irreducible component $A \subseteq W \cap T$ there is $\mathbf{T} \in \tau(W)$ such that $A \subseteq W \cap \mathbf{T}$.

(iv)
$\text{lin.d}_{\,{}^*\mathbb{Q}}(X/2\pi i\mathbb{Z}) + \text{tr.d}\,(\exp X/\mathbb{C}) - \text{mlt.rk}\,\exp X \geq 0$

Consider $L \subseteq \mathbb{C}^n$ m-generated $\mathbb{Q}$-module. Then $L(^*\mathbb{C}) \subseteq {}^*\mathbb{C}^n$ is m-generated $^*\mathbb{Q}$-module. So,

$$\mathrm{lin.d}_{\,^*\mathbb{Q}}(X/2\pi i\mathbb{Z}) \leq m, \text{ for all } X \subset_{\mathrm{fin}} L(^*\mathbb{C}).$$

Consider $L \subseteq \mathbb{C}^n$ m-generated $\mathbb{Q}$-module. Then $L({}^*\mathbb{C}) \subseteq {}^*\mathbb{C}^n$ is m-generated ${}^*\mathbb{Q}$-module. So,

$$\mathrm{lin.d}\,_{{}^*\mathbb{Q}}(X/2\pi i\mathbb{Z}) \leq m, \text{ for all } X \subset_{\text{fin}} L({}^*\mathbb{C}).$$

Proposition The following are equivalent:

(i) $\mathrm{tr.d}\ \exp X + c(L) - \mathrm{mlt.rk}\ \exp X \geq 0$
for all $X \subset_{\text{fin}} L({}^*\mathbb{C})$.

(ii) the geometry of $\exp L$ is linear (locally modular) in the field $\mathbb{C}$.

(iii) (Mordell-Lang) For every algebraic variety $W \subseteq \mathbb{C}^n$ over $\mathbb{C}$, $W \cap \exp L$ is equal to a finite union of cosets of subgroups $T \cap \exp L$, T tori in $\mathbb{C}^n$.

Consider $L \subseteq \mathbb{C}^n$ m-generated $\mathbb{Q}$-module. Then $L({}^*\mathbb{C}) \subseteq {}^*\mathbb{C}^n$ is m-generated ${}^*\mathbb{Q}$-module. So,

$$\text{lin.d}_{\,{}^*\mathbb{Q}}(X/2\pi i\mathbb{Z}) \leq m, \text{ for all } X \subset_{\text{fin}} L({}^*\mathbb{C}).$$

Proposition The following are equivalent:

(i) $\text{tr.d} \exp X + c(L) - \text{mlt.rk} \exp X \geq 0$
for all $X \subset_{\text{fin}} L({}^*\mathbb{C})$.

(ii) the geometry of $\exp L$ is linear (locally modular) in the field $\mathbb{C}$.

(iii) (Mordell-Lang) For every algebraic variety $W \subseteq \mathbb{C}^n$ over $\mathbb{C}$, $W \cap \exp L$ is equal to a finite union of cosets of subgroups $T \cap \exp L$, T tori in $\mathbb{C}^n$.

Corollary CIT implies Mordell-Lang.

B.Poizat (2000) used the condition on
$G \leq \mathbf{F}^*$

$$(k+1) \cdot \mathrm{tr.d}\, Y - k \cdot \mathrm{mlt.rk}\, Y \geq 0, \quad Y \subset_{\mathrm{fin}} G$$

to define a G of model theoretic dimension equal to $\frac{\mathrm{mtdim}\, \mathbf{F}}{k+1}$.

B.Poizat (2000) used the condition on
$G \leq \mathbf{F}^*$

$$(k+1)\cdot \mathrm{tr.d}\, Y - k\cdot \mathrm{mlt.rk}\, Y \geq 0, \quad Y \subset_{\mathrm{fin}} G$$

to define a G of model theoretic dimension equal to $\frac{\mathrm{mtdim}\,\mathbf{F}}{k+1}$.

Theorem (2002) The weak Schanuel conjecture

$$2\cdot \mathrm{tr.d}\, Y - \mathrm{mlt.rk}\, Y \geq 0, \quad Y \subset_{\mathrm{fin}} \exp(\alpha\mathbb{R})$$

implies

$$\mathrm{mtdim}\,\mathbb{R} = \frac{\mathrm{mtdim}\,\mathbb{C}}{2}$$

B.Poizat (2000) used the condition on
$G \leq \mathbf{F}^*$

$$(k+1)\cdot \mathrm{tr.d}\, Y - k\cdot \mathrm{mlt.rk}\, Y \geq 0, \quad Y \subset_{\mathrm{fin}} G$$

to define a G of model theoretic dimension equal to $\frac{\mathrm{mtdim}\,\mathbf{F}}{k+1}$.

Theorem (2002) The weak Schanuel conjecture

$$2\cdot \mathrm{tr.d}\, Y - \mathrm{mlt.rk}\, Y \geq 0, \quad Y \subset_{\mathrm{fin}} \exp(\alpha\mathbb{R})$$

implies

$$\mathrm{mtdim}\,\mathbb{R} = \frac{\mathrm{mtdim}\,\mathbb{C}}{2}$$

Proposition Assume Schanuel's conjecture for the p-adic exponentiation.
Then, for every k there is $\alpha \in \mathbb{Q}_p^{\mathrm{alg}}$, $|\alpha|_p = 1$, such that

$$\frac{k+1}{k}\cdot \mathrm{tr.d}\, Y - \mathrm{mlt.rk}\, Y \geq 0,$$

for all $Y \subset_{\mathrm{fin}} \exp(\alpha p\mathbb{Z}_p)$.

Corollary $\mathrm{mtdim}\,\mathbb{Z}_p = 0$, if defined.

The Uniform Schanuel conjecture

Theorem(2001) CIT+SCH′ implies
Uniform SCH′: *Given an algebraic subvariety* $W \subseteq \mathbb{C}^{2n}$ *over* $\mathbb{Q}$ *with* $\dim W < n$ *there is a positive integer* N *such that*

$$\langle x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \rangle \in W \Rightarrow$$

$$\bigvee_{|m_i| \leq N} \exp(m_1 x_1 + \ldots + m_n x_n) = 1 \ \& \ \bigvee_i m_i \neq 0.$$

The Uniform Schanuel conjecture

Theorem(2001) CIT+SCH$'$ implies
Uniform SCH$'$: *Given an algebraic subvariety $W \subseteq \mathbb{C}^{2n}$ over $\mathbb{Q}$ with $\dim W < n$ there is a positive integer N such that*

$$\langle x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \rangle \in W \Rightarrow$$

$$\bigvee_{|m_i| \leq N} \exp(m_1 x_1 + \ldots + m_n x_n) = 1 \ \& \ \bigvee_i m_i \neq 0.$$

Theorem (2004, with J.Kirby)
SCH($\mathbb{R}_{\text{exp}}$) *is uniform. That is* SCH($\mathbb{R}_{\text{exp}}$) *is equivalent to:*
Given an algebraic subvariety $W \subseteq \mathbb{R}^{2n}$ over $\mathbb{Q}$ with $\dim W < n$ *there is a positive integer N such that*

$$\langle x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \rangle \in W \Rightarrow$$

$$\bigvee_{|m_i| \leq N} m_1 x_1 + \ldots + m_n x_n = 0 \ \& \ \bigvee_i m_i \neq 0.$$

The proof is based on the *analytic cell decomposition* result (T.L.Loi) for $\mathbb{R}_{\text{exp}}$ (which follows from Wilkie's Theorem).

The Weierstrass function

The case of the Weierstrass function $\mathbf{p}_\omega(x)$, for a fixed lattice is very similar.

The 'full' Weierstrass function

The Weierstrass function $\mathbf{p}(\tau, x)$ as a function of two variables

$$\mathbf{p}(\tau, x) = \frac{1}{x^2} + \sum_{\lambda \in \langle \tau,\ 1 \rangle \setminus (0)} [\frac{1}{(x-\lambda)^2} - \frac{1}{\lambda^2}].$$

The 'full' Weierstrass function

The Weierstrass function $\mathbf{p}(\tau, x)$ as a function of two variables

$$\mathbf{p}(\tau, x) = \frac{1}{x^2} + \sum_{\lambda \in \langle \tau,\ 1 \rangle \setminus (0)} [\frac{1}{(x-\lambda)^2} - \frac{1}{\lambda^2}].$$

For every $\tau \in \mathcal{H}$ define the field k_τ as $\mathbb{Q}$ or $\mathbb{Q}(i_\tau)$, if the corresponding elliptic curve has complex multiplication i_τ.

The 'full' Weierstrass function

The Weierstrass function $\mathbf{p}(\tau, x)$ as a function of two variables

$$\mathbf{p}(\tau, x) = \frac{1}{x^2} + \sum_{\lambda \in \langle \tau,\ 1\rangle \setminus (0)} [\frac{1}{(x-\lambda)^2} - \frac{1}{\lambda^2}].$$

For every $\tau \in \mathcal{H}$ define the field k_τ as $\mathbb{Q}$ or $\mathbb{Q}(i_\tau)$, if the corresponding elliptic curve has complex multiplication i_τ.

The corresponding 'Schanuel conjecture' must take into account the trivial geometry on $\mathcal{H}$ (with the action of $\mathrm{SL}_2(\mathbb{Q})$) and the linear geometry along each elliptic curve. Thus it takes the form:
given $\tau_1, \ldots, \tau_m \in \mathcal{H}$ and $x_1, \ldots, x_n \in \mathbb{C}$,

$$\mathrm{tr.d}\,(\{\tau_i\},\ \{x_j\},\ \{\mathbf{p}(\tau_i, x_j)\}) - \sum_{\tau_i/\mathrm{SL}_2(\mathbb{Q})} \mathrm{lin.d}_{\,k_{\tau_i}}\{x_j\} \geq 0$$