Perfect infinities and finite approximation

Boris Zilber
University of Oxford

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In this paper we present an analysis of uses of *infinity* in "applied mathematics", by which we mean mathematics as a tool for understanding the real world (whatever the latter means). This analysis is based on certain developments in Model Theory, and lessons and question related to these developments.

Model theory occupies a special position in mathematics, with its aim from the very outset being to study real mathematical structures from a logical point of view and, more ambitiously, to use its unorthodox methods and approaches in search of solutions to problems in core mathematics. Model-theorists made an impact and gained experience and some deep insights in many areas of mathematics: number theory, various fields of algebra, algebraic geometry, real and complex geometry, the theory of differential equations, real and complex analysis, measure theory. The present author believes that model theory is well-equipped to launch an attack on some prolems of modern physics. This article, in particular, discusses what sort of problems and challenges of physics can be tackled model-theoretically. Another topic of the discussion, in our view intrinsically related to the first one, is the way mathematical infinities arise from finite structures, the concept of limit and its variations.

1 Continuity and its alternatives

1.1 The mathematically best developed form of actual infinity is the notion of a continuous line and a continuous space. As we all well know this concept didn't look indisputable to the ancient Greeks and has only become "intuitively obvious" perhaps since Newton made it a part of his Physics. In fact, the assumption that Newtonian physics takes place in a Euclidean space (and in a smooth form) is a powerful axiom from which most of the physics follows. Modern physics has moved away from the assumption that the space is Euclidean to a space being a manifold, but it is still the same idea of continuity. This causes a lot of trouble, e.g. showing up in non-convergent integral expressions that are dealt with by various heuristic tricks having no justification in continuous mathematics (see e.g. a very interesting example of such a calculation in [**?**]). These days a considerable proportion of physicists believe that the assumption of a continuous universe is false. But the formulation of an alternative paradigm will have to wait at least till the solution of the problem of quantum gravity.

So, why continuity is so crucial? The answer lies in the practices of physics. Continuity organises the structure of the physical world and gives it a certain *regularity*, as opposed to a potential chaos. Indeed, if we assume that the trajectory of a particle is smooth we can predict its position in the (near) future based on the knowledge of the past. The alternative seems to be destroying any prospect of having a predictive theory at all.

1.2 A few words about spaces as manifolds: these are patched together from standard canonical pieces of a Euclidean space, such as an interval of a real line, a cube in 3-space and the higher-dimensional versions of these (or their complex analogues). A defining feature of the construction is its high degree of **homogeneity**: *a manifold M looks the same in a small neighbourhood of any of its points.* This can be expressed in more rigorous terms by saying that *there is a local isomorphism between neighbourhoods of any two points of M.*

1.3 In fact, an alternative to continuity does exist and is well-known in mathematics and increasingly being used in physics. This is based on a topology of a different kind, the Zariski topology, which is coarse and in general does not allow metrisation.

Zariski topology enters mathematics and physics in at least two possible ways. One is by formally restricting one's study to the context of algebraic varieties and schemes (thus allowing only charts which are zero-sets of systems of polynomial equations over an abstract field, and maps which are rational). This seems to be unreasonably restrictive to physics although methods developed by Grothendieck's school allow mimicking many notions of analysis in this context and calculate very delicate cohomological invariants that alternatively can be visualised in complex geometry. The other appearance of algebraic geometry comes via complex analysis and the study of complex manifolds. A crucial feature of complex functions is that differentiabily (even just once) implies a very strong form of smoothness – the function becomes analytic. This eventually entails that the behaviour of compact complex manifolds is very close to that of general algebraic varieties. A manifestation of this is the theorem by Riemann stating that any one-dimensional compact complex manifold can be realised as a complex algebraic curve. As a matter of fact this is a corollary of a stronger theorem by Riemann about compact real surfaces with a Riemannian metric (Riemann surfaces): every such surface can be identified with a complex algebraic curve with a metric induced by the metric on the complex numbers. These sort of connections with metric geometry led physicists to appreciate the relevance of algebraic geometry. Recall that Calabi-Yau manifolds which, according to string theory, underpin the structure of physics are objects of the same dual nature. By some definition (slightly stronger than the usual one, see [2]) they turn out to be algebraic.

1.4 The shift from continuous geometry based on the reals towards algebraic geometry and an even more general algebraic and category-theoretic mathematical setting in physics is characteristic of our time. Moreover, there is a growing realisation of the need to reconsider the mathematical constructs at the foundations of physics. In [3] we read: "Indeed, there has always been a school of thought asserting that quantum theory itself needs to be radically changed/developed before it can be used in a fully coherent quantum theory of gravity. This iconoclastic stance has several roots, of which, for us, the most important is the use in the standard quantum formalism of certain critical mathematical ingredients that are taken for granted and yet which, we claim, implicitly assume certain properties of space and time. Such an a priori imposition of spatio-temporal concepts would be a major error if they turn out to be fundamentally incompatible with what is needed for a theory of quantum gravity. A prime example is the use of the continuum which, in this context, means the real and/or complex numbers. "

We suggest a way to approach this issue by aiming at identifying what *logically perfect mathematical structures* should be. Once this is achieved, these should be taken as background structures for physics. A successful definition of logical perfection must entail a degree of regularity for structures enjoying the property that makes them classifiable enough to have a good mathematical theory and flexible enough to model physical systems.

The idea of having a perfect structure as a mathematical basis for physics is not very original. Certainly, the Euclidean space in the background for Newtonian physics is perfect enough a structure. An even more characteritic example is provided by the perfect spheres underlying Ptolemaic astronomy, which later led to the introduction of more sophisticated structures, epicycles, approximating the motion of planets quite well. These may look totally inadequate today but one must keep in mind that the approximation by epicycles is essentially Fourier analysis, fully respectable in modern physics.

We use a model-theoretic approach built around the analysis of interaction of a mathematical structure and its description in a formal language (C.Isham and A.Döring in [3] start by discussing the type of language that can lie at the foundations of physics).

2 In search of logically perfect structures

2.1 The main developments in model theory in recent decades have been centered around stability theory and the core of stability theory is the theory of categoricity in uncountable cardinals.

It is well-known that the first-order description of a structure $\mathbf{M}$ can be (absolutely) categorical if and only if $\mathbf{M}$ is finite, which is a quite trivial situation (unless we put a restriction on the size of the first order axiomatisation), hence the need for a more subtle definition.

A structure $\mathbf{M}$ *is said to be categorical in cardinality* λ *if there is exactly one, up to isomorphism, structure* $\mathbf{M}$ *of cardinality* λ *satisfying the (first-order) theory* $\mathrm{Th}(\mathbf{M})$.

In other words, if we add to $\mathrm{Th}(\mathbf{M})$ the (non first-order) statement that the cardinality of its universe is λ the description becomes categorical. Of special interest is the case of uncountable cardinality λ. In his seminal work [4] on categoricity M.Morley proved that the categoricity of a theory in one uncountable λ implies the categoricity in all uncountable cardinalities, so in fact the actual value of λ does not matter. What we have is a large structure (of an uncountable cardinality) which has a concise (countable) categorical description.

There are purely mathematical arguments towards accepting the above for a definition of perfection. First, we note that the theory of the field of complex numbers (in fact any algebraically closed field) is uncountably categorical. So, the field of complex numbers is a perfect structure, and so are all objects of complex algebraic geometry by virtue of being definable in the field.

It is also remarkable that Morley's theory of categoricity (and its extensions) exhibits strong regularities in models of categorical theories generally. First, the models have to be highly *homogeneous*, in the sense technically different from one discussed for manifolds in 1.2 but similar in spirit (in fact, it follows from results of complex geometry that any compact complex manifold is ω-stable of finite Morley rank; many such manifolds are categorical). Moreover, a notion of *dimension* (the Morley rank) is applicable to definable subsets in categorical structures, which gives one a strong sense of working with curves, surfaces and so on in this very abstract setting. A theorem of the present author states more precisely that an uncountably categorical structure M is either reducible to a 2-dimensional "pseudo-plane" with at least a 2-dimensional family of curves on it (so is non-linear), or is reducible to a linear structure like an (infinite-dimensional) vector space, or to a simpler structure like a G-set for a discrete group G. This led to a Trichotomy conjecture, [9], which specifies that the non-linear case is reducible to algebraically closed fields, that effectively implies that M in this case is an object of algebraic geometry over an algebraically closed field.

There remains the question of whether the restriction to the first-order language is natural. Although we would like to keep this open there are good reasons that the ultimate logical perfection must be first-order. The use of first-order languages was effectively suggested by Hilbert for reasons of its finitarity, the ability (and the restriction) to use only expressions of finite length, which agrees well with practicalities of physics.

Alternatively, we could extend the idea to dealing with very large finite structures as if they are infinite with (uncountably) categorical theories. This could be formalised provided a right notion of approximation is found. We develop this approach in the second part of the paper.

2.2 The arguments above sugest that because of the high degree of logical perfection exhibited by uncountably categorical structures they must already be in the centre of mathematics. Certainly, abstract mathematics being based on pure logic finds a special interest in objects having a concise and complete description. Physics, supposedly based on objective reality, is different in this regard but many recent discussions about the possible interaction between human intelligence and the structure of physical reality as we perceive it (anthropic principle) may suggest a similar relevance of the notion of categoricity to physics. In particular, the notion of *algorithmic compressibility* (the idea and the term comes from [5]) seems to be in close relation to categoricity as expression for "concise and complete". In [6] we read that "the existence of regularities [in the real world] may be expressed by saying that the world is algorithmically compressible." And further on, "The fundamental laws of physics seem to be expressible as succinct mathematical statements. ... does this fact tell us something important about the structure of the brain, or the physical world, or both?".

2.3 Although we now know that the Trichotomy conjecture is technically false, it is believed to be (in words of D.Marker) "morally true" . The significance and the limits of the conjecture are now understood much better.

E.Hrushovski showed that the Trichotomy conjecture in its full generality is false, producing a series of counter-examples, beginning in [8]. Since then other variations of counter-examples appeared exhibiting various possibilities of how the Trichotomy conjecture may fail. Remarkably, after more than 20 years since [8] Hrushovski's construction is the only source of counterexamples. So, what does this construction demonstrate – the failure of the philosophy behind the conjecture and existence of chaotic, pathological structures consistent with categoricity, or incompleteness of the conjecture itself?

In [10] this author showed that Hrushovski's construction when applied in the right mathematical context produces in fact quite perfect structures. The structure in [10] is an analogue of the field of complex numbers with exponentiation, $(\mathbb{C}, +, \cdot, \exp)$. It turns out that the $L_{\omega_1,\omega}(Q)$-theory of this structure (call it a *perfect exponentiation*) is categorical in uncountable cardinalities, so there is exactly one, up to isomorphism, such structure of cardinality continuum. On the other hand, all comparisons between the structure with perfect exponentiation and the actual complex numbers with exponentiations suggest that the one with perfect exponentiation may be isomorphic to the genuine one. This suggestion taken as a hypothesis has a number of important consequences, including the remarkable Schanuel conjecture that, extended appropriately, covers practically all which is known or conjectured in the transcendental number theory.

Today this pattern of linking Hrushovski's counter-examples to classical analytic structures (based on classical transcendental functions) is supported by more case studies including the study of the Weierstrass $\mathfrak{P}$-function and the modular function j (the j-invariant for elliptic curves). So, one may suggest that although the Trichotomy version of the conjecture is false, there is another, more credible mathematical interpretation of the general principle of "logically perfect structures" with algebraic geometry replaced by its appropriate analytic extension.

In any case there is a feeling that something serious is going on around the principle of logical perfection. The above mentioned categoricity theorem for perfect exponentiation is not trivial. Its proof requires a considerable input from model theory (mainly Shelah's theory of *excellence* for *abstract elementary classes*, [11]), but also a serious amount of results from transcendental and Diophantine number theory. A sense of magic is present when in order to prove the categoricity you identify the need of a number-theoretic fact not known to you prior to this work and you learn from experts that the fact indeed holds, but has been proven just a few years ago. In [12] we proved the equivalence of categoricity for some type of structures (universal covers of abelian varieites) to a conjunction of number-theoretic statements (among these the so called "hard theorem of Serre"). Most of the statements are known, either as facts or as conjectures. Some have been proved just recently.

To sum up this line of developments around categoricity we would like to stress a remarkable phenomenon. The assumption of categoricity led one to a construction (of perfect exponentiation) that is not based on continuity but nevertheless turned out to possess all the features observed for a classical analytic transcendental function and, moreover, predicted other properties that have been independently conjectured.

2.4 Topological structures. The crucial, in this author's view, improvement to the notion of logical perfection has been introduced in [13] by Hrushovski and this author (a similar but weaker notion was already present in the paper [14] by Pillay and Srour). The idea is to account for a topological ingredient in logic, essentially by giving special significance to *positive* formulas, assuming that in the given structure positively definable sets give rise to a (coarse) topology. In [15] and in the second part of this paper such structures are called *topological structures*.

Of course, the syntax of an axiomatisation has always been of importance in logic and must have been part of any notion of logical perfection, but in model theory of the 1960 it was found convenient to abstract away from the syntax of formulas to the framework of Boolean algebras of definable sets. This now needs a correction, especially if one approaches the subject with a view of applications in physics. Clearly, a formulation of a physical low is expected to have the form of an equation rather than its negation. To even start thinking about the idea of *approximation* one needs to assume the possibility that certain type of statement, e.g. equations, can happen to fail in reality but be considered true in approximation.

Now looking back at our examples one can rephrase that model theory of algebraically closed fields becomes algebraic geometry if we pay special attention to positively definable sets, i.e. the sets closed in Zariski topology. This motivated the terminology in [13] where we called a 1-dimensional categorical topological structure a Zariski structure provided the topological dimension agrees with Morley rank and the topology satisfies certain "dimension theorem" which holds in algebaic geometry on smooth varieties: in an n-dimensional space M every irreducible component of the intersection of two closed irreducible sets S_1 and S_2 is of dimension at least $\dim S_1 + \dim S_2 - n$ (we call the latter property *presmoothness* now).

In [15] this definition has been lifted to aribitrary dimensions. Note also that the Zariski topology in [13] and generally in first-order categorical structures where it can be defined is Noetherian. The field with perfect exponentiation of subsection 2.3 can also be treated as a topological structure with a Zariski-type topology which is not Noetherian, and some of key questions about the topology on this field remain unanswered.

2.5 The arguments above suggest a choice for the notion we have been looking for. From now on logically perfect structures will be identified as (Noetherian) Zariski structures.

This agrees with the hierarchy of classes of structures (theories) developed by Shelah's classification theory. Zariski structures rightly can be placed in the centre of the classification picture surrounded by "less perfect" classes, such as classes of ω-stable, superstable, stable theories and so on. The theory of formally real fields has its place in the classification outside the class of stable theories but not very far from it.

Does this classification indicate an order of "importance" of mathematical structures? Certainly not. Real analysis on complex algebraic varieties provides an invaluable insight in the mathematics of purely algebraically defined object. Yet, as far as physics is concerned, there may be good reasons to see certain structures (or certain choice of languages) as basic, and other structures as auxiliary. This would agree with now broadly accepted Heisenberg's programme of basing the theory of quantum physics on "the relationships between magnitudes that are in principle observable".

2.6 By reducing our analysis of logical perfection to Zariski geometries we achieve at least one meaningful gain. Firstly, this class is rich in mathematically significant examples, e.g. compact complex manifolds in their natural language are Zariski geometries. And secondly, this class allows a fine classification theory and, in particular, essentially satisfies the Trichotomy principle (that is the Trichotomy conjecture within the class is proven to be true).

The following is the main classification result by Hrushovski and this author [13]

Theorem. *For a non-linear Zariski geometry* $\mathbf{M}$ *there is an algebraically closed field* F *and a nonconstant Zariski-continuous "meromorphic" function* $\psi : \mathbf{M} \to \mathrm{F}$.

In particular, if $\dim M = 1$ *then there is a smooth algebraic curve* C_M *and a Zariski-continuous finite covering map*

$$p : \mathbf{M} \to C_M(\mathrm{F}),$$

the image of any relation on $\mathbf{M}$ *is just an algebraic relation on* C_M.

The remarkable message of this theorem is that the purely logical criteria that lead to the definition of Zariski geometries materialise in an, in fact, uniquely defined by $\mathbf{M}$ algebraically closed field F. The proof effectively reconstructs main ingredients of algebraic geometry starting from the most abstract ones and leading consequently to the reconstruction of the field itself. Note that the linear case which is not covered by the theorem, is reasonably classifiable, although the full account of this case has not been given yet.

The word "essentially" in the reference to the Trichotomy principle above is to indicate that the reduction to algebraic geometry is not as straightforward as one imagined when Zariski geometries were first introduced. Indeed, there is an example of *non-classical* Zariski geometries $\mathbf{M}$ of dimension 1 which is not reducible to an algebraic curve, but is only "finite over" the algebraic curve C_M. In other words, $\mathbf{M}$ can be obtained by "inserting" a finite structure over each point of C_M in some uniform way, but so that the construction is not reducible to a direct product in any sensible form.

Note that one-dimensional Zariski geometries that originate from compact complex manifolds are classical also in the sense above due to the classification of compact Riemann surfaces; they all can be identified as complex algebraic curves, that is $\mathrm{F} = \mathbb{C}$ and the covering map $p : \mathbf{M} \to C_M(\mathrm{F})$ is the identity.

2.7 The defect of non-classicality seemed insignificant in the beginning, partly because it didn't affect a number of applications that followed and partly because "finite" sounds almost as "trivial" in model theory.

But the situation is much more interesting if one tries to understand the non-classical examples from the geometer's and even the physicst's point of view.

The most comprehensive modern notion of a *geometry* is based on the consideration of a *co-ordinate algebra* of the geometric object. The classical meaning of a coordinate algebra comes from the algebra of *co-ordinate functions* on the object, that is functions $\psi : \mathbf{M} \to \mathrm{F}$ as in 2.6, of a certain class. The most natural algebra of functions for Zariski geometries seems to be the algebra $\mathrm{F}[\mathbf{M}]$ of Zariski-continuous functions. But in a non-classical case by virtue of construction $\mathrm{F}[\mathbf{M}]$ is naturally isomorphic to $\mathrm{F}[C_M]$, the algebra of Zariski-continuous (definable) functions on the algebraic curve C_M. That is the only geometry which we see by looking into $\mathrm{F}[\mathbf{M}]$ is the geometry of the algebraic curve C_M.

In [16], in order to see the rest of the structure we extended F[**M**] by introducing auxiliary *semi-definable* functions, which satisfy certain *equations* but are not uniquely defined by these equations. The F-algebra $\mathcal{H}(\mathbf{M})$ of semi-definable functions contains the necessary information about **M** but is not canonically defined. On the other hand it is possible to define an F-algebra $\mathcal{A}(\mathbf{M})$ of linear operators on the linear space $\mathcal{H}(\mathbf{M})$ in a canonical way, depending on **M** only. Moreover, using a specific auxiliary function one can define a natural involutive mapping $X \to X^*$ on generators of $\mathcal{A}(\mathbf{M})$ thus defining a weak version of adjoints. We wrote down explicit lists of generators and defining relations for algebras $\mathcal{A}(\mathbf{M})$ for some examples and demonstrated that $\mathcal{A}(\mathbf{M})$ as an abstract algebra with involution contains all the information needed to recover the "hidden" part of structure **M**.

Later studies in [17], [18] and yet unpublished theses of V.Solanki and D.Sustretov confirm that this is a typical situation. There are lessons that one learns from it:

- The class of Zariski geometries extends algebraic geometry over algebraically closed fields into the domain of non-commutative, quantum geometry.
- For large classes of quantum algebras Zariski geometries serve as counterparts in the duality "co-ordinate algebra – geometric object" extending the canonical duality of commutative geometry.
- The non-commutative co-ordinate algebras for Zariski geometries emerge essentially for the same reasons as they did in quantum physics.

2.8 In [16] one more important observation was made. The examples of non-classical Zariski geometries come in uniform families with variations within a family given by the size of the fibre of the covering map p. It is natural to ask what can be seen when the size of the fibre tends to infinity. Is there a well-defined limit structure? For examples studied in [16] it is possible to introduce a discrete metric on the Zariski structures so that there is a well-defined limit structure that was identified as a real differential manifold with a non-trivial gauge field on it.

This example demonstrates that one can study non-classical Zariski objects seeing them "from afar" in quite a classical way, as real metric, even differentiable manifolds. This agrees with the general principle of physics of how quantised theories must behave: *when quantum mechanics is applied to big structures, it must give the results of classical mechanics.*

Along with this the question arises, *what is the appropriate version of approximation (limit)?* The one used in [16] is the Hausdorff limit for sequences of metric space. But to apply the Hausdorff (or Gromov-Hausdorff) limit one needs that the quantum structures already have a natural metric on them. Is it possible that a sequence of non-metrisable structures have a limit structure with a nice metric?

The questions above have also practical significance. As a matter of fact the structures in finite fibres hold key information, in essense they are finite-dimensional representations of the operator algebras involved, see [17]. The problem of limit to a large degree amounts to a choice of discrete (even finite) models of physical processes in question. Providing a solution to this problem one possibly will be solving problems with non-convergent calculations mentioned in 1.1.

In the second part of this paper we introduce a notion of a *structural approximation* that we believe has a potential to give a mathematically rigorous formalisation to often heuristic approximation procedures used by physicists. (A beautiful and honest account by a mathematician attemting to translate the physicist's vision of "matrix algebras converging to the sphere" into a mathematical concept provides the introductory section of [20].)

The notion of structural approximation is closely linked to the idea of treating structures as topological, in topology induced by the choice of the language, see 2.4. The example in [16] mentioned above demonstrate that there exists a possibility that beautiful and mathematically rich differential-geometric structures of physics could be just limits of discrete logically perfect ones. We prove in 5.1 that algebraically closed fields, that shape the "classical part" of a logically perfect structure (see 2.6), are approximable by finite fields. So one can even make a suggestion that structures central to physics have "perfect" finite approximations, or even that they **are** finite. In this regard it is worth mentioning an attempt to build a physical theory based on a large finite field instead of the reals (P. Kustaanheimo and others). A non-trivial argument in 5.1 proves that this wouldn't be possible. The complex numbers is the only locally compact field that can be approximated by finite fields preserving structural properties. One can say that if the physical world is indeed parametrised by a huge finite field, then it would look from afar as an object of complex algebraic geometry. Which agrees well with trends in modern mathematical physics.

In the last subsection of the second part we discuss an open problem regarding structural approximation of compact Lie groups, such as SO(3), by finite groups. We provide some references in literature that links this problem to key issues in quantum field theory.

3 Structural approximation

3.1 Topological structures

Following [15] we consider structures $\mathbf{M}$ endowed with a topology in a language $\mathcal{C}$. We say that the language is topological meaning that it is a relational language which will be interpreted so that any n-ary $P \in \mathcal{C}$ (basic $\mathcal{C}$-predicate) defines a closed subset $P(\mathbf{M})$ of M^n in the sense of a topology on M^n, all $n \in \mathbb{N}$. Not every closed subset of the topology in question is necessarily assumed to have the form $P(\mathbf{M})$, so those which are will be called **$\mathcal{C}$-closed**.

We assume that the equality is closed and all structures in question satisfy the $\mathcal{C}$-theory which ascertains that

- if $S_i \in \mathcal{C}$, $i = 1, 2$, then $S_1 \& S_2 \equiv P_1$, $S_1 \vee S_2 \equiv P_2$, for $P_1, P_2 \in \mathcal{C}$;
- if $S \in \mathcal{C}$, then $\forall x S \equiv P$, for some $P \in \mathcal{C}$;

We say that a $\mathcal{C}$-structure $\mathbf{M}$ is **complete** if, for each $S(x, y) \in \mathcal{C}$ there is $P(y) \in \mathcal{C}$ such that $\mathbf{M} \vDash \exists x S \equiv P$.

Note that we can always make $\mathbf{M} = (M, \mathcal{C})$ complete by extending $\mathcal{C}$ with relations P_S corresponding to $\exists x S$ for all S in the original $\mathcal{C}$. We will call such an extension of the topology **the formal completion** of $\mathbf{M}$.

We say $\mathbf{M}$ is **quasi-compact** (often just **compact**) if $\mathbf{M}$ is complete, every point in M is closed and for any filter of closed subsets of M^n the intersection is nonempty.

Remark The family of $\mathcal{C}$-closed sets forms a basis of a topology, the closed sets of which are just the infinite interestions of filters of $\mathcal{C}$-closed sets (the topology generated by $\mathcal{C}$).

If the topology generated by $\mathcal{C}$ is Noetherian then its closed sets are exactly the ones which are $\mathcal{C}$-closed.

3.2 Structural approximation

Definition. Given a structure $\mathbf{M}$ in a topological language $\mathcal{C}$ and structures $\mathbf{M}_i$ in the same language we say that $\mathbf{M}$ is **approximated** by a family $\{\mathbf{M}_i : i \in I\}$ along an ultrafilter D on I if, for some elementary extension $\mathbf{M}^D \succcurlyeq \prod_D \mathbf{M}_i$, there is a surjective homomorphism

$$\lim : \mathbf{M}^D \to \mathbf{M}.$$

3.3 Proposition. *Suppose every point of* $\mathbf{M}$ *is closed and* $\mathbf{M}$ *is approximated by the sequence* $\{\mathbf{M}_i = \mathbf{M} : i \in I\}$ *for some* I *along an ultrafilter* D *on* I*, such that* $\mathbf{M}^D$ *is saturated. Then the formal completion of* $\mathbf{M}$ *is quasi-compact.*

Proof Consider the $\mathbf{M}_i$ and $\mathbf{M}$ formally completed, that is in the extended toplogy. Note that the given $\lim : \mathbf{M}^D \to \mathbf{M}$ is still a homomorphism in this language, since a homomorphism preserves positive formulas.

Closedness of points means that for every $a \in \mathbf{M}$ there is a positive one-variable $\mathcal{C}$-formula P_a with the only realisation a in $\mathbf{M}$. Under the assumptions for ${}^*\mathbf{M} \succ \mathbf{M}$, setting for $a \in \mathbf{M}$, $i(a)$ to be the unique realisation $\hat{a} \in {}^*\mathbf{M}$ of P_a we get an elementary embedding $i : \mathbf{M} \prec {}^*\mathbf{M}$. Now lim becomes a specialisation onto $\mathbf{M}$. This implies by [22] (see also a proof in [15]) that $\mathbf{M}$ is quasi-compact. $\square$

In agreement with the proposition we will consider only approximations to quasi-compact structures.

3.4 Proposition *Suppose* $\mathbf{M}$ *is a quasi-compact topological* $\mathcal{C}$*-structure and* $\mathbf{N}$ *is an* $|M|$*-saturated* $\mathcal{C}$*-structure such that* $\mathbf{N}$ *is complete and for every positive* $\mathcal{C}$*-sentence* σ

$$\mathbf{N} \vDash \sigma \Rightarrow \mathbf{M} \vDash \sigma.$$

Then there is a surjective homomorphism $\lim : \mathbf{N} \to \mathbf{M}$.

Proof Given $A \subseteq N$, a partial *strong homomorphism* $\lim_A : A \to \mathbf{M}$ is a map defined on A such that for every $a \in A^k$, $\hat{a} = \lim_A a$ and $S(x, y) \in \mathcal{C}$ such that $\mathbf{N} \vDash \exists y S(a, y)$, we have $\mathbf{M} \vDash \exists y\, S(\hat{a}, y)$.

When $A = \emptyset$ the map is assumed empty but the condition still holds, for any sentence of the form $\exists y\, S(y)$. So it follows from our assumptions that $\lim_\emptyset$ does exist.

Claim 1. Suppose for some $A \subseteq N$ there is a partial strong homomorphism $\lim_A : A \to \mathbf{M}$, and $b \in N$. Then $\lim_A$ can be extended to a partial strong homomorphism $\lim_{Ab} : Ab \to \mathbf{M}$.

Proof of Claim. Let $\mathbf{N} \vDash \exists z\, S(a, b, z)$, for $S(x, y, z)$ a positive formula and a a tuple in $\mathbf{N}$. Then $\mathbf{N} \vDash \exists yz\, S(a, y, z))$ and hence $\mathbf{M} \vDash \exists yz S(\hat{a}, y, z))$.

It follows that the family of closed sets in $\mathbf{M}$ defined by $\{\exists z S(\hat{a}, y, z) : \mathbf{N} \models \exists z\, S(a, b, z)\}$ is a filter. By quasi-compactness of $\mathbf{M}$ there is a point, say $\hat{b}$ in the intersection. Clearly, letting $\lim_{Ab} : b \to \hat{b}$, we preserve formulas of the form $\exists z\, S(x, y, z)$. Claim proved.

Claim 2. For $A \subset N$, $|A| \leq |M|$, assume $\lim_A$ exists and let $\hat{b} \in M \setminus A$. Then there is a $b \in N$ and an extension $\lim_{Ab} : b \mapsto \hat{b}$.

Proof. Consider the type over A,

$$p = \{\neg\exists z\, S(a, y, z) : \mathbf{M} \models \neg\exists z\, S(\hat{a}, \hat{b}, z) : \hat{a} = \lim_A a, a \subset A, S \in \mathcal{C}\}.$$

This is consistent in $\mathbf{N}$ since otherwise

$$\mathbf{N} \models \forall y \bigvee_{i=1}^{k} \exists z_i\, S_i(a, y, z_i)$$

for some finite subset of the type. The formula on the right is equivalent to $P(a)$, some $P \in \mathcal{C}$, so

$$\mathbf{M} \models \forall y \bigvee_{i=1}^{k} \exists z_i\, S_i(\hat{a}, y, z_i)$$

$$\mathbf{M} \models \bigvee_{i=1}^{k} \exists z_i\, S_i(\hat{a}, \hat{b}, z_i),$$

the contradiction. Claim proved.

To finish the proof of the proposition consider a maximal partial strong homomorphism $\lim = \lim_A : A \to M$. By Claim 1, $A = N$, so lim is a total map on N. By Claim 2, lim is surjective.□

3.5 Theorem. *Let $\mathbf{M}_i$, $i \in I$, be a family of formally completed topological $\mathcal{C}$-structures. Let $\mathbf{M}$ be a formally complete quasi-compact $\mathcal{C}$-structure. Then the following two conditions are equivalent,*

(i) there is an unltrafilter D on I such that $\lim_D \mathbf{M}_i = \mathbf{M}$,

(ii) for every sentence $P \in \mathcal{C}$ such that $\mathbf{M} \models \neg P$ there is an $i \in I$, $\mathbf{M}_i \models \neg P$.

Proof. (i) implies (ii) since positive formulas are preserved by homomorphisms.

We now prove (ii)⇒(i). For a given sentence $P \in \mathcal{C}$ let

$$I_P = \{i \in I : \mathbf{M}_i \models \neg P\}.$$

Let

$$D_{\mathbf{M}} = \{I_P : \mathbf{M} \models \neg P\}.$$

$D_{\mathbf{M}}$ is a filter. Indeed, every element of $D_{\mathbf{M}}$ is nonempty by (ii). Also, the intersection of two elements of $D_{\mathbf{M}}$ is an element of $D_{\mathbf{M}}$, since $P_1 \vee P_2 \equiv P \in \mathcal{C}$, for any $P_1, P_2 \in \mathcal{C}$, by definition.

Take D to be any ultrafilter on I extending $D_{\mathbf{M}}$. The statement follows from Proposition 3.4.

4 Examples

In this section we assume for simplicity that $\mathbf{M}^D = \prod \mathbf{M}_i/D$.

4.1 Metric spaces

Let $\mathbf{M}$ and $\mathbf{M}_i$ be metric spaces in the language of binary predicates $d_r^{\leq}(x,y)$ and $d_r^{\geq}(x,y)$, all $r \in \mathbb{Q}$, $\quad r \geq 0$, with the interpretation $\mathrm{dist}(x,y) \leq r$ and $\mathrm{dist}(x,y) \geq r$ correspondingly. The sets given by positive existential formulas in this language form our class $\mathcal{C}$.

Proposition. Assume $\mathbf{M}$ is compact and

$$\mathbf{M} = \text{GH-}\lim\nolimits_D \mathbf{M}_i,$$

the Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter D on I. Then

$$\mathbf{M} = \lim_D \mathbf{M}_i$$

Proof By definition, for any n there is an $X_n \in D$ such that $\mathrm{dist}(M_i, M) \leq \frac{1}{n}$, in a space containing both all the M_i for $i \in X_n$ and M. For any $\alpha \in \prod_i M_i$ define $\hat{\alpha}$ to be an element of M^I such that $\hat{\alpha}(i)$ is an element of M at a minimal distance from $\alpha(i)$ (choose one if there is more than one at the minimal distance). Let a_α be the limit point of the sequence $\{\hat{\alpha}(i) : i \in I\}$ along D in M. We define

$$\lim_D \alpha := a_\alpha.$$

It follows from the construction that, for $\alpha, \beta \in \prod_i M_i$,

$$\{i \in I : M_i \vDash d_r(\alpha(i), \beta(i))\} \in D \Rightarrow M \vDash d_r(\lim_D \alpha, \lim_D \beta).$$

4.2 Cyclic groups in profinite topology

Consider the coset-topology on $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. The compactification of $\mathbb{Z}$ is then $\hat{\mathbb{Z}}$, the profinite completion. Choose a non-principal ultrafilter D on $\mathbb{N}$ so that $m\mathbb{N} \in D$ for every positive integer m (a *profinite ultrafilter*).

Claim.

$$\prod_D \mathbb{Z}/n\mathbb{Z} \cong \hat{\mathbb{Z}} \dot{+} \mathbb{Q}^\kappa \dot{+} T, \text{ some cardinal } \kappa \text{ and the torsion subgroup } T \qquad (1)$$

Proof Follows from the Eklof-Fisher classification of saturated models of Abelian groups [23].□

Proposition. *The group $\hat{\mathbb{Z}}$ is approximated by $\mathbb{Z}/n\mathbb{Z}$ in the profinite topology. That is there is a surjective homomorphism*

$$\lim : \prod_D \mathbb{Z}/n\mathbb{Z} \to \hat{\mathbb{Z}}.$$

Proof. Define $\lim : \hat{\mathbb{Z}} \dot{+} \mathbb{Q}^\kappa \dot{+} T \to \hat{\mathbb{Z}}$ to be the projection (with kernel $\mathbb{Q}^\kappa \dot{+} T$).□

As an example, consider the element (sequence) $\gamma(n)$ such that $\gamma(n) = \frac{n}{2}\ \mathrm{mod} n$, all $n \in 2\mathbb{N}$. Then $2\gamma = 0$ in $\prod_D \mathbb{Z}/n\mathbb{Z}$, a torsion element, so $\lim \gamma = 0$.

4.3 The ring of p-adic integers

Consider the sequence of finite rings $\mathbb{Z}/p^n\mathbb{Z}$ and its ultraproduct

$$R_p := \prod_D \mathbb{Z}/p^n\mathbb{Z}$$

over a non-principal ultrafilter. Let $J_p \subset R_p$ be the ideal of divisible elements, that is the maximal ideal with the property $kJ_p = J_p$ for every integer k.

Claim. R_p/J_p is an integral domain.

Indeed, $a \cdot b \in J_p$ if and only if $a \cdot b$ is p^∞-divisible, if and only if a or b is p^∞-divisible, if and only if $a \in J_p$ or $b \in J_p$. Claim proved.

Introduce a metric on $\bar{R}_p = R_p/J_p$ setting the distance $d(a,b) \leq p^{-k}$ if $a - b \in p^k\bar{R}_p$. Then

$$d(a,b) \leq p^{-k} \text{ for all } k \;\text{ iff } a-b \text{ is } p^\infty\text{-divisible iff } a = b.$$

Clearly, the diameter of the metric space is 1 and it follows (using the saturatedness of R_p) that $\bar{R}_p$ is compact in the corresponding topology.

It follows that

$$R_p/J_p \cong \mathbb{Z}_p,$$

the quotient is isomorphic to the ring of p-adic integers.

We thus have proved

Proposition. *The ring $\mathbb{Z}_p$ of p-adic integers is approximated by the finite rings $\mathbb{Z}/p^n\mathbb{Z}$. That is there is a surejective homomorphism of rings*

$$\lim : \prod_D \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p.$$

4.4 Compactified groups

Call a compact topological structure $\mathbf{M}$ a **compactified group** if there is a closed subset $P \subset M^3$ and an open dense subset $G \subset M$ such that the restriction of P to G is a graph of a group operation on G, $P(g_1, g_2, g_3) \equiv g_1 \cdot g_2 = g_3$, and $P \cap G \times M^2$ defines an action of G on M.

We usually write such an $\mathbf{M}$ as $\bar{G}$.

Examples.

(1) The structure $\check{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ with the ternary relation $S(x,y,z)$, defined as the closure of the graph of addition in the metric of the real line (the two-point compactification of $\mathbb{Z}$). By this definition $\models \forall z\, S(-\infty, +\infty, z)$, $\models \forall z\, S(z, +\infty, +\infty)$ and $\models \forall z\, S(z, -\infty, -\infty)$.

(2) The projective space $\mathbf{P}^{n^2}(\mathrm{F})$, for F an algebraically closed field, is a compactified group $\mathrm{GL}_n(\mathrm{F})$ in the Zariski topology of the projective space.

(3) The projective space $\mathbf{P}^n(\mathrm{F})$, for F an algebraically closed field, is a compactified additive group F^n, in the Zariski topology of the projective space, and in the metric topology if $\mathrm{F} = \mathbb{C}$.

In particular, for $n = 1$, example (2) is a 2-point compactification of the multiplicative group of the fields, and (3) is a 1-point compactification of the additive group.

4.5 Cyclic groups in metric topology and their compactifications

The compactification of $\mathbb{Z}$ in the metric topology corresponding to the usual embedding of integers into the Riemann sphere $\mathbf{P}^1(\mathbb{C})$ is obviously $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$, with the addition relation $S(x, y, z)$ (see 4.4(1)) extended to the extra element: $\models \forall x\, S(x, \infty, \infty)$ and $\models \forall z\, S(\infty, \infty, z)$. We still write $x + y = z$ instead of $S(x, y, z)$.

For a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ define a metric as the metric of the regular n-gon with side 1 on the plane, induced by the metric of the plane.

We identify elements of $\prod_D \mathbb{Z}/n\mathbb{Z}$ with sequences $a = \{a(n) \in \mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\}$ modulo D, any given non-principal ultrafilter.

Define

$$\lim a = \begin{cases} m, & \text{if } \{n \in \mathbb{N} : a(n) = m + n\mathbb{Z}\} \in D \\ \infty, & \text{otherwise} \end{cases}$$

In other words, all bounded elements of $\prod_D \mathbb{Z}/n\mathbb{Z}$, which have to be eventually constant, specialise to their eventual value, and the rest go into ∞.

This is a surjective homomorphism onto $\bar{Z}$ in the language $\{S\}$ and so lim also preserves (the topology of) the positively definable subsets. But lim is not a homomorphism in the language for metric, since for an unbounded element $a \in \prod_D \mathbb{Z}/n\mathbb{Z}$ we have $\lim a = \infty = \lim(a + 1)$ but $\models d_1^{\gtrsim}(a, a + 1)$ while $\models \neg d_1^{\gtrsim}(\infty, \infty)$.

The downside of the 1-point compactification is that $\bar{\mathbb{Z}}$ "believes" that every its element is divisible, that is

$$\bar{\mathbb{Z}} \models \forall x \exists y\, x = \underbrace{y + \ldots + y}_{m}$$

as one can always take ∞ for y.

4.6 2-ends compactification of $\mathbb{Z}$.

Consider the additive group $\mathbb{Z}$ with its natural embedding into the reals. A natural compactification of the real line adds two points, $+\infty$ and $-\infty$ with the obvious interpretation. It induces a compactification $\check{\mathbb{Z}}$ of $\mathbb{Z}$,

$$\check{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\},$$

with the graph $S(x, y, z)$ of addition compactified so that

- $\models S(x, -\infty, -\infty)$, for all $x \neq +\infty$;
- $\models S(x, +\infty, +\infty)$, for all $x \neq -\infty$;
- $\models S(+\infty, -\infty, x)$, for any x.

The basic relations of language $\mathcal{C}$ are the relations defined from S by positive $\exists$-formulas.

Note that among the latter relations there are unary predicates, for all $n > 0$,

$$P_n(x) \equiv \exists y\, x = \underbrace{y + \ldots + y}_{n}.$$

Note that, for $n > 1$, $\neg P_n(q)$ holds.

Now we investigate for what ultrafilter D on $\mathbb{N}$ the family of finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, approximates $\check{\mathbb{Z}}$ along D. That is when

$$\lim_D \mathbb{Z}/n\mathbb{Z} = \check{\mathbb{Z}}. \qquad (2)$$

4.7 Proposition. (2) *holds if and only if for any natural number* m,

$$\{n \in \mathbb{N} : m|n\} \in D \tag{3}$$

Proof Suppose the negation of (3) for some m, that is m does not divide n along the ultrafilter. Let $m = m_1 m_2$ such that $m_1|n$ and $(m_2, n) = 1$ for all $n \in X$, some $X \in D$, $m_2 \neq 1$. We may assume $m = m_2$. For all $n \in X$, let u_i, v_i be the integers such that $u_n m + v_n n = 1$. Correspondingly,

$$u_n m \equiv 1 \bmod n.$$

It follows

$$\prod_D \mathbb{Z}/n\mathbb{Z} \models \forall x P_m(x)$$

holds, in contrast with $\check{\mathbb{Z}} \models \neg P_m(1)$. So there is no homomorphism from the ultraproduct onto $\check{\mathbb{Z}}$.

Conversely, suppose (3) holds. Consider the ultrapower ${}^*\mathbb{Z} := \mathbb{Z}^N/D$ as an ordered additive group, $\mathbb{Z} \preccurlyeq {}^*\mathbb{Z}$, $\mathbb{Z}$ is convex subgroup of ${}^*\mathbb{Z}$. Define, for $\eta \in {}^*\mathbb{Z}$,

$$\lim \eta = \begin{cases} +\infty, & \text{if } \eta > \mathbb{Z} \\ -\infty, & \text{if } \eta < \mathbb{Z} \\ m, & \text{if } \eta = m \in \mathbb{Z} \end{cases}$$

This clearly is a homomorphism onto $\check{\mathbb{Z}}$ with respect to S and so all the relations in $\mathcal{C}$.

Note that D is a profinite ultrafilter by (3). By 4.2, factoring by the torsion subgroup we get a surjective group homomorphism

$$\phi : \prod_D \mathbb{Z}/n\mathbb{Z} \to {}^*\mathbb{Z}.$$

Now we use the surjective homomorphism

$$\lim : {}^*\mathbb{Z} \to \check{\mathbb{Z}}$$

constructed above and finally the composition $\lim \circ \phi$ is a requred limit map. □

5 Approximation by some finite structures

5.1 Approximation by finite fields. According to 3.1 we discuss the approximation of a compactification $\bar{K} = K \cup \{\infty\} = \mathbf{P}^1(K)$, when speaking of an approximation of a field K. The standard topology that we will assume for $\bar{K}$ is the topology *generated by the Zariski topology on* $\bar{K}$, that is the smallest quasi-compact topology $\mathcal{T}$ extending the Zariski topology. Equivalently, by [22], these are the fields K such that for any elementary extension ${}^*K \succ K$ there is a specialisation (place) $\pi : {}^*\bar{K} \to \bar{K}$.

Remark One may also consider the two-point compactification $\mathbb{R} \cup \{-\infty, +\infty\}$ of the field of reals. But if this is approximable, then so is $\bar{\mathbb{R}}$, since there exists an obvious surjective homomorphism $\mathbb{R} \cup \{-\infty, +\infty\} \to \bar{\mathbb{R}}$ taking $\pm\infty$ to ∞.

Conjecture. For an infinite field, $\bar{K}$ is quasi-compact iff K is algebraically closed or K is isomorphic to one of the known non algebraically closed locally compact fields: $\mathbb{R}$ or finite extension of $\mathbb{Q}_p$ or $\mathrm{F}_p\{t\}$.

5.2 Proposition. (i) Any algebraically closed field K with respect to the Zariski language is approximable by finite fields.

(ii) No locally compact field, other than algebraically closed, is approximable by finite fields.

Proof (i) $\prod_D M_n = \mathrm{F}$, for M_n finite fields, is a pseudofinite field. Choose M_n and D so that $\mathrm{char}\,\mathrm{F} = \mathrm{char}\,K$. Let ${}^*\mathrm{F} \succ \mathrm{F}$ be a large enough elementary extension.

We will construct a total surjective specialisation $\pi : {}^*\mathrm{F} \to \bar{K} = K \cup \{\infty\}$. Obviously there is a partial specialisation, in fact embedding, of the prime field F_0 of $\mathrm{char}\mathrm{F}$ into K. So we have constructed partial $\pi : {}^*\mathrm{F} \to \bar{K}$. It is easy to extend π to the transcendence basis B of ${}^*\mathrm{F}$ so that $\pi(B) = \bar{K}$, since algebraically independent elements satisfy no nontrivial Zariski closed relation.

Now note that any partial π into $\bar{K}$, for K algebraically closed, can be extended to a total one. This is the case when π is a partial specialisation from $\bar{K}'$ to $\bar{K}$ for K' algrbraically closed, since $\bar{K}$ s quasi-compact (see e.g. [15], Prop. 2.2.7) But ${}^*\mathrm{F} \subseteq K'$ for some algebraically closed field, so the statement follows.

(ii) It is known ([24]) that F is a *pseudo-algebraically closed* field, that is any absolutely irreducible variety C over F has an F-point.

First we are going to prove (ii) for the case $K = \mathbb{R}$, the field of reals.

Claim. The affine curve C given by the equations

$$x^2 + y^2 + 1 = 0; \quad \frac{1}{x^2} + z^2 + 2 = 0$$

is irreducible over $\mathbb{C}$ and so is absolutely irreducible.

Proof. It is well known that $x^2 + y^2 + a = 0$, for $a \neq 0$, with any of the point removed is biregularly isomorphic to $\mathbb{C}$, and so irreducible. For the same reason the subvariety of $\mathbb{C}^2$ given by $\frac{1}{x^2} + z^2 + 2 = 0$ is also irreducible. We also note that the natural embeddings of both varieties into $\mathbf{P}^2$ are smooth.

The curve C projects into (x, y)-plane as the curve C_{xy} given by $x^2+y^2+1 = 0$ and into the (x, z)-plane as the curve C_{xz} given by $\frac{1}{x^2} + z^2 + 2 = 0$.

Suppose towards a contradiction that $C = C_1 \cup C_2$ with C_1 an irreducible curve, $C_1 \neq C$, and C_2 Zariski closed. We denote $\bar{C}$, $\bar{C}_1$ and $\bar{C}_2$ the corresponding closures in the projective space $\mathbf{P}^3$.

Consider the projection $\mathrm{pr}_{xy} : \bar{C}_1 \to \bar{C}_{xy}$. This is surjective and the order of the projection is either 1 or 2. In the second case $\mathrm{pr}_{xy}^{-1}(a) \cap \bar{C}_1 = \mathrm{pr}_{xy}^{-1}(a) \cap \bar{C}$ for all $a \in \bar{C}_{xy}$, so $C = C_1$ and we are left with the first case only. In this case pr_{xy} is an isomorphism between $\bar{C}_1$ and $\bar{C}_{xy}$. It is also clear in this case that C_2 must be a curve, and pr_{xy} also an isomorphism from $\bar{C}_2$ to $\bar{C}_{xy}$. The points of intersection of C_1 and C_2 are the points over $a \in C_{xy}$ where $|\mathrm{pr}_{xy}^{-1}(a) \cap C| = 1$. One immediately sees that this can only be the points where $z = 0$, $x^2 = -\frac{1}{2}$, $y^2 = -\frac{1}{2}$.

We can apply the same arguments to the projection pr_{xz} onto $\bar{C}_{xz}$ and find that the points of intersection of C_1 and C_2 must satisfy $y = 0$, $x^2 = 1$ and $z^2 = -2$. The contradiction. Claim proved.

Now we prove that the existence of a total specialisation $\pi : {}^*\mathrm{F} \to \mathbb{R} \cup \{\infty\}$ leads to a contradiction.

By above there exist a point (x, y, z) in $C({}^*\mathrm{F})$. Then either $\pi(x)$ or $\pi(\frac{1}{x}) \in \mathbb{R}$ (are finite). Let us assume $\pi(x) \in \mathbb{R}$. Then necessarily $\pi(y) \neq \infty$, since $\pi(x)^2 + \pi(y)^2 + 1 = 0$, but the latter contradicts that $x^2 + y^2 \geq 0$ in $\mathbb{R}$. So (ii) for the reals is proved.

Now we prove (ii) for the remaining cases, that is locally compact nonarchimedean valued fields K. If L is a residue field for a valued field K, then the residue map $\bar{K} \to \bar{L}$ is a place. So assuming there is a sujective place ${}^*\mathrm{F} \to \bar{K}$ we get a surjective place ${}^*\mathrm{F} \to \bar{L}$. This is not possible for a PAC-field, by [24], Corollary 11.5.5. □

5.3 Approximation by finite groups. In physics interesting gauge field theories are based on compact Lie groups such as the orthogonal group SO(3) or SU(N). On the other hand, since calculations in this theory and the analytic justification of the theory encounters enormous difficulties, there have been numerous attempts to develop a gauge field theory with finite group, see [25], or earlier [26] where an approximation of SU(3) by its finite subgroups was discussed.

The following, we believe, is crucial.

Problem

1. Is the group SO(3) approximable by finite groups in the group language?

2. More generally, let G be a compact simple Lie group. Is G approximable by finite groups in the group language? Equivalently (assuming for simplicity the continuum hypothesis), is there a sequence of finite groups G_n, $n \in \mathbb{N}$, an ultrafilter D on $\mathbb{N}$ and a surjective group homomorphism from the ultraproduct onto G,

$$\prod_D G_n \to G.$$

Remark 1. This problem has an easy solution (in fact, well-known to physicists) if we are content with G_n to be quasi-groups, that is omit the requirement of associativity of the group operation:

For each n, choose an $\frac{1}{n}$-dense finite subset $G(n) \subset G$ of points. For $a, b \in G(n)$ set $a * b$ to be a point in $G(n)$ which is at a distance less than $\frac{1}{n}$ from the actual product $a \cdot b$ in G. Now set, for $\gamma \in \prod_n G_n$,

$$\lim \gamma = g \text{ iff } \{n \in \mathbb{N} : \ \mathrm{dist}(\gamma(n), g) \leq \frac{1}{n}\} \in D,$$

which is in fact the standard part map. Then clearly

$$\lim(\gamma_1 * \gamma_2) = \lim \gamma_1 \cdot \lim \gamma_2,$$

that is the map is a homomorphism.

Remark 2. By Theorem 3.5 for a given compact Lie group G the problem reduces to proving that for any positive sentence σ in the group language, such that $G \models \neg\sigma$, there is a finite group G_n such that $G_n \models \neg\sigma$.

Note that any compact simple Lie group G is definable in the field of reals in an explicit way, and hence the first order theory of G is decidable. This implies that the list of positive sentences σ such that $G \models \neg\sigma$, is recursive.

Finally, I would like to mention that in the last 2 years the problem was discussed with many people including M.Sapir, J.Wilson, C.Drutu and A.Muranov who made some valuable remarks, but no solution found as yet.

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