

Quantum Harmonic Oscillator as Zariski Geometry

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1 Introduction

We describe the structure $\text{QHO} = \text{QHO}_N$ (dependent on the positive integer number N) on the universe L which is a finite cover, of order N , of the projective line $\mathbb{P} = \mathbb{P}(\mathbb{F})$, \mathbb{F} an algebraically closed field of characteristic 0. We prove that QHO is a complete irreducible Zariski geometry of dimension 1. We also prove that QHO is *not classical* in the sense that the structure is not interpretable in an algebraically closed field and, for the case $\mathbb{F} = \mathbb{C}$, is not a structure on a complex manifold.

There are several reasons that motivate our interest in this particular example. First, this Zariski geometry differs considerably from the series of examples in [HZ] which all are based on the actions of certain kinds of noncommutative groups as the groups of Zariski automorphisms of the structures constructed. In the present case we represent the well-known noncommutative algebra with generators P and Q satisfying the relation

$$QP - PQ = i, \tag{1}$$

as the bundle of eigenspaces of the Hamiltonian $H = \frac{1}{2}(P^2 + Q^2)$, the system known as *the simple harmonic oscillator*.

The algebra considered is the famous Heisenberg algebra, historically the first example of a *quantisation* of a classical Hamiltonian system and an important source of *noncommutative geometry*. In [Zil2] the first author developed a construction that puts in correspondence to an arbitrary *quantum algebra at roots of unity* a Zariski structure, similar to the correspondence between polynomial algebras and affine varieties. This construction is a source of many new non-classical Zariski structures, but there the new examples start from dimension 2. The Heisenberg algebra differs from the class of algebras considered in [Zil2] by virtue of it not being an algebra at roots of unity. Indeed, whereas the irreducible modules for the algebras considered in [Zil2] were finite-dimensional, irreducible modules for the Heisenberg algebra are necessarily infinite-dimensional. Consequently, this paper constitutes an extension of the construction and method of proof in [Zil2] to a wider class of noncommutative algebras.

Some remarks about the methods of noncommutative geometry and possible interactions with model theory are in order. Noncommutative geometry provides something of a union of operator theory (specifically C^* -algebras) and algebraic topology. To this end, noncommutative geometers invoke a dictionary by which concepts in topology can be translated into concepts in operator theory and vice versa. A feature of noncommutative geometry is that concrete constructions of geometric counterparts to the algebras studied isn't carried out, the operator methods in themselves sufficing for any "geometric" arguments one may need to produce. The Gelfand-Naimark and Serre-Swan correspondences provide the means by which such a philosophy is justified.

It is our belief, that though operator methods are very powerful in their own right, the absence of geometric counterparts corresponding to (noncommutative) operator algebras results in a picture that is incomplete. This paper, and the paper [Zil2] represent steps taken in the direction towards filling this gap. Furthermore, given that the notion of a Zariski geometry provides an abstract characterization

of the geometry on an algebraic variety, model-theoretically one has the means of proving that a non-classical geometric structure associated to a specific operator algebra is ‘rich’ (i.e. that one has a means of developing algebraic geometry on the structure).

It should be noted that the Heisenberg algebra is a $*$ -algebra: it is an algebra equipped with an additional operation $*$ which associates to each element X of the algebra an element X^* , seen (by analogy with Hilbert space theory) as the *adjoint* to X . It is not a C^* -algebra: any representation of the Heisenberg algebra as an algebra of operators on a Hilbert space must, by the nature of the defining relations, result in at least one of the operators being unbounded. Of course, an important theorem in the representation theory of C^* -algebras is that any C^* -algebra can be represented as an algebra of *bounded* operators on some Hilbert space. Consequently, one does not have many of the methods available to non-commutative geometers to study this algebra directly. The *Weyl algebra* (the ‘exponential’ of the Heisenberg algebra) is a C^* -algebra and is consequently the favoured object of study. What is interesting about the approach developed in this paper is that this apparent issue with the Heisenberg algebra has not manifested itself geometrically: the corresponding geometry is still rich.

By the postulates of quantum mechanics, P and Q are considered to be self-adjoint (self-adjoint operators have real eigenvalues). Consequently, so is H . The Zariski structure considered does not originally witness the $*$ -structure on the Heisenberg algebra, and so it produces, for $\mathbb{F} = \mathbb{C}$, essentially a (non-classical) complex geometry. The assumption of self-adjointness, in the canonical commutative context, leads to *cutting out the real part* of a complex variety. The result of the same operation with our structure QHO is the discrete substructure, (the finite cover of) the infinite set

$$\left\{n + \frac{1}{2} : n = 0, 1 \dots\right\} \quad (2)$$

Namely the energy levels of the Hamiltonian are quantized. Our Zariski geometry should therefore be seen as *the complexification of (2)*, obtained from the same noncommutative coordinate algebra. This complexification exposes the true geometry of the discrete structure.

Finally, we would like to note that although the construction of QHO represents the eigenstates of H and the creation and annihilation of these, it is still a rather limited example as far as mathematical physics is concerned. One fails to see the interdependence between eigenstates of H , P and Q , expressed mathematically in the form of inner product. Such issues will be addressed in the near future.

1.1 Background

In this subsection, we outline the appropriate background concerning the analysis of the quantum harmonic oscillator. Any physical system has a corresponding Hamiltonian H given by:

$$H = \frac{P^2}{2m} + V(Q)$$

The first quantity on the right-hand side is kinetic energy. The latter quantity, the potential $V(Q)$, is typically a polynomial expression in Q . The Hamiltonian, as a physical quantity, is conserved and represents the total energy of the physical system. In the case of the quantum harmonic oscillator, P and Q satisfy the canonical commutation relation (1) (written $[Q, P] = i$, taking $\hbar = 1$). The Hamiltonian H takes the form:

$$H = \frac{1}{2}(P^2 + Q^2)$$

We redefine the algebra in terms of two operators \mathbf{a} and \mathbf{a}^\dagger and these satisfy the following relations:

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1 \quad H = \mathbf{a}\mathbf{a}^\dagger + \frac{1}{2}$$

Define:

$$\mathbf{N} := \mathbf{H} - \frac{1}{2} = \mathbf{a}^\dagger \mathbf{a}.$$

One easily sees that

$$[\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger \text{ and } [\mathbf{N}, \mathbf{a}] = -\mathbf{a}.$$

It follows that if e_a is an eigenvector of \mathbf{N} with eigenvalue a , then

$$\mathbf{N} \mathbf{a} e_a = (\mathbf{a} \mathbf{N} - \mathbf{a}) e_a = (a - 1) e_a,$$

$$\mathbf{N} \mathbf{a}^\dagger e_a = (\mathbf{a}^\dagger \mathbf{N} + \mathbf{a}^\dagger) e_a = (a + 1) e_a.$$

For this reason, \mathbf{a} and \mathbf{a}^\dagger are referred to as *ladder operators* (respectively annihilation and creation operators in the broader context of quantum field theory). When \mathbf{a}^\dagger (respectively \mathbf{a}) acts on an eigenvector e_a , it gives a new eigenvector with an eigenvalue $a + 1$ (respectively $a - 1$).

If this algebra is represented as an algebra of linear operators on a Hilbert space, with \mathbf{P} and \mathbf{Q} assumed self-adjoint and the eigenvectors e_a normalised, then \mathbf{a}^\dagger is adjoint to \mathbf{a} and the inner product satisfies

$$(e_a, \mathbf{a} \mathbf{a}^\dagger e_a) = (e_a, \{\mathbf{a}^\dagger \mathbf{a} + 1\} e_a) = (e_a, \{\mathbf{N} + 1\} e_a) = a + 1.$$

In other words

$$\mathbf{a}^\dagger e_a = b e_{a+1}, \quad b^2 = a + 1.$$

Similarly,

$$\mathbf{a} e_a = b e_{a-1}, \quad b^2 = a.$$

Now observe that since \mathbf{H} is the sum of squares of self-adjoint operators, its eigenvalues $a + \frac{1}{2}$ are real non-negative. But \mathbf{a} , applied to an eigenvector e_a lowers its eigenvalue by 1. It must therefore follow that after finitely many applications of \mathbf{a} one obtains an eigenvector e_0 so that $\mathbf{N} e_0 = 0$ (referred to as the *ground state*). So the spectrum of \mathbf{N} consists of all the non-negative integers and the spectrum of the Hamiltonian \mathbf{H} is the set (2) above.

2 The structure

Definition 2.1. We consider the two-sorted theory T_N with sorts L and \mathbb{F} in the language $\mathcal{L} = \mathcal{L}_r \cup \{\infty, \pi, \cdot, \mathbf{A}, \mathbf{A}^\dagger\}$ subject to the following axioms:

1. \mathbb{F} is an algebraically closed field of characteristic 0.
2. \mathbb{P} is the projective line over \mathbb{F} .
3. $\pi : L \rightarrow \mathbb{P}$ is surjective.
4. We have a free and transitive group action $\cdot : \mathbb{F}[N] \times L \rightarrow L$ on each of the fibers $\pi^{-1}(a)$ for $a \in \mathbb{P}$.
5. The ternary relations $\mathbf{A}, \mathbf{A}^\dagger$ (on $L^2 \times \mathbb{F}$) obey the following property:

$$(\forall a \in \mathbb{F})(\forall e \in \pi^{-1}(a))(\exists b \in \mathbb{F})(\exists e' \in \pi^{-1}(a + 1))(b^2 = a \wedge \mathbf{A}(\gamma \cdot e, \gamma \cdot e', b) \wedge \mathbf{A}^\dagger(\gamma \cdot e', \gamma \cdot e, b))$$

for every $\gamma \in \mathbb{F}[N]$.

6. For N even, we postulate the following additional properties for $\mathbf{A}, \mathbf{A}^\dagger$:

$$\mathbf{A}(e, e', b) \rightarrow \mathbf{A}(\gamma \cdot e, -\gamma \cdot e', -b)$$

$$\mathbf{A}^\dagger(e', e, b) \rightarrow \mathbf{A}^\dagger(\gamma \cdot e', -\gamma \cdot e, -b)$$

Evidently the theory T_N is first-order. We will denote models of T_N by QHO_N . The additional constant symbol ∞ is required to define the equivalence relation to be introduced below. As is well-known, one can identify \mathbb{P} with $\mathbb{F} \cup \{\infty\}$. Each of the fibers $\pi^{-1}(a)$ has size N : take $x \in \pi^{-1}(a)$ and ϵ a generator of $\mathbb{F}[N]$. Then each of the $\epsilon^k \cdot x$ for $0 \leq k < N$ are distinct (as the action is free) which implies that $|\pi^{-1}(a)| \geq N$. If $|\pi^{-1}(a)| > N$ then there would be $y \in \pi^{-1}(a)$ such that $y \neq \epsilon^k \cdot x$ for any $0 \leq k < N$, contradicting transitivity.

We now use this structure to define a line bundle over \mathbb{F} (not claiming local triviality), achieved by introducing the following equivalence relation on $L \times \mathbb{P}$:

$$(e, x) \sim (e', x) \Leftrightarrow (\pi(e) = \pi(e') \wedge (\exists \gamma \in \mathbb{F}[N])(\gamma \cdot e = e' \wedge \gamma^{-1}x = x') \vee x = x' = 0 \vee x = x' = \infty)$$

We have a structure which is induced from QHO_N and has universe $(L \times \mathbb{P}) / \sim$ (so it lives in QHO_N^{eq}), which we refer to as Q_N . One can define the following additional operations of addition and scalar multiplication on Q_N (by abuse of notation, we write (e, x) for the equivalence classes $[(e, x)]$):

$$\lambda(e, x) := (e, \lambda x) \quad (e, x) + (e, y) := (e, x + y)$$

One sees that we have compatibility with the group action: $(\gamma \cdot e, x) = (e, \gamma x) = \gamma(e, x)$ for $\gamma \in \mathbb{F}[N]$. If we set $V_a = \{[(e, x)] : \pi(e) = a \wedge x \in \mathbb{F}\}$ then we see that for each $a \in \mathbb{F}$, V_a is a one-dimensional vector space over \mathbb{F} with the above operations. Put $\mathcal{H} = \bigcup_{a \in \mathbb{F}} V_a$.

We can now introduce the linear maps $\mathbf{a}, \mathbf{a}^\dagger$ on Q_N : for each $a \in F$ and $e \in \pi^{-1}(a)$, $\mathbf{a}(e, 1) := (e', b)$ where $\mathbf{A}(e, e', b)$ in the structure QHO_N and we extend this linearly. Similarly $\mathbf{a}^\dagger(e', 1) := (e, b)$ where $\mathbf{A}^\dagger(e', e, b)$, also extended linearly.

Remark 2.1. *Suppose we have that $\mathbf{a}(e, 1) = (e', b)$. Then for any $\gamma \in \mathbb{F}[N]$, it follows that $\mathbf{a}(\gamma \cdot e, 1) = \gamma \mathbf{a}(e, 1) = \gamma(e', b) = (\gamma \cdot e', b)$. So we should have that $\mathbf{A}(\gamma \cdot e, \gamma \cdot e', b)$ in the structure QHO_N . Similarly for \mathbf{a}^\dagger .*

Remark 2.2. *Suppose that N is even and that $\mathbf{a}(e, 1) = (e', 1)$. We then have that $-1 \in \mathbb{F}[N]$ and $\mathbf{a}(-e, 1) = \mathbf{a}(e, -1) = (e', -b)$. As $(-b)^2 = a$, this explains why we stipulated the additional condition on \mathbf{A} for the even case. Similarly for \mathbf{A}^\dagger .*

Proposition 2.1. *The theory T_N is consistent and, for even N , is categorical in uncountable cardinals. Moreover, if \mathbb{F} and \mathbb{F}' correspond to the field sort in two models QHO and QHO' of theory T_N and there exists $i : \mathbb{F} \rightarrow \mathbb{F}'$, a ring isomorphism, then i can be extended to an isomorphism $\hat{i} : \text{QHO} \rightarrow \text{QHO}'$. In particular the only relations on \mathbb{F} induced from QHO are the initial relations corresponding to the field structure.*

Proof. First we construct a model of T_N . For each $a \in \mathbb{F}$ choose $e_a \in L(\text{QHO})$ such that $\mathbf{p}(e_a) = a$ and choose arbitrarily \sqrt{a} , a square root of a . Now for every a define $\mathbf{a}e_a := \sqrt{a}e_{a+1}$ and $\mathbf{a}^\dagger e_a := \sqrt{a-1}e_{a-1}$. Extend this linearly to maps $V_a \rightarrow V_{a+1}$ and $V_a \rightarrow V_{a-1}$ correspondingly. This is well defined for all a and so defines \mathbf{a} and \mathbf{a}^\dagger on a model Q_N according to the axioms of T_N . One then sees that we have a corresponding model QHO of T_N .

To prove categoricity, consider two models of T_N with isomorphic fields. We may assume that $\mathbb{F} = \mathbb{F}'$ and i is the identity. Partition \mathbb{P} into the orbits of the action of the additive subgroup $\mathbb{Z} \subseteq \mathbb{F}$:

$$\mathbb{P} = \bigcup_{s \in S} s + \mathbb{Z}$$

where S is some choice of representatives, one for each orbit ($\infty + m = \infty$ for each $m \in \mathbb{Z}$, so we have a one 1-element orbit). For each $s \in S$ choose first $e_s \in L(\text{QHO})$ and $e'_s \in L(\text{QHO}')$. Now for each $n \in \mathbb{Z}$

choose arbitrarily $(s+n)^{\frac{1}{2}}$. By the axioms there is $e \in L \cap \mathbf{p}^{-1}(s+1)$ such that $\mathbf{a}e_s = \epsilon(s+n)^{\frac{1}{2}}e$, for some $\epsilon \in \{1, -1\}$. Define $e_{s+1} := \epsilon e$ which is in L since N is even and $\epsilon \in \mathbb{F}[N]$. Similarly define $e'_{s+1} \in L(\text{QHO}')$. By induction we can define e_{s+n} and e'_{s+n} for all $n \geq 0$ so that in the induced Q_N , Q'_N of the two models, $\mathbf{a}e_{s+n} := (s+n)^{\frac{1}{2}}e_{s+n+1}$ (and the corresponding relations for e'_{s+n}). Note that by axioms we also have, for all $n > 0$, $\mathbf{a}^\dagger e_{s+n} = (s+n-1)^{\frac{1}{2}}e_{s+n-1}$.

By the similar inductive procedure for all $n > 0$ define e_{s-n} so that $\mathbf{a}^\dagger e_{s+1-n} = (s-n)^{\frac{1}{2}}e_{s-n}$ and the same in the second model. Again by axioms this determines the action of \mathbf{a} on e_{s-n} and e'_{s-n} , for all $n > 0$. Hence we have constructed the bijective correspondence $e_a \mapsto e'_a$, $a \in \mathbb{F}$ so that it extends to the action of the linear maps \mathbf{a} and \mathbf{a}^\dagger on Q_N in the corresponding structures, therefore inducing an isomorphism $\text{QHO} \rightarrow \text{QHO}'$. \square

Lemma 2.1. *Assume that $\mathbb{F} = \mathbb{C}$. Then for each N one can construct QHO_N definable in \mathbb{R} .*

Proof. Consider \mathbb{C} as $\mathbb{R} + i\mathbb{R}$, definable in \mathbb{R} . Choose an \mathbb{R} -definable complex function $z \mapsto z^{\frac{1}{2}}$ satisfying $(z^{\frac{1}{2}})^2 = z$. Define L to be \mathbb{C} and \mathbf{p} to be the map $x \mapsto x^N$. One can now define Q_N and then define \mathbf{a} as the only linear map $V_z \rightarrow V_{z+1}$ such that for each $e_z \in \mathbf{p}^{-1}(z)$ there is $e_{z+1} \in \mathbf{p}^{-1}(z+1)$ such that $\mathbf{a}e_z := z^{\frac{1}{2}}e_{z+1}$. As observed before this also defines \mathbf{a}^\dagger . \square

Proposition 2.2. *QHO_N is not definable in an algebraically closed field.*

Proof. Suppose towards a contradiction it is definable in an algebraically closed field \mathbb{F}' . Since any infinite field definable in an algebraically closed field is definably isomorphic to it, and in case of characteristic zero by a unique isomorphism, we may assume that $\mathbb{F}' = \mathbb{F}$. We may also assume that \mathbb{F} is of infinite transcendence degree.

Let \mathbb{F}_0 be the minimal (finitely generated) subfield of \mathbb{F} which contains parameters for the definition of L , \mathbf{p} , \mathbf{a} , \mathbf{a}^\dagger and other operations in the definition of QHO as well as all elements of $\mathbb{F}[N]$.

Let a be a generic element of \mathbb{F} over \mathbb{F}_0 and consider an element $e \in \mathbf{p}^{-1}(a)$ (identified with $(e, 1)$) which by the assumption of definability can be identified with a tuple in \mathbb{F} and also $\mathbb{F}_0(e) \supseteq \mathbb{F}_0(a)$. The orbit $O(e)$ of e under the action of the Galois group $\text{Gal}(\mathbb{F}_0(e) : \mathbb{F}_0(a))$ is a subset of $\mathbf{p}^{-1}(a)$ and also if $\epsilon e \in O(e)$ for $\epsilon \in \mathbb{F}[N]$ then for all $e' \in O(e)$, $\epsilon e' \in O(e)$. Since by definition each $e' \in O(e)$ is of the form ϵe , it follows that $O(e)$ is also the orbit under the action of a subgroup Γ of $\mathbb{F}[N]$. Moreover, if $\sigma \in \text{Gal}(\mathbb{F}_0(e) : \mathbb{F}_0(a))$ fixes e then it fixes ϵe for all $\epsilon \in \mathbb{F}[N]$, so $\text{Gal}(\mathbb{F}_0(e) : \mathbb{F}_0(a)) \cong \Gamma$ and is cyclic.

Let k be the order of Γ , which is also the order of the cyclic Galois extension $(\mathbb{F}_0(e) : \mathbb{F}_0(a))$. Since all roots of 1 of order k are in \mathbb{F}_0 , by theory of cyclic extensions there exists an element $b \in \mathbb{F}_0(e)$ such that $b^k = a$ and $\mathbb{F}_0(e) = \mathbb{F}_0(b)$. In particular $e = f(b)$ for some rational function f over \mathbb{F}_0 and also $b = g(e)$ for some other such function. It follows that we can assume that in our interpretation of QHO in \mathbb{F} for all but finitely many $a \in \mathbb{F}$ the set $\mathbf{p}^{-1}(a)$ contains all k solutions of the $\mathbb{F}_0(a)$ -irreducible equation $x^k = a$ and e is one of these. Also, the action $e \mapsto \gamma e$ for $\gamma \in \Gamma$ is the action by Galois automorphisms, so for the solutions of $x^k = a$ can be identified with the multiplication $e \mapsto \epsilon e$, $\epsilon \in \mathbb{F}[k]$.

By the axioms of T_N there is $e' \in \mathbf{p}^{-1}(a+1)$ and $\rho \in \mathbb{F}$ such that

$$\mathbf{a}(e, 1) = (e', r), \quad \mathbf{a}^\dagger(e', 1) = (e, r), \quad r^2 = a.$$

Clearly $r = r(e, e')$ is a definable, hence rational, function of e, e' . Since $\{y \in \mathbb{F} : y^k = a+1\} \subseteq \mathbf{p}^{-1}(a+1)$ by axioms $e' = \epsilon b$, for some $b \in E$, $\epsilon \in \mathbb{F}[N]$, $b^k = a+1$. Redefining $q(e, b) := \epsilon r(e, e')$ and writing x, y for e, b we have

$$\mathbf{a}(x, 1) = (y, q(x, y)), \quad \mathbf{a}^\dagger(y, 1) = (x, q(x, y)\epsilon^{-2}) \quad q(x, y)^2 = \epsilon^2 x^k. \quad (3)$$

Denote by C the set of all the $(x, y) \in \mathbb{F}^2$ which satisfy (3). Up to finitely many points C is a plane curve over \mathbb{F}_0 . We see that for generic x there is an y such that $y^k = x^k + 1$ and $(x, y) \in C$.

Since $y^k = x^k + 1$ defines an irreducible curve we conclude that all but finitely many points of this curve belong to C . In particular, for every $\gamma \in \mathbb{F}[k]$, the point $(x, \gamma y)$ is in $y^k = x^k + 1$ and so in C .

For $\gamma^2 \neq 1$ this contradicts the first two equations of (3). So, we conclude $k = 1$ or $k = 2$ and $\epsilon^2 = 1$.

Now in the first case $q(x, x+1)^2 = x$ for all generic x – contradiction. In the second case $q(x, y)^2 = x^2$ on the curve $y^2 = x^2 + 1$. This implies that $q(x, y) \equiv x$ or $q(x, y) \equiv -x$ on the curve. But $q(x, -y) = -q(x, y)$ by the first equation of (3) – contradiction. \square

Corollary 2.1. *QHO $_N(\mathbb{C})$ is not Zariski-isomorphic to a structure on a complex space, with relations given by analytic subsets.*

Proof. If QHO $_N(\mathbb{C})$ were a complex space it will have to be an unramified finite cover of the projective line $\mathbf{P}(\mathbb{C})$, so a 1-dimensional compact manifold. But every such manifold is biholomorphically isomorphic (so Zariski isomorphic) to a complex algebraic curve – contradiction. \square

Remark 2.3. *The real part of QHO $_N(\mathbb{C})$. Consider the extra assumption that \mathbf{P} and \mathbf{Q} are self-adjoint operators. Then the analysis in section 1.1 shows that the eigenvalues of $\mathbf{a}^\dagger \mathbf{a}$ must be non-negative and so the only points in \mathbb{P} that survive this extra condition are the non-negative integers \mathbb{N} . The corresponding points in L form the N -cover of \mathbb{N} , so we get the discrete structure QHO $_N(\mathbb{N})$ as the real part of QHO $_N(\mathbb{C})$. Conversely, the latter is the complexification of the former.*

3 Definable sets

We can view Q_N as a two-sorted structure $(\mathcal{H}, \mathbb{F})$, where \mathcal{H} is defined as before. Introduce the projection map $\mathbf{p} : \mathcal{H} \rightarrow \mathbb{F}$ where $\mathbf{p} : (e, x) \mapsto \pi(x)$. Note that \mathbf{p} is definable. We wish to pick “canonical basis” elements in each fiber V_a which we regard as having modulus one. In our terminology, these canonical basis elements are exactly the elements $(e, 1)$ in each fiber. Note that there are N possible choices in each fiber: if $\gamma \in \mathbb{F}[N]$, then $(e, \gamma) = (\gamma \cdot e, 1)$. We introduce a predicate $E(f, \alpha)$ on $\mathcal{H} \times \mathbb{F}$ which says “ f is a canonical basis element of the fiber $\mathbf{p}^{-1}(\alpha)$ ”.

We follow the analysis in [Zil2]. Suppose that f is an s -tuple of variables from \mathcal{H} . Let $\Sigma \subseteq \{(i, j) : 1 \leq i, j \leq s\}$. We suppose that g is tuple of variables from \mathcal{H} of length $|\Sigma|$, α is an s -tuple of variables from \mathbb{F} , γ is a tuple of variables from \mathbb{F} of length $|\Sigma|$, as is b . Suppose further that e is an n -tuple of variables from \mathcal{H} , λ is an n -tuple of variables from \mathbb{F} and that a is an m -tuple of variables from \mathbb{F} . Define the following formulas:

$$G^\Sigma(f, g, b, \gamma, \alpha) := \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)} \wedge \bigwedge_{(i,j) \in \Sigma} (E(g_{(i,j)}, \alpha_j) \wedge \mathbf{a}^{n(i,j)} f_i = b_{(i,j)} g_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)} = b_{(i,j)} f_i \wedge g_{(i,j)} = \gamma_{(i,j)} f_j) \right)$$

$$A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) := \bigwedge_{i=1}^s E(f_i, \alpha_i) \wedge G^\Sigma(f, g, b, \gamma) \wedge \bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} e_{ij} = \lambda_{ij} f_i$$

It is clear from the definition that each $c^{(i,j)}$ is an $n_{(i,j)}$ tuple of variables from the sort \mathbb{F} . The first conjunct of $G^\Sigma(f, g, b, \gamma)$ ensures that the $b_{(i,j)}$ are products of square roots, i.e. that the $g_{(i,j)}$ is the canonical basis element chosen in accordance with repeated applications of the map \mathbf{a} to f_i . It will become clear that G^Σ only becomes significant in the case where f_i, f_j lie in the same coset of the additive subgroup \mathbb{Z} .

Note that we have arranged a particular enumeration of the tuple of variables $e = (e_1, \dots, e_n)$: the n elements are enumerated as $\{e_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$ where $s_1 + \dots + s_s = n$. We call such an

enumeration a **partitioning enumeration**. It is evident from the notation established that we have a corresponding partitioning enumeration for λ . Let $R(\alpha, \gamma, b, \lambda, a)$ define a Zariski constructible set in \mathbb{F}^q where $q = s + 2|\Sigma| + n + m$ (over a parameter set $C \subseteq \mathbb{F}$). We define a **core formula** over R to be:

$$\exists f \exists g \exists \alpha \exists \gamma \exists b \exists \lambda (A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge R(\alpha, \gamma, b, \lambda, a))$$

So this is a formula with free variables (e, a) over $C \subseteq \mathbb{F}$. Denote by $ctp(e, a/C)$ the set of core formulas over R (over C) with free variables (e, a) .

Proposition 3.1. *If $ctp(e^1, a^1/C) = ctp(e^2, a^2/C)$ then $tp(e^1, a^1/C) = tp(e^2, a^2/C)$.*

Proof. Assume that Q_N is \aleph_0 -saturated. We show that there is an automorphism σ such that $\sigma : (e^1, a^1) \rightarrow (e^2, a^2)$. Fix a partitioning enumeration of e^1 so that $\mathbf{p}(e_{ij}^1) = \mathbf{p}(e_{kl}^1)$ if and only if $i = k$. Then $1 \leq i \leq s$ for some s and there exist $\alpha_i^1 \in \mathbb{F}$ such that $\mathbf{p}(e_{ij}^1) = \alpha_i^1$. By re-enumerating the α_i^1 if necessary, we suppose that $\alpha_i^1 < \alpha_j^1$ for $i < j$. Now we construct a subset $\Sigma \subseteq \{(i, j) : 1 \leq i, j \leq s\}$ as follows: put $(i, j) \in \Sigma$ if there is an $n_{(i,j)} > 0$ such that $\alpha_i^1 + n_{(i,j)} = \alpha_j^1$ (so α_i^1 and α_j^1 lie in the same coset of the additive subgroup \mathbb{Z}). We can choose a canonical basis element f_i^1 in each fiber $\mathbf{p}^{-1}(\alpha_i^1)$ and so there exist λ_{ij}^1 in \mathbb{F} such that:

$$Q_N \models \bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} e_{ij}^1 = \lambda_{ij}^1 f_i^1$$

For $(i, j) \in \Sigma$, by repeated application of \mathbf{a} to f_i^1 ($n_{(i,j)}$ times) we obtain $b_{(i,j)}^1$ and $g_{(i,j)}^1$, where the latter is a canonical basis element of the fiber $\mathbf{p}^{-1}(\alpha_j^1)$. As f_j^1 is also a canonical basis element of the fiber $\mathbf{p}^{-1}(\alpha_j^1)$, there is $\gamma_{(i,j)}^1 \in \mathbb{F}[N]$ such that $g_{(i,j)}^1 = \gamma_{(i,j)}^1 f_j^1$. Doing this for each $(i, j) \in \Sigma$, we obtain tuples g^1, b^1, γ^1 such that $Q_N \models G^\Sigma(f^1, g^1, b^1, \gamma^1)$. It follows that:

$$Q_N \models A^\Sigma(f^1, g^1, e^1, \alpha^1, \gamma^1, b^1, \lambda^1)$$

We consider the following type in variables $f, e, \alpha, \gamma, b, \lambda$, which by assumption is consistent:

$$q = \{A^\Sigma(f, g, e^2, \alpha, \gamma, b, \lambda) \wedge R(\alpha, \gamma, b, \lambda, a^2) : Q_N \models A^\Sigma(f^1, g^1, e^1, \alpha^1, \gamma^1, b^1, \lambda^1) \wedge R(\alpha^1, \gamma^1, b^1, \lambda^1, a^1)\}$$

By \aleph_0 -saturation, q is realized by $f^2, g^2, \alpha^2, \gamma^2, b^2, \lambda^2$. In particular, by quantifier elimination for algebraically closed fields, in the language of rings $tp^{\mathbb{F}}(\alpha^1, \gamma^1, b^1, \lambda^1, a^1) = tp^{\mathbb{F}}(\alpha^2, \gamma^2, b^2, \lambda^2, a^2)$. So by saturation of \mathbb{F} there is an automorphism σ of \mathbb{F} such that:

$$\sigma : (\alpha^1, \gamma^1, b^1, \lambda^1, a^1) \mapsto (\alpha^2, \gamma^2, b^2, \lambda^2, a^2)$$

Partition $\mathbb{F} = \bigcup_{r \in R} r + \mathbb{Z}$ where the set of representatives R contains as many of the α_i^1 as possible. For each i , we extend σ to the fiber $\mathbf{p}^{-1}(\alpha_i^1)$ by $\sigma : \mu f_i^1 \mapsto \sigma(\mu) f_i^2$. Clearly, it then follows that $\sigma(e_{ij}^1) = e_{ij}^2$. Take some $\alpha_i \in R$. By the axioms, there is $c \in \mathbb{F}$ and $h \in \mathbf{p}^{-1}(\alpha_i^1 + 1)$ such that $\mathbf{a}f_i^1 = ch$. Similarly, there is $d \in \mathbb{F}$ and $l \in \mathbf{p}^{-1}(\sigma(\alpha_i^1) + 1)$ such that $\mathbf{a}f_i^2 = dl$. As $c^2 = \alpha_i^1$ and $d^2 = \alpha_i^2$ we have that $\sigma(c) = \epsilon d$ where $\epsilon \in \{-1, 1\}$. So we extend σ to $\mathbf{p}^{-1}(\alpha_i^1 + 1)$ by mapping $\mu h \mapsto \sigma(\mu) \epsilon l$ (note that we have assumed N is even). We continue this process inductively to extend σ to every fiber $\mathbf{p}^{-1}(\alpha_i^1 + n)$ for $n > 0$ as in the proof of categoricity. If we have $\alpha_j^1 = \alpha_i^1 + n_{(i,j)}$ for some $n_{(i,j)} > 0$ everything still works by construction. Similarly, extend σ in the other direction and repeat the construction for each coset. \square

It follows (by compactness) that any formula with free variables (e, a) over $C \subseteq \mathbb{F}$ is a finite disjunction of a conjunction of core formulas and their negations. Denote a core formula over R by $\exists f R$. For further purposes, we would like to determine the effects of conjunction and negation on core formulas. Indeed, for core formulas $\exists f R_1, \exists f R_2$ we would like to show that:

- $\exists f(R_1 \wedge R_2) \equiv \exists fR_1 \wedge \exists fR_2$.
- $\exists f(\neg R_1) \equiv \neg \exists fR_1$.

Fix a core formula $\exists fR$ where $R(\alpha, \gamma, b, \lambda, a)$ defines a Zariski constructible set in \mathbb{F}^q . For $\delta = (\delta_1, \dots, \delta_s) \in \mathbb{F}[N]^s$, we wish to define $R^\delta(\alpha, \gamma, b, \lambda, a)$ (which we regard as the action of δ on R). First assume that R defines an irreducible set. Put:

$$V_R := \{\alpha \in \mathbb{F}^s : \exists \gamma \exists b \exists \lambda \exists a R(\alpha, \gamma, b, \lambda, a)\}$$

We define R^δ to be the Zariski closure of the following set:

$$\{(\alpha, \gamma, b, \lambda, a) : \alpha \in V_R \wedge \exists \gamma' \exists \lambda' \left(\bigwedge_{(i,j) \in \Sigma} \gamma'_{(i,j)} = \delta_i \gamma_{(i,j)} \delta_j^{-1} \wedge \bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \lambda'_{ij} = \lambda_{ij} \delta_i^{-1} \right) \wedge R(\alpha, \gamma', b, \lambda', a)\}$$

Remark 3.1. Suppose we have a tuple (e, a) such that $Q_N \models \exists fR(e, a)$. Then we obtain canonical basis elements $f_i \in \mathbf{p}^{-1}(\alpha_i)$ for $1 \leq i \leq s$ and λ_{ij} such that $e_{ij} = \lambda_{ij} f_i$ and similarly for some Σ we also have the relations $g_{(i,j)} = \gamma_{(i,j)} f_j$ holding. We wish to examine the effect of transforming $f_i \mapsto f'_i = \delta_i f_i$ on these relations. We find that the $g_{(i,j)}$ get transformed to $g'_{(i,j)} = \delta_i g_{(i,j)}$. Consequently, $g'_{(i,j)} = \delta_i \gamma_{(i,j)} f_j$ and so we put $\gamma'_{(i,j)} = \delta_i \gamma_{(i,j)} \delta_j^{-1}$ so that the relations $g'_{(i,j)} = \gamma'_{(i,j)} f'_j$ hold. Similarly, $e_{ij} = \lambda'_{ij} f'_i$ where $\lambda'_{ij} = \lambda_{ij} \delta_i^{-1}$. So the set R^δ gives those tuples $(\alpha, \gamma, b, \lambda, a)$ for which R still holds after the transformation of basis elements.

If R is Zariski closed, we can decompose R into a finite union of irreducibles: $R = R_1 \cup \dots \cup R_k$. In this case, we put $R^\delta := R_1^\delta \cup \dots \cup R_k^\delta$. We say that R is $\mathbb{F}[N]$ -invariant if $R^\delta = R$ for every $\delta \in \mathbb{F}[N]^s$.

Lemma 3.1. We may assume that the core formulas in $ctp(e, a/C)$ are over $(R \wedge \neg S)(\alpha, \gamma, b, \lambda, a)$ where R and S are systems of equations and S is $\mathbb{F}[N]$ -invariant.

Proof. Recall the type $p = tp^{\mathbb{F}}(\alpha, \gamma, b, \lambda, a)$ obtained in the proof of quantifier-elimination for core formulas. For $P \in p$, we can assume that either P is a system of equations or the negation of a system of equations. If $P = R$ a system of equations, we are done. So we deal with the case that $P = \neg S$ where S is a system of equations.

If $\bigwedge_{\delta \in \mathbb{F}[N]^s} \neg S^\delta \in p$ then $\bigwedge_{\delta \in \mathbb{F}[N]^s} \neg S^\delta \equiv \neg T$ and for every $\epsilon \in \mathbb{F}[N]^s$ we have $\neg(T^\epsilon) = \neg T$. So T is $\mathbb{F}[N]$ -invariant, $\neg S \models \neg T$ and we can replace P by $\neg T$. So suppose that $\bigwedge_{\delta \in \mathbb{F}[N]^s} \neg S^\delta \notin p$. Then there is a maximal subset $\Delta \subseteq \mathbb{F}[N]^s$ such that:

$$\neg T = \bigwedge_{\delta \in \Delta} \neg S^\delta \in p$$

As $\neg S \in p$, we have that $1 \in \Delta$. Put $Stab(\Delta) = \{\delta \in \mathbb{F}[N]^s : \delta \Delta = \Delta\}$. As Δ is maximal, for any $\delta \in \mathbb{F}[N]^s \setminus Stab(\Delta)$, we have $\neg T^\delta \notin p$. As p is complete, it follows that $T^\delta \in p$ and so:

$$\bigwedge_{\delta \in \mathbb{F}[N]^s \setminus Stab(\Delta)} T^\delta \in p$$

Now note that:

$$\bigvee_{\delta \in \mathbb{F}[N]^s} \neg T^\delta \wedge \bigwedge_{\delta \in \mathbb{F}[N]^s \setminus Stab(\Delta)} T^\delta \models \bigvee_{\delta \in Stab(\Delta)} \neg T^\delta$$

The first disjunct is clearly in p and the last disjunct is equivalent to $\neg T \models \neg S$ as $1 \in \Delta$. So we take $R = \bigwedge_{\delta \in \mathbb{F}[N]^s \setminus Stab(\Delta)} T^\delta$ and $\neg S_1 = \bigvee_{\delta \in \mathbb{F}[N]^s} \neg T^\delta$, S_1 is $\mathbb{F}[N]$ -invariant, and replace P by $R \wedge S_1$. \square

We may also assume that R is $\mathbb{F}[N]$ -invariant by replacing R with $R_1 = \bigvee_{\delta \in \mathbb{F}[N]^s} R^\delta$. It is clear that $\exists f(R \wedge \neg S)$ implies $\exists f(R_1 \wedge \neg S)$. We use the above lemma to show the converse: suppose that $Q_N \models \exists f(R_1 \wedge \neg S)(e, a)$. Then there are $f, g, \alpha, \gamma, b, \lambda$ such that $Q_N \models A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda)$ and $Q_N \models R^\delta(\alpha, \gamma, b, \lambda, a)$. So $Q_N \models R(\alpha, \gamma', b, \lambda', a)$ (as in the definition of R^δ) and by the remark following the definition we also have $Q_N \models A^\Sigma(f', g', e, \alpha, \gamma', b, \lambda')$. By the $\mathbb{F}[N]$ invariance of S we then obtain that $Q_N \models \exists f(R \wedge \neg S)(e, a)$.

Lemma 3.2. *Suppose that $\exists f R_1$ and $\exists f R_2$ are core formulas and that R_2 is $\mathbb{F}[N]$ -invariant. Then:*

1. $\exists f(R_1 \wedge R_2) \equiv \exists f R_1 \wedge \exists f R_2$.
2. $\exists f(\neg R_2) \equiv \neg \exists f R_2$.

Proof. Left-to-right in 1 is obvious. Conversely, suppose that $Q_N \models \exists f R_1(e, a)$ and $Q_N \models \exists f R_2(e, a)$. Then there are $f^1, g^1, \alpha, \gamma^1, b, \lambda^1$ witnessing $\exists f R_1$ and $f^2, g^2, \alpha, \gamma^2, b, \lambda^2$ witnessing $\exists f R_2$. There exists $\delta \in \mathbb{F}[N]^s$ such that $f^1 = \delta f^2$ and so carrying out this transformation and noting that $R_2^\delta = R_2$, we obtain $Q_N \models \exists f(R_1 \wedge R_2)(e, a)$.

For 2, right-to-left is obvious. If $Q_N \models \exists f(\neg R_2)$ then there are some elements witnessing this. If there was some witness to $\exists f R_2$, then by $\mathbb{F}[N]$ -invariance of R_2 we could transform the latter elements into the former, resulting in contradiction. \square

Combining the previous two lemmas with the previous proposition, we get that any formula with free variables (e, a) over parameters $C \subseteq \mathbb{F}$ is equivalent to a finite disjunction of core formulas over R_i where the R_i are $\mathbb{F}[N]$ -invariant.

We now consider a more general class of formulas over parameters in \mathcal{H} and $C \subseteq \mathbb{F}$. This time e is an $(n+r)$ -tuple of variables from \mathcal{H} and we define A^Σ on the first n elements of e as before. Suppose that $h = (h_1, \dots, h_t)$ is a tuple of parameters from \mathcal{H} where each h_i is a canonical basis element. Suppose that μ is an r -tuple of variables from \mathbb{F} . For a partitioning enumeration of the remaining r variables in e , $\{e_{s+i,j} : 1 \leq i \leq t : 1 \leq j \leq t_i\}$ We define:

$$B(e, h, \mu) := \bigwedge_{i=1}^t \bigwedge_{j=1}^{t_i} e_{n+i,j} = \mu_{i,j} h_i$$

Suppose further that $\Delta_1 \subseteq \{(i, j) : 1 \leq i \leq s, 1 \leq j \leq t\}$ and $\Delta_2 \subseteq \{(i, j) : 1 \leq i \leq t, 1 \leq j \leq s\}$. We define the following formulas:

$$D_1(f, p, h, m, \delta) := \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = m_{(i,j)} \wedge \bigwedge_{(i,j) \in \Delta_1} (E(p_{(i,j)}, \mathbf{p}(h_j)) \wedge \mathbf{a}^{n(i,j)} f_i = m_{(i,j)} p_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} p_{(i,j)} = m_{(i,j)} f_i \wedge p_{(i,j)} = \delta_{(i,j)} h_j) \right)$$

$$D_2(f, q, h, o, \epsilon, \alpha) := \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(h_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = o_{(i,j)} \wedge \bigwedge_{(i,j) \in \Delta_2} (E(q_{(i,j)}, \alpha_j) \wedge \mathbf{a}^{n(i,j)} h_i = o_{(i,j)} q_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} q_{(i,j)} = o_{(i,j)} h_i \wedge q_{(i,j)} = \epsilon_{(i,j)} f_j) \right)$$

So p is a tuple of variables of length $|\Delta_1|$ from \mathcal{H} , m, δ are tuples of variables of length $|\Delta_1|$ from \mathbb{F} and $\mathbb{F}[N]$ respectively, q is a tuple of variables of length $|\Delta_2|$ from \mathcal{H} and o, ϵ are tuples of variables of

length $|\Delta_2|$ from \mathbb{F} , $\mathbb{F}[N]$ respectively.

Put $D := D_1 \wedge D_2$. Suppose that $R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a)$ defines a Zariski constructible subset of \mathbb{F}^q where $q = s + 2(|\Sigma| + |\Delta_1| + |\Delta_2|) + n + m + r$. We define a **general core formula** over R with parameters h to be:

$$\exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu (A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge B(e, h, \mu) \wedge R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a))$$

Proposition 3.2. *Every formula with parameters in \mathcal{H} is equivalent to a finite disjunction of general core formulas.*

Proof. Suppose that ϕ is a formula with free variables (e, a) over a finite tuple of parameters $l = (l_1, \dots, l_p)$ in \mathcal{H} . Then ϕ is equivalent to $\psi(e, l, a)$ where ψ contains no parameters in \mathcal{H} . By the previous proposition, $\psi(e, v, a)$ (where v is a tuple of variables replacing the parameters l) is equivalent to a finite disjunction of core formulas. So it suffices to prove that a core formula (with free variables (e, v, a)) is equivalent to a finite disjunction of general core formulas after the substitution $v := l$.

We carry out the substitution and re-name the v variables as e_{ij} and arrange a partitioning enumeration of the e variables so that the substitution occurs in the variables $\{e_{ij} : s < i \leq s + t, 1 \leq j \leq q_i\}$ where $q_i \leq p_i$ and we have that $\mathbf{p}(l_{ij}) = \mathbf{p}(l_{km})$ if and only if $i = k$. There exists β_i such that $\mathbf{p}(l_{ij}) = \beta_i$ for each $s < i \leq s + t$. For each fiber $\mathbf{p}^{-1}(\beta_i)$, there are only finitely many possible choices of canonical basis elements h_i . Once a h_i has been chosen, we obtain fixed λ_{ij}^1 and for those h_i, h_j for which β_i, β_j lie in the same coset of \mathbb{Z} , we have fixed $b_{(i,j)}^1, \gamma_{(i,j)}^1$ and $g_{(i,j)}^1$. It follows that $\exists f R^{v:=l}$ is equivalent to:

$$\bigvee_{h_i \in \mathbf{p}^{-1}(\beta_i) \wedge E(h_i, \beta_i)} \exists f \exists g \exists \alpha \exists \gamma \exists b \exists \lambda (A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge R(\alpha, \gamma, b, \lambda, a))^{(e_i, \alpha_i, g_i, b_i, \gamma_i) := (l, \beta, g^1, b^1, \gamma^1)}$$

Here, by α_i we mean those α variables corresponding to l , and similarly for the other variables. After substitution, we rename the remaining λ variables λ_{ij} for $s < i \leq s + t, q_i < j \leq p_i$ as $\mu_{i-s, j-q_i}$. R then becomes a constructible predicate in $\alpha, \gamma, b, \lambda, \mu, a$ over a parameter set $C \cup \{\beta, \gamma^1, b^1\}$. We now deal with the formula $A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda)^{(e_i, \alpha_i, g_i, b_i, \gamma_i) := (l, \beta, g^1, b^1, \gamma^1)}$. Some conjuncts trivially hold, i.e. $E(h_i, \beta_i)$, all $l_{ij} = \lambda_{ij}^1 h_i$ and:

$$\exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)}^1 \wedge \right.$$

$$\left. \bigwedge_{(i,j) \in \Sigma_1} (E(g_{(i,j)}^1, \beta_j) \wedge \mathbf{a}^{n(i,j)} h_i = b_{(i,j)}^1 g_{(i,j)}^1 \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)}^1 = b_{(i,j)}^1 h_i \wedge g_{(i,j)}^1 = \gamma_{(i,j)}^1 h_j) \right)$$

Here we have some $\Sigma_1 \subseteq \{(i, j) : s < i, j \leq s + t\}$. So we delete these. The remaining conjuncts are then:

$$E(f_i, \alpha_i) \tag{4}$$

$$\bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} e_{ij} = \lambda_{ij} f_i \tag{5}$$

$$\bigwedge_{i>s} \bigwedge_{j>q_i} e_{ij} = \mu_{i-s, j-q_i} h_{i-s} \tag{6}$$

$$\begin{aligned} & \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)} \wedge \right. \\ & \bigwedge_{(i,j) \in \Sigma_2} (E(g_{(i,j)}, \alpha_j) \wedge \mathbf{a}^{n(i,j)} f_i = b_{(i,j)} g_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)} = b_{(i,j)} f_i \wedge g_{(i,j)} = \gamma_{(i,j)} f_j) \end{aligned} \quad (7)$$

$$\begin{aligned} & \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)} \wedge \right. \\ & \bigwedge_{(i,j) \in \Delta_1} (E(g_{(i,j)}, \alpha_j) \wedge \mathbf{a}^{n(i,j)} f_i = b_{(i,j)} g_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)} = b_{(i,j)} f_i \wedge g_{(i,j)} = \gamma_{(i,j)} h_j) \end{aligned} \quad (8)$$

$$\begin{aligned} & \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(h_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)} \wedge \right. \\ & \bigwedge_{(i,j) \in \Delta_2} (E(g_{(i,j)}, \alpha_j) \wedge \mathbf{a}^{n(i,j)} h_i = b_{(i,j)} g_{(i,j)} \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)} = b_{(i,j)} h_i \wedge g_{(i,j)} = \gamma_{(i,j)} f_j) \end{aligned} \quad (9)$$

We combine (4), (5) and (7) to give $A^{\Sigma_2}(f, g, e, \alpha, \gamma, b, \lambda)$ for some $\Sigma_2 \subseteq \{(i, j) : 1 \leq i, j \leq s\}$. We recognize (6) as $B(e, h, \mu)$ after re-indexing h_{i-s} as h_i and $\mu_{i-s, j-q_i}$ as μ_{ij} . In (8), the index set $\Delta_1 \subseteq \{(i, j) : 1 \leq i \leq s, s < j \leq s+t\}$. We replace the occurrences of α_j in $E(g_{(i,j)}, \alpha_j)$ with $\mathbf{p}(h_j)$ and after the re-indexing of h_j , Δ_1 becomes a subset of $\{(i, j) : 1 \leq i \leq s, 1 \leq j \leq t\}$. By replacing $g_{(i,j)}$, $b_{(i,j)}$ and $\gamma_{(i,j)}$ with $p_{(i,j-s)}$, $m_{(i,j-s)}$ and $\delta_{(i,j-s)}$ respectively, we obtain $D_1(f, p, h, m, \delta)$. Similarly in (9), after re-indexing of the h_j we get that $\Delta_2 \subseteq \{1 \leq i \leq t, 1 \leq j \leq s\}$ and we replace $g_{(i,j)}$, $b_{(i,j)}$ and $\gamma_{(i,j)}$ with $q_{(i-s,j)}$, $o_{(i-s,j)}$ and $\epsilon_{(i-s,j)}$ respectively to get $D_2(f, q, h, o, \epsilon, \alpha)$. Furthermore, note that by renaming the variables in (8), (9), R becomes a constructible predicate $R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a)$. \square

A similar analysis to that carried out for core formulas will give that we can assume R to be $\mathbb{F}[N]$ -invariant: one specifies additional conditions for $(i, j) \in \Delta_1, \Delta_2$ in the definition of R^η , namely the following:

$$\bigwedge_{(i,j) \in \Delta_1} \delta'_{(i,j)} = \eta_i \delta_{(i,j)} \wedge \bigwedge_{(i,j) \in \Delta_2} \epsilon'_{(i,j)} = \epsilon_{(i,j)} \eta_j^{-1}$$

We introduce a topology on Q_N by defining the **basic closed sets** as those defined by general core formulas over R where R are $\mathbb{F}[N]$ -invariant systems of polynomial equations. We will denote the corresponding basic closed set by \hat{R} . Closed sets are defined as finite unions and arbitrary intersections of basic closed sets.

Proposition 3.3. *The topology on Q_N is Noetherian.*

Proof. Suppose that $\hat{R}_1 \supseteq \hat{R}_2 \supseteq \dots$ is a descending chain of basic closed sets.

Claim: If $\exists f R_1$ and $\exists f R_2$ are general core formulas over parameters h_1, h_2 respectively and $\exists f R_2 \rightarrow \exists f R_1$, then we can assume that $h_1 = h_2$.

Proof. Indeed, there is $\eta \in \mathbb{F}[N]^t$ such that $h_1 = \delta h_2$. If $\exists f R_2(e, a)$ holds, then for some μ_{ij} we have $e_{s+i,j} = \mu_{ij} h_{1i}$. Consequently, by putting $\mu'_{ij} = \delta_i \mu_{ij}$ we get $e_{s+i,j} = \mu'_{ij} h_{2i}$. Similar considerations apply to o and δ . Consequently, we replace R_2 by another predicate $R'_2(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a) := R_2(\alpha, \gamma, \eta \cdot \delta, \epsilon, b, m, \eta \cdot o, \lambda, \eta \cdot \mu, a)$ and we see that $\exists f R'_2(e, a)$ holds. \square

We can therefore assume that the general core formulas $\exists f R_i$ have the same parameters from \mathcal{H} . As there are only finitely many ways of partitioning the first n e -variables, we can assume that each of the $\exists f R_i$ have the same partitioning enumeration. Consequently, the question of whether the sequence of closed sets stabilizes reduces to the question of whether the sequence $R_1 \supseteq R_2 \supseteq \dots$ stabilizes, which holds as algebraic varieties are Noetherian topological spaces. \square

We know (by quantifier elimination) that the projection of a constructible set is constructible. For further purposes, we require more information about what the formulas that define these projective sets look like.

Lemma 3.3. *Suppose we have a general core formula $\exists f R$ (with previously established notation). Let $k \in \{i : 1 \leq i \leq s + t\}$ and let l be a subset of indices from $\{j : 1 \leq j \leq s_k\}$ if $k \leq s$ or a subset of indices from $\{j : 1 \leq j \leq t_k\}$ if $k > s$. Then the formula $\exists e_{kl} \exists f R$ is a general core formula over R' with R' equivalent to one of the following:*

1. $\exists \mu_{k-s,l} R$.
2. $\exists m_{(i_1, j_1)} \exists \delta_{(i_1, j_1)} \exists o_{(i_2, j_2)} \exists \epsilon_{(i_2, j_2)} \exists \mu_{k-s,l} R$ where $(i_1, j_1) \in \Delta_1 \setminus \Delta'_1$ and $(i_2, j_2) \in \Delta_2 \setminus \Delta'_2$ for some $\Delta'_1 \subseteq \Delta_1$ and $\Delta'_2 \subseteq \Delta_2$.
3. $\exists \lambda_{kl} R$.
4. $\exists \alpha_k \exists b_{(i_1, j_1)} \exists \gamma_{(i_1, j_1)} \exists m_{(i_2, j_2)} \exists \delta_{(i_2, j_2)} \exists o_{(i_3, j_3)} \exists \epsilon_{(i_3, j_3)} \exists \lambda_{kl} R$ where $(i_1, j_1) \in \Sigma \setminus \Sigma_1$, $(i_2, j_2) \in \Delta_1 \setminus \Delta'_1$ and $(i_3, j_3) \in \Delta_2 \setminus \Delta'_2$ for some $\Sigma_1 \subseteq \Sigma$, $\Delta'_1 \subseteq \Delta_1$ and $\Delta'_2 \subseteq \Delta_2$.

Proof. The proof divides into four cases:

1. Suppose that $k > s$ and that the indices l are not all of $1 \leq l \leq t_k$. The variables e_{ij} do not occur in A^Σ , D or R , so the formula $\exists e_{kl} \exists f R$ is equivalent to:

$$\exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu (A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge \exists e_{kl} B(e, h, \mu) \wedge R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a))$$

Now $\exists e_{kl} B(e, h, \mu)$ is equivalent to $B(e', h, \mu')$ where e' contains the variables $e \setminus \{e_{kl}\}$ and μ' contains the variables $\mu \setminus \{\mu_{k-s,l}\}$. So the above formula is equivalent to:

$$\exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu (A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge B(e, h, \mu) \wedge \exists \mu_{k-s,l} R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a))$$

To obtain this, we have noted that R is the only part of the formula that contains the variables $\mu_{k-s,l}$ that we have deleted from B , so we can move the existential quantification over these variables to R .

2. Suppose that $k > s$ and that the the l are all of $1 \leq l \leq t_k$. We then obtain a further reduction, namely that any formulas involving the parameter h_k now become redundant. So we get a new formula $D(f, p', q', h', m', o', \delta', \epsilon', \alpha')$ where h' is $h \setminus \{h_k\}$, p', m', δ' contain $p_{(i,j)}$, $m_{(i,j)}$, $\delta_{(i,j)}$ respectively, where $(i, j) \in \Delta'_1$ for some $\Delta'_1 \subseteq \Delta_1$ and similarly q', o', ϵ' contain $q_{(i,j)}$, $o_{(i,j)}$, $\epsilon_{(i,j)}$ respectively, where $(i, j) \in \Delta'_2$ for some $\Delta'_2 \subseteq \Delta_2$. So we get that $\exists e_{kl} \exists f R$ is a general core formula over $\exists m_{(i_1, j_1)} \exists \delta_{(i_1, j_1)} \exists o_{(i_2, j_2)} \exists \epsilon_{(i_2, j_2)} \exists \mu_{k-s,l} R$ where $(i_1, j_1) \in \Delta_1 \setminus \Delta'_1$ and $(i_2, j_2) \in \Delta_2 \setminus \Delta'_2$ (note that it is possible that some of the α_j also get deleted from D , but we can't shift existential quantifications over these α_j to R because these α_j still occur in A^Σ).

3. Suppose that $k \leq s$ and that the l are not all of $1 \leq l \leq s_k$. The variables e_{kl} only occur in A^Σ so $\exists e_{kl} \exists f R$ is equivalent to:

$$\exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu (\exists e_{kl} A^\Sigma(f, g, e, \alpha, \gamma, b, \lambda) \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge B(e, h, \mu) \wedge R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a))$$

As in the previous cases, we can eliminate the variables e_{kl} to obtain $A^\Sigma(f, g, e', \alpha, \gamma, b, \lambda')$ where e' is $e \setminus \{e_{kl}\}$ and λ' is $\lambda \setminus \{\lambda_{kl}\}$. Then the above formula is equivalent to:

$$\exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu (A^\Sigma(f, g, e', \alpha, \gamma, b, \lambda') \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge B(e, h, \mu) \wedge \exists \lambda_{kl} R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a))$$

4. If $k \leq s$ and the l range over all of $1 \leq l \leq s_k$, then analogously to case 2 we have that f_k becomes redundant. So we delete the conjuncts $E(f_k, \alpha_k)$, $e_{kl} = \lambda_{kl} f_k$ and conjuncts involving f_k in G^Σ to obtain another formula $A^{\Sigma_1}(f', g', e', \alpha', \gamma', b', \lambda')$ where e', λ' are as in case 3, α' is $\alpha \setminus \{\alpha_k\}$, f' is $f \setminus \{f_k\}$ and the g', γ', b' contain exactly those $g_{(i,j)}, b_{(i,j)}, \gamma_{(i,j)}$ for $(i,j) \in \Sigma_1$ for some $\Sigma_1 \subseteq \Sigma$. We also delete any expressions involving f_k from D to obtain $D(f', p', q', h, m', o', \delta', \epsilon', \alpha')$ where p', m', δ' contain $p_{(i,j)}, m_{(i,j)}, \delta_{(i,j)}$ respectively, where $(i,j) \in \Delta_1$ for some $\Delta_1' \subseteq \Delta_1$ and similarly q', o', ϵ' contain $q_{(i,j)}, o_{(i,j)}, \epsilon_{(i,j)}$ respectively, where $(i,j) \in \Delta_2$ for some $\Delta_2' \subseteq \Delta_2$. We therefore get that $\exists e_{kl} \exists f R$ is a general core formula over:

$$\exists \alpha_k \exists b_{(i_1, j_1)} \exists \gamma_{(i_1, j_1)} \exists m_{(i_2, j_2)} \exists \delta_{(i_2, j_2)} \exists o_{(i_3, j_3)} \exists \epsilon_{(i_3, j_3)} \exists \lambda_{kl} R$$

where $(i_1, j_1) \in \Sigma \setminus \Sigma_1$, $(i_2, j_2) \in \Delta_1 \setminus \Delta_1'$ and $(i_3, j_3) \in \Delta_2 \setminus \Delta_2'$.

□

Now consider two general core formulas $\exists f R$ and $\exists f S$ with $R, S \mathbb{F}[N]$ -invariant and with possibly different partitioning enumerations of the first n e -variables. We assume that the parameters h_1, \dots, h_t are the same in both formulas. Take $\exists f R$. Linearly enumerate the first n e -variables as $\{e_1, \dots, e_n\}$. We introduce an equivalence relation on $\{1, \dots, n\}$: that $k \sim_R l$ if and only if k corresponds to some (i, j) and l corresponds to some (i, j') in the old enumeration of the e -variables. Let I_R be a set of representatives for this equivalence relation. We do the same for $\exists f S$: linearly enumerate the first n e -variables and introduce the equivalence relation \sim_S . Let I_S be a set of representatives.

We will define a formula \tilde{R} which will be seen to be equivalent to $\exists f R$. Suppose that $\Sigma' \subseteq \{(i, j) : 1 \leq i, j \leq n\}$, $\Delta_1' \subseteq \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq t\}$ and $\Delta_2' \subseteq \{(i, j) : 1 \leq i \leq t, 1 \leq j \leq n\}$. Then we define \tilde{R} to be:

$$\begin{aligned} \exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu & \left(\bigwedge_{i=1}^n E(f_i, \alpha_i) \wedge G^{\Sigma'}(f, g, b, \gamma) \wedge \bigwedge_{i=1}^n e_i = \lambda_i f_i \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge \right. \\ & B(e, h, \mu) \wedge R(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a) \wedge \bigwedge_{i \sim_R j} (f_i = f_j \wedge \alpha_i = \alpha_j) \wedge \\ & \bigwedge_{i \sim_R k} \bigwedge_{j \sim_R l} (g_{(i,j)} = g_{(k,l)} \wedge b_{(i,j)} = b_{(k,l)} \wedge \gamma_{(i,j)} = \gamma_{(k,l)}) \wedge \\ & \left. \bigwedge_{i \sim_R j} (p_{(i,k)} = p_{(j,k)} \wedge m_{(i,k)} = m_{(j,k)} \wedge \delta_{(i,k)} = \delta_{(j,k)}) \wedge \right) \end{aligned}$$

$$\bigwedge_{i \sim_{Rj}} (q_{(k,i)} = q_{(k,j)} \wedge o_{(k,i)} = o_{(k,j)} \wedge \epsilon_{(k,i)} = \epsilon_{(k,j)})$$

D is over the indexing set $\Delta'_1 \cup \Delta'_2$. The main difference now is that f and α are n -tuples of variables. It is easy to see that we can find Σ' , Δ'_1 and Δ'_2 so that $\exists f R$ is equivalent to \tilde{R} : one carries out the stated linear re-enumeration of the first n e -variables and then duplicates and re-enumerates the remaining variables according to the established equivalence relation. We also have an analogous formula \tilde{S} equivalent to $\exists f S$.

Lemma 3.4. *The formula $\tilde{R} \wedge \tilde{S}$ is equivalent to the formula \tilde{T} defined as:*

$$\begin{aligned} & \exists f \exists g \exists \alpha \exists \gamma \exists \delta \exists \epsilon \exists b \exists p \exists q \exists m \exists o \exists \lambda \exists \mu \left(\bigwedge_{i=1}^n E(f_i, \alpha_i) \wedge G^{\Sigma}(f, g, b, \gamma) \wedge \bigwedge_{i=1}^n e_i = \lambda_i f_i \wedge D(f, p, q, h, m, o, \delta, \epsilon, \alpha) \wedge \right. \\ & B(e, h, \mu) \wedge (R \wedge S)(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a) \wedge \bigwedge_{i \sim_{RSj}} (f_i = f_j \wedge \alpha_i = \alpha_j) \wedge \\ & \bigwedge_{i \sim_{RSk} j \sim_{RSl}} (g_{(i,j)} = g_{(k,l)} \wedge b_{(i,j)} = b_{(k,l)} \wedge \gamma_{(i,j)} = \gamma_{(k,l)}) \wedge \\ & \bigwedge_{i \sim_{RSj}} (p_{(i,k)} = p_{(j,k)} \wedge m_{(i,k)} = m_{(j,k)} \wedge \delta_{(i,k)} = \delta_{(j,k)}) \wedge \\ & \left. \bigwedge_{i \sim_{RSj}} (q_{(k,i)} = q_{(k,j)} \wedge o_{(k,i)} = o_{(k,j)} \wedge \epsilon_{(k,i)} = \epsilon_{(k,j)}) \right) \end{aligned}$$

where \sim_{RS} is the transitive closure of the composition of \sim_R and \sim_S .

Proof. The implication from right-to-left is trivial. Conversely, suppose that $(\tilde{R} \wedge \tilde{S})(e, a)$ holds. Then we obtain elements $f^r, g^r, \alpha^r, \gamma^r, \delta^r, \epsilon^r, b^r, p^r, q^r, m^r, o^r, \lambda^r, \mu^r$ for $r \in \{1, 2\}$ such that the following formulas hold:

$$\begin{aligned} & \bigwedge_{i=1}^n E(f_i^1, \alpha_i^1) \wedge G^{\Sigma_1}(f^1, g^1, b^1, \gamma^1) \wedge \bigwedge_{i=1}^n e_i = \lambda_i^1 f_i^1 \wedge D(f^1, p^1, q^1, h, m^1, o^1, \delta^1, \epsilon^1, \alpha^1) \wedge \\ & B(e, h, \mu^1) \wedge R(\alpha^1, \gamma^1, \delta^1, \epsilon^1, b^1, m^1, o^1, \lambda^1, \mu^1, a) \wedge \bigwedge_{i \sim_{Rj}} (f_i^1 = f_j^1 \wedge \alpha_i^1 = \alpha_j^1) \wedge \\ & \bigwedge_{i \sim_{Rk} j \sim_{Rl}} (g_{(i,j)}^1 = g_{(k,l)}^1 \wedge b_{(i,j)}^1 = b_{(k,l)}^1 \wedge \gamma_{(i,j)}^1 = \gamma_{(k,l)}^1) \wedge \\ & \bigwedge_{i \sim_{Rj}} (p_{(i,k)}^1 = p_{(j,k)}^1 \wedge m_{(i,k)}^1 = m_{(j,k)}^1 \wedge \delta_{(i,k)}^1 = \delta_{(j,k)}^1) \wedge \\ & \bigwedge_{i \sim_{Rj}} (q_{(k,i)}^1 = q_{(k,j)}^1 \wedge o_{(k,i)}^1 = o_{(k,j)}^1 \wedge \epsilon_{(k,i)}^1 = \epsilon_{(k,j)}^1) \end{aligned}$$

and:

$$\begin{aligned} & \bigwedge_{i=1}^n E(f_i^2, \alpha_i^2) \wedge G^{\Sigma_2}(f^2, g^2, b^2, \gamma^2) \wedge \bigwedge_{i=1}^n e_i = \lambda_i^2 f_i^2 \wedge D(f^2, p^2, q^2, h, m^2, o^2, \delta^2, \epsilon^2, \alpha^2) \wedge \\ & B(e, h, \mu^2) \wedge S(\alpha^2, \gamma^2, \delta^2, \epsilon^2, b^2, m^2, o^2, \lambda^2, \mu^2, a) \wedge \bigwedge_{i \sim_{Sj}} (f_i^2 = f_j^2 \wedge \alpha_i^2 = \alpha_j^2) \wedge \end{aligned}$$

$$\begin{aligned} & \bigwedge_{i \sim_S k} \bigwedge_{j \sim_S l} (g_{(i,j)}^2 = g_{(k,l)}^2 \wedge b_{(i,j)}^2 = b_{(k,l)}^2 \wedge \gamma_{(i,j)}^2 = \gamma_{(k,l)}^2) \wedge \\ & \bigwedge_{i \sim_S j} (p_{(i,k)}^2 = p_{(j,k)}^2 \wedge m_{(i,k)}^2 = m_{(j,k)}^2 \wedge \delta_{(i,k)}^2 = \delta_{(j,k)}^2) \wedge \\ & \bigwedge_{i \sim_S j} (q_{(k,i)}^2 = q_{(k,j)}^2 \wedge o_{(k,i)}^2 = o_{(k,j)}^2 \wedge \epsilon_{(k,i)}^2 = \epsilon_{(k,j)}^2) \end{aligned}$$

where the D in the first formula is over $\Delta_1 \cup \Delta_2$ and the D in the second formula is over $\Delta'_1 \cup \Delta'_2$. Note that $\mathbf{p}(e_i) = \alpha_i^1 = \alpha_i^2 = \alpha$ for $1 \leq i \leq n$ and so $\alpha^1 = \alpha^2$. Similarly we obtain that $\mu^1 = \mu^2$. For each $1 \leq i \leq n$, as $\mathbf{p}(f_i^1) = \mathbf{p}(f_i^2)$ there is $\eta_i \in \mathbb{F}[N]$ such that $f_i^2 = \eta_i f_i^1$. So we carry out the transformation $f_i^1 \mapsto f_i^2$ and the elements $g^1, \gamma^1, \lambda^1, p^1, q^1, \delta^1, \epsilon^1$ also get transformed so that \tilde{R} still holds (by the $\mathbb{F}[N]$ -invariance of R) of the new elements. So for $i \in I_R$, we can assume that $f_i^1 = f_i^2$. By a symmetrical argument, we can assume that $f_i^1 = f_i^2$ for $i \in I_S$. It remains to prove that we can assume $f_i^1 = f_i^2$ for $1 \leq i \leq n$.

Suppose that $I_R \cup I_S \subseteq \{1, \dots, n-1\}$ and that we have (by induction) $f_i^1 = f_i^2$ for $1 \leq i \leq n-1$. There is some $k \in I_R$ such that $k \sim_R l$ and there is some $k \in I_S$ such that $n \sim_S l$. So we get that $f_n^1 = f_k^1 = f_k^2$ and $f_n^2 = f_l^2 = f_l^1$. Also we get that $\alpha_l = \alpha_n = \alpha_k$ and so there is $\eta \in \mathbb{F}[N]$ such that $f_n^1 = \eta f_n^2$. Put $J_l = \{i \leq n : f_i^1 = f_l^1\}$ and $J'_l = \{i \leq n : f_i^2 = f_l^1\}$. Then $n \in J'_l$ but $n \notin J_l$. So we apply the transformations $f_i^1 \mapsto \eta f_i^1$ for $i \in J_l$ and $f_j^2 \mapsto \eta f_j^2$ for $j \in J'_l$ (the reason we do this is because $f_n^2 = f_l^1$ implies that $f_n^1 = \eta f_n^2 = \eta f_l^1$ and to preserve the respective equalities, we also have to transform every f_i^1 such that $f_i^1 = f_l^1$ by η). By the $\mathbb{F}[N]$ -invariance of R and S we can therefore assume that $f_n^1 = f_n^2$ as required.

Suppose that $(i, j) \in \Sigma_1$. Then the following conjunct holds:

$$\begin{aligned} & \exists c^{(i,j)} \left(\bigwedge_{k=1}^{n(i,j)} (c_k^{(i,j)})^2 = \pi(f_i + k) \wedge \prod_{k=1}^{n(i,j)} c_k^{(i,j)} = b_{(i,j)}^1 \wedge \right. \\ & \left. \bigwedge_{(i,j) \in \Sigma_1} (E(g_{(i,j)}^1, \alpha_j) \wedge \mathbf{a}^{n(i,j)} f_i^1 = b_{(i,j)}^1 g_{(i,j)}^1 \wedge (\mathbf{a}^\dagger)^{n(i,j)} g_{(i,j)}^1 = b_{(i,j)}^1 f_i^1 \wedge g_{(i,j)}^1 = \gamma_{(i,j)}^1 f_i^1) \right) \end{aligned}$$

As we are assuming that $f_i^1 = f_i^2$ we can add (i, j) to Σ_2 (otherwise, if $(i, j) \in \Sigma_2$ already, then $b_{(i,j)}^1 = b_{(i,j)}^2$, $g_{(i,j)}^1 = g_{(i,j)}^2$ and $\gamma_{(i,j)}^1 = \gamma_{(i,j)}^2$). So we can assume that $(i, j) \in \Sigma_2$. By symmetry, we can take $\Sigma_1 = \Sigma_2$. Analogous arguments will give that we can take $\Delta_1 = \Delta'_1$ and $\Delta_2 = \Delta'_2$. The formula stated in the lemma now holds. \square

Suppose that $\exists fR$ and $\exists fS$ define two basic closed sets. We obtain formulas \tilde{R}, \tilde{S} where $\exists fR$ is equivalent to \tilde{R} and $\exists fS$ is equivalent to \tilde{S} . By the above lemma, $\tilde{R} \wedge \tilde{S}$ is equivalent to \tilde{T} . One can transform \tilde{T} into a general core formula by first picking a set of representatives for \sim_{RS} . We then pick an enumeration for the first n e -variables as $\{e_{ij} : 1 \leq i, j \leq s\}$ where for two tuples (i, j) and (k, l) , $i = k$ if and only if (i, j) corresponds to some m , (k, l) corresponds to some m' in the old enumeration, and $m \sim_{RS} m'$. Pick f_i for each coset representative and delete the remaining f_i and $g_{(i,j)}, p_{(i,j)}, q_{(i,j)}$ in accordance with the equivalences corresponding to \sim_{RS} . After some re-enumeration we get a general core formula $\exists fT$ where T is:

$$\begin{aligned} & (R \wedge S)(\alpha, \gamma, \delta, \epsilon, b, m, o, \lambda, \mu, a) \wedge \bigwedge_{i \sim_{RS} j} (\alpha_i = \alpha_j) \wedge \\ & \bigwedge_{i \sim_{RS} k} \bigwedge_{j \sim_{RS} l} b_{(i,j)} = b_{(k,l)} \wedge \gamma_{(i,j)} = \gamma_{(k,l)} \wedge \end{aligned}$$

$$\bigwedge_{i \sim_{RS} j} m_{(i,k)} = m_{(j,k)} \wedge \delta_{(i,k)} = \delta_{(j,k)} \wedge \\ \bigwedge_{i \sim_{RS} j} o_{(k,i)} = o_{(k,j)} \wedge \epsilon_{(k,i)} = \epsilon_{(k,j)})$$

One sees that this is $\mathbb{F}[N]$ -invariant. For a basic closed set \hat{R} , we associate to \hat{R} a canonical basic closed set \hat{P} (obtained as that basic closed subset corresponding to the finest partition of variables by repeated applications of the previous lemma) such that $\hat{R} = \hat{P}$. We call $P(\mathbb{F})$ the **variety associated with \hat{P}** . For canonical P , we define:

$$\dim(\hat{P}) := \dim(P(\mathbb{F}))$$

For constructible S , we define the dimension of S to be the dimension of its closure.

Theorem 3.1. *The structure Q_N and its substructure QHO_N are Zariski geometries, i.e. dimension notion introduced satisfies the properties of good dimension and that (SP), (sPS) and (EU) hold (providing that the algebraically closed field \mathbb{F} is uncountable). Furthermore, QHO_N is a one-dimensional complete Zariski geometry.*

Proof. Exactly as in [Zil2], which we do not replicate: all statements that need to be verified translate to corresponding statements about algebraic varieties. In particular, it follows that Q_N is one-dimensional. That QHO_N is a Zariski geometry follows from the observation that QHO_N is a definable substructure of Q_N (so the two structures are inter-definable) with closed one-dimensional irreducible universe and closed induced defining relations. \square

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