Non-commutative geometry and new stable structures

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This paper grew out of an observation that some new stable structures discovered in the 1990' as counterexamples to well-known conjectures in pure model theory might be related to non-commutative geometry.

The general meaning of the conjectures was that "very good", or more technically, very stable structures must be in a certain way reducible to algebraic geometry over algebraically closed fields or to linear structures (Trichotomy conjecture and Algebraicity conjecture for groups, see [Z0]). This proved to be true to some extent (see [HZ]) but still two types of counterexamples signal the necessity to reconsider the connection between model theoretic classification principles and classical mathematics.

The first class of counterexamples shows that nonlinear one-dimensional Zariski geometries are not necessarily algebraic curves. Given a smooth algebraic curve C with big enough group of regular automorphisms one can produce a "smooth" Zariski curve $\tilde{C}$ along with a finite cover $p : \tilde{C} \to C$. $\tilde{C}$ can not be identified with any algebraic curve because the construction produces an unusual subgroup of the group of regular automorphisms of $\tilde{C}$ ([HZ, section 10). The main theorem of [HZ] states that it is the biggest deviation from an algebraic curve that can happen to a Zariski curve. Typical example of an unusual subgroup of a such $\tilde{C}$ is the nilpotent group of two generators $\mathbf{U}$ and $\mathbf{V}$ with the central commutator $\epsilon = [\mathbf{U}, \mathbf{V}]$ of finite order N. So, the defining relations are

$$\mathbf{UV} = \epsilon\, \mathbf{VU}, \quad \epsilon^N = 1.$$

This, of course, hints towards the known structure of non-commutative geometry, the non-commutative (quantum) torus at the Nth root of unity. We call this example T_N.

The other example is of a different nature. B.Poizat constructed in [P] a multiplicative subgroup $\mathcal{G}$ of an algebraically closed field (we may assume this to be the field $\mathbb{C}$ of complex numbers) such that $(\mathbb{C}, +, \cdot, \mathcal{G})$ has ω-stable theory of rank half of that of $\mathbb{C}$ (so called "bad field", related to the Algebraicity conjecture). The present author has shown in [Z2] that, assuming Schanuel's conjecture, one can construct $\mathcal{G}$ by means of real analytic geometry. More specifically one can consider $\mathcal{G}$ of the form $\mathcal{G} = \exp(\alpha\mathbb{Z}) \cdot \exp(\beta\mathbb{R})$, α and β linearly independent over $\mathbb{R}$, $\beta \notin \mathbb{R} \cup \mathrm{i}\mathbb{R}$, and see that $(\mathbb{C}, +, \cdot, \mathcal{G})$ is superstable of dimension half of that of $\mathbb{C}$. We then note that the structure on the quotient $\mathbb{C}^*/\mathcal{G}$ is geometrically the same as what one gets in the quotient

$$T_h^2 = (\mathcal{S} \times \mathcal{S})/L_h$$

of the square of the unit circle $\mathcal{S} \subseteq \mathbb{C}^*$ by a Kronecker foliation L_h (setwise this is the same as the group $\mathbb{R}/\langle 1, \mathrm{h}\rangle$). This is a basic example and motivation of A.Connes [C] for introducing non-commutative geometry.

Of course, one of the biggest challenges in relating non-commutative geometry to model theory comes from the difference in the way objects are represented in each of the approaches. Geometers tend to replace a structure M by the dual object, the algebra $\mathbb{C}[M]$ of functions on M, or even more abstract non-commutative algebra of "observables" which take the role of the algebra of functions. Generally, non-commutative geometry does not assume that one has a reverse procedure of getting a structure back from the algebra of observables. Yet it is desirable to have a manifold-kind structure underlying a given algebra of observables. Yu.Manin makes this point in [Man] I.1.4 as a foundational problem.

In the present paper we undertake a thorough study of both classes of examples. We try to give answers to the following questions:

1. What are the "algebras of functions" for T_N and T_h^2? Can these structures be identified as objects of non-commutative geometry?

2. What is the structure that non-commutative geometry "sees" on T_N and T_h^2?

3. Is there a uniform representation of both types of structures?

By virtue of construction the algebra of Zariski continuous (regular) functions $\mathrm{T}_N \to \mathbb{C}$ is the same as that of $\mathbb{C}^*$, that is $\mathbb{C}[t, t^{-1}]$, so does not reflect enough of the structure T_N. We show that specifically to the structures under question one can introduce the algebra of *semi-definable* functions. These are not uniquely defined but the commutative algebra $\mathcal{H}$ they generate is determined uniquely up to isomorphism. Moreover, uniquely determined is the action on $\mathcal{H}$ of certain linear operators related to the "hidden" structure of T_N. Algebra of these linear operators is the same as that of non-commutative torus at root of unity known to geometers.

One of the semi-definable functions plays a special role in the construction of $\mathbf{U}$ and $\mathbf{V}$, this is the *angular function*

$$\mathrm{ang}_N : \mathbb{C}^* \to \mathbb{C}[N], \qquad N\text{-roots of } 1,$$

satisfying certain conditions. Answering the second question above we show that T_N can be identified with a space of linear functionals $\mathcal{H} \to \mathbb{C}$ of a *positive orientation.* We introduce the orientation in terms of the angular function. Alternatively but equivalently T_N can be identified with the space of N-dimensional irreducible modules of positive orientation over the coordinate algebra.

Then we look for a similar construction that can play a role of the limit structure T_N as N tends to ∞. The usual model-theoretic limit (the ultraproduct) does not quite work here, for the same reasons as the universal cover $\exp : \mathbb{C} \to \mathbb{C}^*$ can not be obtained as the ultraproduct of finite covers $x \mapsto x^N$, $\mathbb{C}^* \to \mathbb{C}^*$. We find a natural construction in terms of the structure of real and complex numbers, dependent on a real parameter h, the *analytic Zariski* structure T_h, which we show to behave as the limit structure in many respects. In particular the irreducible modules are of countable dimension with $\mathbf{U}$-eigenvalues of the form $q^n\mu$, $\;m \in \mathbb{Z}$, $q = \exp 2\pi \mathrm{ih}$, $\mu \in \mathbb{C}^*$, one μ for each module. The corresponding space $\mathcal{H}$ of semi-definable functions on T_h together with the action of $\mathbf{U}$ and $\mathbf{V}$ on it turns out to be a close analogue of the space with an action corresponding to Connes' quantum torus T_h^2. The correspondent angular function gets the form of a function

$$(\mathbb{C}, +) \to \exp(2\pi \mathrm{ih}\mathbb{Z}),$$

behaving similarly to the function $z \mapsto \exp(\mathrm{i}\,\mathrm{Re}\, z)$.

At this point we don't have a full analogy yet, since setwise the space of our irreducible positively oriented modules is $\mathbb{C}/\langle 1, \mathrm{h}\rangle$ rather than $\mathbb{R}/\langle 1, \mathrm{h}\rangle$. Connes specifies, using his $\mathbb{C}^*$-algebras language, that $\mathbf{U}$ and $\mathbf{V}$ must be *unitary* operators. This immediately translates into the fact that the eigenvalues $q^n\mu$ above must lie on the unit circle and so he gets $\mathcal{S}/\langle q\rangle$ while we have $\mathbb{C}^*/\langle q\rangle$. Instead of using the (unstable) $\mathbb{C}^*$-algebras language we note that *the group of regular automorphisms of* T_h *(commuting with* $\mathbf{U}$ *and* $\mathbf{V}$) is exactly the above group $\mathcal{G} = \exp(2\pi \mathrm{ih}\mathbb{Z} + \beta\mathbb{R})$. This implies that the action of $\mathbf{U}$ and $\mathbf{V}$ is well-defined on the quotient $\mathbb{C}^*/\mathcal{G}$ which is definable in our T_h and is representing Connes' T_h^2.

We hence found a way to represent uniformly our T_N's together with Connes' T_h. Moreover, we can see that there exists a *universal object* $\mathcal{U}$ in this uniform representation. Namely, for each $N \in \mathbb{N} \cup \{\mathrm{h}\}$ there is a surjective map

$$e_N : \mathcal{U} \to \mathrm{T}_N$$

which also gives an interpretation of T_N in terms of $\mathcal{U}$.

It is important to mention that the above description of the structures can not be complete without giving a detailed description of the languages involved. In fact there are at least two levels of languages. The basic language is the language of the example in [HZ], and we prove that T_h is superstable in this language (probably is analytic Zariski of dimension 1 see [PZ] and [Z1]).

We also discuss the language which allows the angular function *ang*. The conditions defining *ang* do not constitute a complete theory, so it is natural to choose a complete extension which axiomatises the *existentially closed* structures. In fact such a choice amounts to choosing *ang* in a uniformly random way. We conjecture that under this choice the theory is *supersimple*. This has been proven by D.Evans in a basic case. It seems both feasible and mathematically meaningful to undertake a detailed analysis of the structure of definable sets in the theory, and develop a probabilisitc measure theory on the sets.

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1 Non-algebraic Zariski geometries

1.1 Recall the following theorem C of [HZ].

Theorem *There exist irreducible pre-smooth Zariski structures (in particular of dimension* 1*) which are not interpretable in an algebraically closed field.*

The construction

Let M be an irreducible pre-smooth Zariski structure, $G \leq \mathrm{ZAut}\, M$ acting freely on M and for some $\tilde{G}$ with **finite** H :

$$1 \to H \to \tilde{G} \to^{p_0} G \to 1.$$

Consider a set $S \subseteq M$ of representatives of G-orbits: for each $a \in M$, $G \cdot a \cap S$ is a singleton.

Consider the formal set

$$M(\tilde{G}) = \tilde{M} = \tilde{G} \times S$$

and the projection map

$$p : (g, s) \mapsto p_0(g) \cdot s.$$

Consider also, for each $f \in \tilde{G}$ the function

$$f : (g, s) \mapsto (fg, s).$$

Claim 1. *The structure*

$$(\tilde{M},\ \{f\}_{f\in\tilde{G}},\ p^{-1}(\textit{Zariski relations on } M))$$

is an irreducible pre-smooth Zariski structure, its isomorphism type is determined by M and $\tilde{G}$ only and $\dim \tilde{M} = \dim M$.

Proof. One can use obvious automorphisms of the structure to prove quantifier elimination. The statement of the claim then follows by checking the definitions. The detailed proof is given in [HZ] Proposition 10.1.

Claim 2. *Suppose H does not split, for every proper $G_0 < \tilde{G}$*

$$G_0 \cdot H \neq \tilde{G}.$$

Then, every equidimensional Zariski expansion $\tilde{M}'$ of $\tilde{M}$ is irreducible.

Indeed. Let $C = \tilde{M}'$ is an $|H|$-cover of the variety M, so $\dim C = \dim M$ and C has at most $|H|$ distinct irreducible components, say C_i, $1 \leq i \leq n$. For generic $y \in M$ the fiber $p^{-1}(y)$ intersects every C_i (otherwise $p^{-1}(M)$ is not equal to C).

Hence H acts transitively on the set of irreducible components. So, $\tilde{G}$ acts transitively on the set of irreducible components, so the setwise stabiliser G^0 of C_1 in $\tilde{G}$ is of index n in $\tilde{G}$ and also $H \cap \tilde{G}^0$ is of index n in H. Hence,

$$\tilde{G} = G^0 \cdot H, \text{ with } H \nsubseteq G^0$$

contradicting our assumptions. Claim proved.

Claim 3. $\tilde{G} \leq \mathrm{ZAut}\, \tilde{M}$, that is $\tilde{G}$ is a subgroup of the group $\mathrm{ZAut}\, M$ of Zariski-continuous bijecions of M.

Immediate by the construction.

Lemma. *Suppose M is a rational or elliptic curve (over an algebraically closed field* F *of characteristic zero), H does not split, $\tilde{G}$ is nilpotent and for some big enough integer μ there is a non-abelian subgroup G_0*

$$|\tilde{G} : G_0| \geq \mu.$$

Then $\tilde{M}$ is not interpretable in an algebraically closed field.

Proof First we show.

Claim 4. Without loss of generality we may assume that $\tilde{G}$ is infinite.

Recall that G is a subgroup of the group $\mathrm{ZAut}\, M$ of rational (Zariski) automorphisms of M. Every algebraic curve is birationally equivalent to a smooth one, so G embeds into the group of birational transformations of a smooth rational curve or an elliptic curve. Now remember that any birational transformation of a smooth algebraic curve is biregular. If M is rational then the group $\mathrm{ZAut}\, M$ is $\mathrm{PGL}(2, \mathrm{F})$. Choose a semisimple (diagonal) $s \in \mathrm{PGL}(2, \mathrm{F})$ be an automorphism of infinite order such that $\langle s \rangle \cap G = 1$

and G commutes with s. Then we can replace G by $G' = \langle G, s\rangle$ and $\tilde{G}$ by $\tilde{G}' = \langle \tilde{G}, s\rangle$ with the trivial action of s on H. One can easily see from the construction that the $\tilde{M}'$ corresponding to $\tilde{G}'$ is the same as $\tilde{M}$, except for the new definable bijection corresponding to s.

We can use the same argument when M is an elliptic curve, in which case the group of automorphisms of the curve is given as a semidirect product of a finitely generated abelian group (complex multiplication) acting on the group on the elliptic curve $E(\mathrm{F})$.

Now, assuming that $\tilde{M}$ is definable in an algebraically closed field F' we will have that F is definable in F'. It is known to imply that F' is definably isomorphic to F, so we may assume that $\mathrm{F}' = \mathrm{F}$.

Also, since $\dim \tilde{M} = \dim M = 1$, it follows that $\tilde{M}$ up to finitely many points is in a bijective definable correspondence with a smooth algebraic curve, say $C = C(\mathrm{F})$.

$\tilde{G}$ then by the argument above is embedded into the group of rational automorphisms of C.

The automorphism group is finite if genus of the curve is 2 or higher, so by Claim 4 we can have only rational or elliptic curve for C.

Consider first the case when C is rational. The automorphism group then is $\mathrm{PGL}(2, \mathrm{F})$. Since $\tilde{G}$ is nilpotent its Zariski closure in $\mathrm{PGL}(2, \mathrm{F})$ is an infinite nilpotent group U. Let U^0 be the connected component of U, which is a normal subgroup of finite index. By Malcev's Theorem (see [Merzliakov], 45.1) there is a number μ (dependent only on the size of the matrix group in question but not on U) such that some normal subgroup U^0 of U of index at most μ is a subgroup of the unipotent group

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

this is Abelian, contradicting the assumption that $\tilde{G}$ has no abelian subgroups of index less than μ.

In case C is an elliptic curve the group of automorphisms is a semidirect product of a finitely generated abelian group (complex multiplication) acting freely on the abelian group of the elliptic curve. This group has no nilpotent non-abelian subgroups. This finishes the proof of the Lemma and of the theorem. □

In general it is harder to analyse the situation when $\dim M > 1$ since the group of birational automorphisms is not so immediately reducible to the group of biregular automorphisms of a smooth variety in higher dimensions. But nevertheless the same method can prove the useful fact that the construction produces examples essentially of non algebro-geometric nature.

Proposition (i) Suppose M is an abelian variety, H does not split and $\tilde{G}$ is nilpotent not abelian. Then $\tilde{M}$ can not be an algebraic variety with $p: \tilde{M} \to M$ a regular map.

(ii) Suppose M is the (semi-abelian) variety $(\mathrm{F}^\times)^n$. Suppose also that $\tilde{G}$ is nilpotent and for some big enough integer $\mu = \mu(n)$ has no abelian subgroup G_0 of index bigger than μ. Then $\tilde{M}$ can not be an algebraic variety with $p: \tilde{M} \to M$ a regular map.

Proof (i) If M is an abelian variety and $\tilde{M}$ were algebraic, the map $p: \tilde{M} \to M$ has to be unramified since all its fibers are of the same order (equal to $|H|$). Hence $\tilde{M}$ being a finite unramified cover must have the same unversal cover as M has. So, $\tilde{M}$ must be an abelian variety as well. The group of automorphisms of an abelian variety $\mathcal{A}$ without complex multiplication is the abelian group $\mathcal{A}(\mathrm{F})$. The contradiction.

(ii) Same argument as in (i) proves that $\tilde{M}$ has to be isomorphic to $(\mathrm{F}^\times)^n$. The Malcev theorem cited above finishes the proof.□

Proposition. *Suppose M is an* F*-variety and, in the construction of $\tilde{M}$, the group G is finite. Then $\tilde{M}$ is definable in any expansion of the field* F *by a total linear order.*

In particular, if M is a complex variety, $\tilde{M}$ is definable in the reals.

Proof Extend the ordering of F to a linear order of M and define

$$S := \{s \in M : \ s = \min G \cdot s\}.$$

The rest of the construction of $\tilde{M}$ is definable.□

Remark In other known examples of non-algebraic $\tilde{M}$ (with G infinite) $\tilde{M}$ is still definable in any expansion of the field F by a total linear order.

Problem (i) Classify Zariski structures definable in the reals.

(ii) Classify Zariski structures definable in the reals as a smooth real manifold.

(iii) Find new Zariski structures definable in $\mathbb{R}_{an}$ as a smooth real manifold.

2 A non-algebraic Zariski curve and its coordinate algebra

2.1 Let F be an algebraically closed field of characteristic 0 and N a positive integer. Consider the groups given by generators and defining relations,

$$G = \langle \mathbf{u}, \mathbf{v} : \ \mathbf{uv} = \mathbf{vu} \rangle,$$

$$\tilde{G} = \langle \mathbf{U}, \mathbf{V} : \ [\mathbf{U}, [\mathbf{U}, \mathbf{V}]] = [\mathbf{V}, [\mathbf{U}, \mathbf{V}]] = 1 = [\mathbf{U}, \mathbf{V}]^N \rangle.$$

Let $a, b \in \mathrm{F}^*$ multiplicatively independent.
G acts on $\mathrm{F}^\times$:

$$\mathbf{u} \cdot x = ax, \ \mathbf{v} \cdot x = bx.$$

Taking M to be $\mathrm{F}^\times$ this determines, by 1.1, a presmooth non-algebraic Zariski curve $\tilde{M}$ which from now on we denote T_N.

Since $[\mathbf{U}, \mathbf{V}]$ is a central element, in every representation of $\tilde{G}$ one can replace $[\mathbf{U}, \mathbf{V}]$ by an $\epsilon \in \mathrm{F}$, a primitive root of unity of order N. So, the defining relation for $\tilde{G}$ becomes just

$$\mathbf{VU} = \epsilon \mathbf{UV},$$

or

$$\mathbf{VUV}^{-1}\mathbf{U}^{-1} = \epsilon.$$

The correspondent definition for the covering map $p : \tilde{M} \to M$ then gives us

$$p(\mathbf{U}t) = ap(t), \quad p(\mathbf{V}t) = bp(t). \tag{1}$$

2.2 Semi-definable functions.

Lemma *Given α, β such that $\alpha^N = a, \ \beta^N = b$, one can define bijections*

$$x_k : T_N \to \mathrm{F}^* \quad k = 0, \dots, N-1$$

so that for any $t \in T_N$ the following **functional equations** *are satisfied,*

$$x_k(t)^N = p(t) \tag{2}$$

$$x_k(\mathbf{U}t) = \alpha \epsilon^k x_k(t), \tag{3}$$

$$x_k(\mathbf{V}t) = \beta x_{k+1}(t), \ \ where\ x_N = x_0, \tag{4}$$

$$\frac{x_{k+1}(t)}{x_k(t)} = \frac{x_k(t)}{x_{k-1}(t)}. \tag{5}$$

Proof First, notice that (3),(4) imply

$$x_k([\mathbf{U},\mathbf{V}]^{-1}t) = \epsilon x_k(t), \tag{6}$$

where $[\mathbf{U},\mathbf{V}]^{-1} = \mathbf{U}^{-1}\mathbf{V}^{-1}\mathbf{U}\mathbf{V}$.

To construct the x_k choose randomly an injection $\sqrt[N]{\ }\ :\ \mathrm{F}^\times \to \mathrm{F}^\times$ with the property

$$(\sqrt[N]{w})^N = w.$$

For any $s \in S$ and $t \in \tilde{G}\cdot s$ of the form $t = \mathbf{U}^m\mathbf{V}^n[\mathbf{U},\mathbf{V}]^l \cdot s$, set

$$x_k(\mathbf{U}^m\mathbf{V}^n[\mathbf{U},\mathbf{V}]^l \cdot s) := \alpha^m\beta^n\epsilon^{mk-l}\sqrt[N]{s}.$$

This satisfies (2)-(5).

To see that each x_k is injective consider $t, t' \in T_N$ such that $x_k(t) = x_k(t')$. We then have, by (2), that $p(t) = p(t')$. Hence $t' = ht$ for some $h \in H$, that is for $h = [\mathbf{U},\mathbf{V}]^j$, some $j \in \{0,\dots,N-1\}$. By (6) this is possible only if $j = 0$, that is $t = t'$.

In order to prove that x_k is surjective we need to solve the equation

$$x_k(t) = \mu$$

for any given $\mu \in \mathrm{F}^\times$. Since p is surjective we can find $t' \in T$ such that $p(t') = \mu^N$, and so by (2) we have $x_k(t') = \epsilon^l\mu$, for some $l \in \mathbb{Z}$. Take now $t = [\mathbf{U},\mathbf{V}]^l t'$ and by (6) this solves the equation. □

2.3 Define the **angular function** on F^* as a function $\mathrm{ang} : \mathrm{F}^\times \to \mathrm{F}[N]$, roots of unity of order N.

Set for $\lambda \in \mathrm{F}^*$,

$$\mathrm{ang}(\lambda) = \frac{x_1(t)}{x_0(t)}, \ \text{if}\ \lambda = x_0(t).$$

This is well-defined since x_0 is a bijection.

Acting by $\mathbf{U}$ on t and using (3) we have

$$\mathrm{ang}\,\alpha\lambda = \epsilon\,\mathrm{ang}\,\lambda \tag{7}$$

We also have

$$\operatorname{ang} \epsilon\lambda = \operatorname{ang} \lambda. \tag{8}$$

since by (6)

$$x_0([\mathbf{U}, \mathbf{V}]^{-1}t) = \epsilon x_0(t) = \epsilon\lambda,$$

and at the same time

$$\operatorname{ang}(\epsilon\lambda) = \frac{x_1([\mathbf{U}, \mathbf{V}]^{-1}t)}{x_0([\mathbf{U}, \mathbf{V}]^{-1}t)} = \frac{x_1(t)}{x_0(t)} = \operatorname{ang} \lambda.$$

Finally, suppose $x_1(t) = \lambda$. Then $x_0(\mathbf{V}t) = \beta\lambda$, by (4), and $x_1(\mathbf{V}t) = \beta x_2(t) = \beta\lambda \cdot \operatorname{ang} \lambda$, by (5). Since $\operatorname{ang} \beta\lambda = x_1(\mathbf{V}t) : x_0(\mathbf{V}t)$, we have

$$\operatorname{ang} \beta\lambda = \operatorname{ang} \lambda. \tag{9}$$

Now we consider the structure

$$(\mathrm{F}, +, \cdot, \operatorname{ang}).$$

It is clear that F is partitioned into N 'sectors' using the angular function:

$$P_\delta = \{\mu \in \mathrm{F}^* : \operatorname{ang} \mu = \delta\}.$$

Proposition T_N is definable in $(\mathrm{F}, +, \cdot, \operatorname{ang})$ using parameters α and β. Moreover, $x_0, \ldots, x_{N-1}$ are definable in the structure as well.

Proof Define $T = \mathrm{F}^\times$ as a set, and for any $t \in \mathrm{F}^\times$ set

$$p(t) = t^N, \quad \mathbf{U}t = \alpha\, t, \quad \mathbf{V}t = \beta \operatorname{ang}(t)\, t.$$

We then have

$$t \to^U \alpha\, t \to^V \alpha\beta \operatorname{ang}(\alpha\, t)t = \alpha\beta \operatorname{ang}(t)\, \epsilon\, t \to^{U^{-1}} \beta \operatorname{ang}(t)\, \epsilon\, t \to^{V^{-1}} \epsilon t.$$

That is

$$\mathbf{V}^{-1}\mathbf{U}^{-1}\mathbf{V}\mathbf{U}t = \epsilon\, t$$

so, the group $\tilde{G}$ acts on the T freely.

It is also clear that

$$p(\mathbf{U}t) = ap(t),\ p(\mathbf{V}t) = bp(t),\ p^{-1}(p(t)) = \{[\mathbf{U}, \mathbf{V}]^{-l}t :\ l = 0, \ldots, N-1\}$$

as required by the description of T_N.

Finally, set $x_k(t) := (\operatorname{ang} t)^k \cdot t$. □

From now on we use notation

$$\check{\mathrm{T}}_N := (\mathrm{F}, +, \cdot, \operatorname{ang}).$$

The interpretation of T_N in the proof of the above proposition we will consider canonical, with respect to α and β.

Remark 1 The isomorphism type of T_N defined by means of $\check{\mathrm{T}}_N$ depends on the isomorphism type (so of the cardinality) of the field F with parameters α, β, ϵ only, and not on the choice of the angular function (equivalently P_δ) since by the construction in 1.1 any two structures $\tilde{M}$ with the same $\tilde{G}$ are isomorphic over M.

Corollary Assuming that $\mathrm{F} = \mathbb{C}$ and $a, b \in \epsilon \cdot \mathbb{R}_{>0}$, $\epsilon = \exp 2\pi i/N$, we have that T_N is definable in the reals using parameters $\alpha, \beta \in \mathbb{R}$ and ϵ such that $\alpha^N = a$, $\beta^N = b$.

Proof It is enough to define an angular function with respect to the chosen parameters. Consider

$$P = \{z \in \mathbb{C}^\times : \ \frac{2\pi}{N} > \arg z \geq 0\}.$$

Define

$$P_{\epsilon^k} := \epsilon^k P, \quad k = 0, \ldots, N-1$$

and

$$\operatorname{ang} \lambda := \epsilon^k \text{ iff } \lambda^N \in \epsilon^k P.$$

This satisfies (7)-(9) by our assumptions.□

2.4 Question Consider a structure $\check{\mathrm{T}}_N$ which is existentially closed in the class of structures satisfying (7) - (9). What is the model-theoretic status of the theory of this structure? Is it supersimple?

Remark Before this paper has been finished D.Evans answered this question in positive.

The fact that $\check{\mathrm{T}}_N$ is supersimple has certain methodological significance. There is a common, albeit informal, understanding that simple structures (theories) come basically from stable structures by introducing a 'random noise'. So, one may think of $\check{\mathrm{T}}_N$ as an algebraic curve with a random angular function.

Problem Study the structure of definable subsets on $\check{\mathrm{T}}_N$. Is there a good probabilistic measure theory on $\check{\mathrm{T}}_N$?

2.5 Systems of semi-definable functions.

Denote $\dot\epsilon = [\mathbf{U}, \mathbf{V}]$ and let $\Gamma = \Gamma_N := \langle \dot\epsilon \rangle$ be the subgroup of $\tilde G$. We denote $\phi : \tilde G \to G = \langle \mathbf{u}, \mathbf{v} \rangle$, the canonical embedding of the subgroup into $\mathrm{F}^\times$.

Lemma *Given α, β such that $\alpha^N = a, \ \ \beta^N = b$, one can define functions*

$$x : G \times T_N \to \mathrm{F}^\times$$

so that for any $g \in G$, $x(g, \cdot) : T_N \to \mathrm{F}^\times$ is a bijection and for any $t \in T_N$ the following **functional equations** *are satisfied,*

$$x(g,t)^N = p(t) \tag{10}$$

$$x(g, \mathbf{U}t) = \alpha \phi(g) x(g,t), \tag{11}$$

$$x(g, \mathbf{V}t) = \beta x(g\mathbf{v}, t), \tag{12}$$

$$x(gf,t)x(gf^{-1},t) = x(g,t)^2 \text{ for any } f \in \tilde G. \tag{13}$$

Proof First, notice that (11),(12) imply

$$x(\gamma \dot\epsilon^{-1} t) = \epsilon x(\gamma, t). \tag{14}$$

To construct x choose randomly an injection $\sqrt[N]{\ } : \mathrm{F}^\times \to \mathrm{F}^\times$ with the property

$$(\sqrt[N]{w})^N = w.$$

For any $s \in S$ and $t \in \tilde G \cdot s$ of the form $t = \mathbf{U}^m \mathbf{V}^n [\mathbf{U}, \mathbf{V}]^l \cdot s$, set

$$x(\gamma, t) := \alpha^m \beta^n \phi(\gamma^m) \epsilon^{-l} \sqrt[N]{s}.$$

This satisfies (10)-(13).

To see that, for each γ, $x(\gamma, \cdot)$ is injective consider $t, t' \in T_N$ such that $x(\gamma, t) = x(\gamma, t')$. We then have, by (10), that $p(t) = p(t')$. Hence $t' =$ for some $\delta \in \Gamma$, that is for $\delta = \dot\epsilon^j$, some $j \in \{0, \ldots, N-1\}$. By (14) this is possible only if $j = 0$, that is $t = t'$.

In order to prove that $x(\gamma, \cdot)$ is surjective we need to solve the equation

$$x(\gamma, t) = \mu$$

for any given $\mu \in \mathrm{F}^{\times}$. Since p is surjective we can find $t' \in T$ such that $p(t') = \mu^N$, and so by (10) we have $x(\gamma, t') = \epsilon^l \mu$, for some $l \in \mathbb{Z}$. Take now $t = \dot{\epsilon}^l t'$ and by (14) this solves the equation. □

Define

$$\xi(t) = \frac{x(\gamma\dot{\epsilon}, t)}{x(\gamma, t)}.$$

By (13) this indeed does not depend on γ.

2.6 The space of semi-definable functions.

Let $\mathcal{H}$ be the F-algebra of semi-definable functions on T_N generated by $x_0, \ldots, x_{N-1}, x_0^{-1}, \ldots, x_{N-1}^{-1}$.

Remark $\mathcal{H}$ is determined as a commutative F-algebra uniquely up to isomorphism by its generators $x_0, \ldots, x_{N-1}$ satisfying the relations (2).

We may also regard it as an F-vector space with some linear operators on them.

We define linear operators $\mathbf{U}^*$ and $\mathbf{V}^*$ on $\mathcal{H}$:

$$\begin{array}{l} \mathbf{U}^* : \ \psi(t) \mapsto \psi(\mathbf{U}t), \\ \mathbf{V}^* : \ \psi(t) \mapsto \psi(\mathbf{V}t). \end{array} \tag{15}$$

Obviously these operators are invertible, so $\mathbf{U}^{*-1}$, $\mathbf{V}^{*-1}$ are the inverses. Denote $\tilde{G}^*$ the group generated by the operators $\mathbf{U}^*$, $\mathbf{V}^*, \mathbf{U}^{*-1}$, $\mathbf{V}^{*-1}$.

$\mathcal{H}$ with the action of $\tilde{G}^*$ on it is determined uniquely up to isomorphism by the defining relation (2)-(6) and so is independent on the arbitrariness in the choices of $x_0, \ldots, x_{N-1}$.

Finally we notice

Lemma The correspondence $\mathbf{U} \mapsto \mathbf{U}^*$, $\mathbf{V} \mapsto \mathbf{V}^*$ generates the anti-isomorphism $\tilde{G} \to \tilde{G}^*$ satisfying the property

$$(g_1 g_2)^* = g_2^* g_1^*, \text{ for any } g_1, g_2 \in \tilde{G}.$$

Proof It can easily be seen if we define the pairing

$$\mathcal{H} \times T \to \mathrm{F}, \quad (\psi, t) \mapsto \psi(t).$$

This allows to consider the adjoint action of any $g \in \tilde{G}$ on $\mathcal{H}$ setting $g^*\psi$ as the unique element of $\mathcal{H}$ such that

$$(g^*\psi, t) = (\psi, gt), \text{ for all } t \in T.$$

We can immediately identify that this definition extends (15). The desired formula follows.□

2.7 Let $\mathrm{Max}(\mathcal{H})$ be the space of maximal ideals of the commutative algebra $\mathcal{H}$.

Lemma 1 $\mathrm{Max}(\mathcal{H})$ consists of ideals $I_{\bar{\mu}}$, $\bar{\mu} = \langle \mu_0, \dots, \mu_{N-1}\rangle$, $\mu_0^N = \dots = \mu_{N-1}^N$,

$$I_{\bar{\mu}} = \langle (x_0 - \mu_0), \dots, (x_{N-1} - \mu_{N-1})\rangle.$$

Proof This is a standard fact of commutative algebra.□

Assuming F is endowed with an angular function $\mathrm{ang} : \mathrm{F}^\times \to \mathrm{F}[N]$ we call $\bar{\mu}$ as above **oriented positively** if $\mu_k = \mathrm{ang}(\mu_0)^k \cdot \mu_0$. Correspondingly, we call an ideal $I_{\bar{\mu}}$, oriented positively if $\bar{\mu}$ is.

$\mathrm{Max}^+(\mathcal{H})$ will denote the subspace of $\mathrm{Max}(\mathcal{H})$ consisting of positively oriented ideals I.

Lemma 2 $\bar{\mu}$ is positively oriented if and only if

$$\langle \mu_0, \dots, \mu_{N-1}\rangle = \langle x_0(t), \dots, x_{N-1}(t)\rangle,$$

for some $t \in T$.

Proof Indeed, since x_0 is a bijection, there is $t \in T$ such that $x_0(t) = \mu_0$. Now apply the definition of natural angular function of 2.3.□

2.8 Lemma

(i) There is a bijective correspondence $\Xi : \ \mathrm{Max}^+(\mathcal{H}) \to T_N$ *between the space of positively oriented maximal ideals and* T_N.

(ii) The action (15) of $\tilde{G}^*$ *on* $\mathcal{H}$ *induces an action on* $\mathrm{Max}(\mathcal{H})$ *and leaves* $\mathrm{Max}^+(\mathcal{H})$ *setwise invariant.*

(iii) The action of $g^* \in \tilde{G}^*$ *on* $\mathrm{Max}(\mathcal{H})$ *(and so on* T_N*) can be identified as*

$$g^* : I_{\langle x_0(t), \dots, x_{N-1}(t)\rangle} \mapsto I_{\langle x_0(g^{-1}t), \dots, x_{N-1}(g^{-1}t)\rangle}.$$

Proof (i). We set

$$\Xi(t) := I_{\bar{\mu}}, \text{ for } \bar{\mu} = \langle x_0(t), \dots, x_{N-1}(t)\rangle.$$

Then $\Xi(t)$ is positively oriented by Remark 2 in 2.7.

Notice that by definition $\bar{\mu}$ is determined uniquely by μ_0. But $x_0 : T_N \to \mathrm{F}^\times$ is bijective, so Ξ is bijective.

(ii)-(iii). For a given $g \in \tilde{G}$, the map $\psi \to g^*\psi$ is an automorphism of the commutative F-algebra $\mathcal{H}$, since $g^*\psi(t) = \psi(gt)$. So, it sends maximal ideals to maximal ideals, namely

$$g : \langle (x_0 - \mu_0), \dots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (x_0^g - \mu_0), \dots, (x_{N-1}^g - \mu_{N-1}) \rangle.$$

Notice that, for the unique $t_\mu \in T_N$ such that $x_0(t_\mu) = \mu_0, \dots, x_{N-1}(t_\mu) = \mu_{N-1}$

$$\langle x_0(\mathbf{U}^{-1}t_\mu), \dots, x_{N-1}(\mathbf{U}^{-1}t_\mu) \rangle = \langle \alpha^{-1}\mu_0, \dots, \alpha^{-1}\epsilon^{1-N}\mu_{N-1} \rangle,$$

by (3). Analogously, by (4)

$$\langle x_0(\mathbf{V}^{-1}t_\mu), \dots, x_{N-1}(\mathbf{V}^{-1}t_\mu) \rangle = \langle \beta^{-1}\mu_{N-1}, \beta^{-1}\mu_0, \dots, \beta^{-1}\mu_{N-2} \rangle.$$

So, by Lemma 2.7.2 both tuples on the right-hand side are positively oriented.

Now notice that by (3) and (4)

$$\mathbf{U} : \langle (x_0 - \mu_0), \dots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (\alpha x_0 - \mu_0), \dots, (\alpha\epsilon^{N-1}x_{N-1} - \mu_{N-1}) \rangle =$$
$$\langle (x_0 - \alpha^{-1}\mu_0), \dots, (x_{N-1} - \alpha^{-1}\epsilon^{1-N}\mu_{N-1}) \rangle =$$
$$= \langle (x_0 - x_0(\mathbf{U}^{-1}t)), \dots, (x_{N-1} - x_{N-1}(\mathbf{U}^{-1}t)) \rangle$$

and

$$\mathbf{V} : \langle (x_0 - \mu_0), \dots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (\beta x_1 - \mu_0), \dots, (\beta x_0 - \mu_{N-1}) \rangle =$$
$$\langle (x_0 - \beta^{-1}\mu_{N-1}), \dots, (x_{N-1} - \beta^{-1}\mu_{N-2}) \rangle =$$
$$= \langle (x_0 - x_0(\mathbf{V}^{-1}t)), \dots, (x_{N-1} - x_{N-1}(\mathbf{V}^{-1}t)) \rangle.$$

This proves that the image of positive $I_{\bar{\mu}}$ under $\mathbf{U}$ and $\mathbf{V}$ is positive. Hence the image under the action of any $g \in \tilde{G}$ is positive, and we have (ii). The above also shows that the action induced by Ξ is anti-isomorphic to the original action and so proves (iii).$\square$

2.9 We may also treat T as the space of F-linear functionals $\mathcal{H} \to \mathrm{F}$ defined by the pairing of 2.6,

$$\mathcal{H}_T^* = \{F_t : \ \psi \mapsto (\psi, t), \quad t \in T\}.$$

Obviously, the kernel of a nonzero functional is a maximal ideal. Moreover,

$$\ker F_t = \{\phi \in \mathcal{H} : \ (\phi, t) = 0\} = I_{\langle x_0(t), \dots, x_{N-1}(t)\rangle}.$$

We also denote $\ker F_t := I^t$.

We call a linear functional F on $\mathcal{H}$ **positive** if $\ker F$ is a positive maximal ideal.

Proposition

(i) The correspondence

$$t \mapsto F_t$$

between T and the space $\mathcal{H}_+^*$ of positive linear functionals on $\mathcal{H}$ is bijective.

(ii) The correspondence transfers isomorphically the natural action of $\tilde{G}$ on T to a natural action of $\tilde{G}$ on $\mathcal{H}_+^*$.

(iii) Consider also the commutative algebra $\mathcal{H}_0$ generated by $p(t)$ and, for each linear functional F_t its restriction F_t^0 on $\mathcal{H}_0$. Then, for any $t_1, t_2 \in T$,

$$F_{t_1}^0 = F_{t_2}^0 \text{ iff } p(t_1) = p(t_2) \text{ iff } F_{t_1} = \epsilon^j F_{t_2}, \text{ for some } j \in \{0, \dots, N-1\},$$

and the correspondence

$$F_t^0 \mapsto p(t)$$

is a bijection between the space $\mathcal{H}_0^*$ of all linear functionals of the form F_t^0 and $\mathrm{F}^\times$.

Proof Let $I \in \mathrm{Max}(\mathcal{H})$. To any such I canonically corresponds the functional

$$F^I : \psi \mapsto \lambda \in \mathrm{F}, \text{ such that } (\psi - \lambda) \in I.$$

We write

$$F(\psi) := \{F, \psi\}.$$

Now, in case $I = I^t = I_{\langle x_0(t), \dots, x_{N-1}(t)\rangle}$ we see that

$$\{F^I, \psi\} = \psi(t) = (\psi, t). \tag{16}$$

The latter establishes the required bijection between $\mathcal{H}_T^*$ and T_N. On the other hand, since functionals of $\mathcal{H}_T^*$ are in bijective correspondens with positive ideals, by Lemma 2.7.2, $\mathcal{H}_T^* = \mathcal{H}_+^*$, the set of all positive functionals. This proves (i).

(ii). Given $F \in \mathcal{H}^*$ and $f \in \tilde{G}^*$ define f^*F as the unique functional such that

$$\{f^*F, \psi\} = (F, f\psi).$$

Then by dualities we have the isomorphism of group with actions on T and $\mathcal{H}^+$ correspondingly

$$g \in \tilde{G} \mapsto g^{**} \in \tilde{G}^{**},$$

$$(\psi, gt) = (g^*\psi, t) = \{F^t, g^*\psi\} = \{g^{**}F^t, \psi\}.$$

(iii). It is immediate from definitions that if F^t evaluates x_0 as $\mu \in \mathrm{F}^\times$, then the function p (as an element of $\mathcal{H}$) is evaluated as μ^N. The statement follows. □

2.10 Comments

1.The space $\mathcal{H}$ is an analogue of the space $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ of all Schwartz functions $\mathbb{R}^2 \to \mathbb{C}$ decaying at infinity along with all its derivatives faster than $\frac{1}{|x|^n}$, any n (see A.Connes),.

2. In mathematical physics linear functionals on certain Hilbert spaces are called **states**.

Assume for a moment that $\mathcal{H}$ is an inner product space. Then any $F \in \mathcal{H}^*$ can be identified with the orthogonal complement $I^\perp$ of the maximal ideal corresponding to F. This is a one-dimensional subspace of $\mathcal{H}$. This provides another version of the notion of states.

3. Even though the present definition of $\mathcal{H}$ considers it a finitely generated commutative ring, it can not treat it as the coordinate ring of an algebraic variety since we consider *positively oriented* ideals only.

We used in 2.9 the natural pairing $\mathcal{H}\times T \to \mathrm{F}$ and the existence of enough functionals on the *linear space* $\mathcal{H}$.

4. Despite the fact that T is in a bijective correspondence with a subset $\mathcal{H}^*_+$ of the space of functionals we can not induce the additive structure on T since $\mathcal{H}^*_+$ is not closed under addition.

3 The limit case

We introduce and study here a structure $\check{\mathrm{T}}_\infty$ which can be seen as the limit version of $\check{T}_N$. It would be important in our view to formulate (and prove) the exact meaning of the transition $N \to \infty$ but we only draw here parts of the possible picture towards this aim.

3.1 Let $\alpha, \beta \in \mathbb{C}^\times$, $\alpha\mathbb{R} + \beta\mathbb{R} = \mathbb{C}$. Set, for $w \in \mathbb{C}$, the α-β - *decomposition* to be the uniquely determined decomposition

$$w = w_a\alpha + w_b\beta, \quad w_a, w_b \in \mathbb{R}.$$

Let $i_a, i_b \in \mathbb{R}$ be the coordinates of the decomposition

$$\mathrm{i} = i_a\alpha + i_b\beta, \text{ here and below } \mathrm{i}^2 = -1.$$

We also choose a real number h and assume that 1, $2\pi i_a$ and $2\pi i_a\mathrm{h}$ are linearly independent over $\mathbb{Q}$.

We define an additive α-β-version of the angular function, which we call **band**

$$\mathrm{bd}_h : \mathbb{C} \to 2\pi\mathrm{ih}\mathbb{Z}, \text{ fixed } \mathrm{h} \in \mathbb{R} \setminus \mathbb{Q}$$

as follows.

First we define the function $r \mapsto [r]_h$ from $\mathbb{R}$ to $\mathbb{Z}$, the **pseudo-integer part of** r with the properties, for all $r \in \mathbb{R}$,

$$[0]_h = 0, \quad [r+1]_h = [r]_h + 1, \tag{17}$$

$$[r + 2\pi i_a]_h = [r]_h, \tag{18}$$

$$[r + 2\pi i_a\mathrm{h}]_h = [r]_h \tag{19}$$

Example Consider a direct sum decomposition

$$\mathbb{R} = \mathbb{R}' \dot{+} 2\pi i_a\mathbb{Q} \dot{+} 2\pi i_a\mathrm{h}\mathbb{Q}, \text{ some subgroup } \mathbb{Q} < \mathbb{R}' < \mathbb{R},$$

and set, for all $r' \in \mathbb{R}'$, $c \in \mathbb{Q}$,

$$[r' + c_1 \cdot 2\pi i_a + c_2 \cdot 2\pi i_a\mathrm{h}]_h := [r' + (c_1 - [c_1]) \cdot 2\pi i_a + (c_2 - [c_2]) \cdot 2\pi i_a\mathrm{h}],$$

$[\,\cdot\,]$ the usual integer part of a real number. This satisfies (17)-(19).

Set

$$\mathrm{bd}_h\, w := 2\pi \mathrm{ih}\, [w_a]_h.$$

We have then, by definition,

$$\mathrm{bd}_h(r\beta + w) = \mathrm{bd}_h\, w, \text{ for every } r \in \mathbb{R}; \tag{20}$$

$$\mathrm{bd}_h(w + 2\pi \mathrm{i}) = \mathrm{bd}_h(w); \tag{21}$$

$$\mathrm{bd}_h(w + 2\pi \mathrm{ih}) = \mathrm{bd}_h\, w. \tag{22}$$

By (17),

$$\mathrm{bd}_h(\alpha + w) = 2\pi \mathrm{ih} + \mathrm{bd}_h\, w. \tag{23}$$

Set,

$$\tilde{\mathbf{U}}: \ w \mapsto \alpha + w,$$

$$\tilde{\mathbf{V}}: \ w \mapsto \beta + w + \mathrm{bd}_h\, w.$$

We have

$$w \mapsto^U \alpha + w \mapsto^V \alpha + \beta + w + \mathrm{bd}_h(\alpha + w) = \alpha + \beta + w + \mathrm{bd}_h\, w + 2\pi \mathrm{ih} \mapsto^{U^{-1}}$$

$$\mapsto^{U^{-1}} \beta + w + 2\pi \mathrm{ih} + \mathrm{bd}_h\, w \mapsto^{V^{-1}} 2\pi \mathrm{ih} + w.$$

That is

$$\tilde{\mathbf{V}}^{-1}\tilde{\mathbf{U}}^{-1}\tilde{\mathbf{V}}\tilde{\mathbf{U}}\, w = w + 2\pi \mathrm{ih}, \tag{24}$$

3.2 Define the additive subgroup of $\mathbb{C}$

$$\mathcal{A}_h = \beta\mathbb{R} + 2\pi \mathrm{ih}\mathbb{Z} + 2\pi \mathrm{i}\mathbb{Z}.$$

Proposition (i) $\mathcal{A}_h$ is the subgroup of all **periods** of bd_h, that is $a \in \mathbb{C}$ such that $\mathrm{bd}_h(a + w) = \mathrm{bd}_h w$.

(ii) $\mathcal{A}_h$ is exactly the subgroup of shifts $w \mapsto a + w$ of $\mathbb{C}$ which are automorphisms of $(\mathbb{C}, \tilde{\mathbf{U}}, \tilde{\mathbf{V}})$.

(iii) $\mathcal{A}_h$ is definable in $(\mathbb{C}, +, \mathrm{bd}_h)$.

Proof (i). Immediate from (20)- (22). For (ii) notice that $\tilde{\mathbf{U}}(a + w) = a + \tilde{\mathbf{U}}w$, for all $a \in \mathbb{C}$ and

$$\tilde{\mathbf{V}}(a + w) = a + \tilde{\mathbf{V}}w \ \text{ iff } \ a \in \mathcal{A}_h.$$

(iv) Immediate by definitions.□

3.3 We consider here the two-sorted structures

$$\big((\mathbb{C},+,\mathrm{bd}_h),\exp,\mathbb{C}^\times\big) \text{ and } \big((\mathbb{C},+,\mathcal{A}_h),\exp,\mathbb{C}^\times\big)$$

where the second sort $\mathbb{C}^\times$ on the nonzero complex numbers comes with the usual language of all Zariski closed relations.

Obviously the functions $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ are definable in $(\mathbb{C},+,\mathrm{bd}_h)$. Conversely, bd_h is definable in $(\mathbb{C},+,\tilde{\mathbf{V}})$ using parameter β.

Proposition 1 The theory of $((\mathbb{C},+,\mathcal{A}_h),\exp,\mathbb{C}^\times)$ is superstable, provided the Schanuel conjecture is true.

Proof It is easy to see that the statement follows if the expansion of $\mathbb{C}^\times$ with the unary predicate for the subgroup $\mathcal{G}_h = \exp(\mathcal{A}_h) = \exp(2\pi \mathrm{i}h\mathbb{Z}+\beta\mathbb{R})$ is superstable. A stronger theorem, stating ω-stability of the theory, for $\mathcal{G} = \exp(\beta\mathbb{R}+\delta\mathbb{Q})$, $\beta \in \mathbb{C}\setminus(\mathbb{R}\cup \mathrm{i}\mathbb{R})$, $\delta \in \mathbb{R}\setminus 2\pi\mathrm{i}\mathbb{Q}$, was proved in [Z2]. The same proof describes the elementary theory of the structure and yields superstability for the present theory. See also [Z3]. □

Notation $\mathcal{G}_h$ will stand for the subgroup $\exp(\mathcal{A}_h)$ of $\mathbb{C}^\times$.

On the other hand $((\mathbb{C},+,\mathrm{bd}_h),\exp,\mathbb{C}^\times)$ defines the following unstable structure on the sort $\mathbb{C}^\times$.

Denote, for $t = \exp w$,

$$\mathrm{ang}_h t := \exp \mathrm{bd}_h w.$$

By (21) this is well-defined, and by (20),(22) we have analogues of (7)-(9), where $q = \exp 2\pi \mathrm{i}h$,

$$\mathrm{ang}_h qt = \mathrm{ang}_h t,$$
$$\mathrm{ang}_h e^\beta t = \mathrm{ang}_h t,$$
$$\mathrm{ang}_h e^\alpha t = q\cdot \mathrm{ang}_h t.$$

Hence, defining

$$\mathbf{U}: t\mapsto e^\alpha\cdot t, \quad \mathbf{V}: t\mapsto e^\beta\cdot t\cdot \mathrm{ang}_h t,$$

we get

$$\mathbf{VU}t = q\mathbf{UV}t, \text{ for all } t \in \mathbb{C}^\times.$$

It is easy to see that also

$$\mathbf{U} \exp w = \exp \tilde{\mathbf{U}} w, \quad \mathbf{V} \exp w = \exp \tilde{\mathbf{V}} w.$$

We define

$$\check{\mathrm{T}}_h := (\mathbb{C}, +, \cdot, \mathrm{ang}_h).$$

This is an obvious analogue of $\check{\mathrm{T}}_N$ defined in 2.3.

Note that the group $\Gamma_h = \exp 2\pi \mathrm{ih}\mathbb{Z} = \mathrm{ang}_h(\mathbb{C}^\times)$ is definable in $\check{\mathrm{T}}_h$.

The full analogy with $\check{\mathrm{T}}_N$ of 2.3 requires also a definition of p_h. We define

$$p_h : \mathbb{C}^\times \to \mathbb{C}^\times/\Gamma_h,$$

the canonical homomorphism. This agrees with 2.3, moreover in the finite case $\mathbb{C}^\times/\langle \epsilon \rangle$ can be definably identified with $\mathbb{C}^\times$ in the full Zariski language, in particular the whole construction is a Zariski structure (obviously, of finite Morley rank).

We also define the maps $\mathbf{u}$ and $\mathbf{v}$ on $\mathbb{C}^\times/\Gamma_h$ by

$$\mathbf{u}\, p_h(t) := p_h(\mathbf{U}t), \quad \mathbf{v}\, p_h(t) := p_h(\mathbf{V}t),$$

that is

$$\mathbf{u} : \ t \cdot \Gamma_h \mapsto e^\alpha \cdot t \cdot \Gamma_h, \quad \mathbf{v} : \ t \cdot \Gamma_h \mapsto e^\beta \cdot t \cdot \Gamma_h.$$

This is obviously well-defined.

Proposition 2 The group of shifts $t \mapsto gt$ on $\mathbb{C}^\times$ commuting with ang_h (and so with $\mathbf{U}$ and $\mathbf{V}$) is $\mathcal{G}_h$. This group is definable in $\check{\mathrm{T}}_h$. The theory of the structure $(\mathbb{C}, +, \cdot, \mathcal{G}_h, \Gamma_h)$ is superstable.

Proof Essentially the same argument as for Proposition 1. The superstability of the weaker structure $(\mathbb{C}, +, \cdot, \Gamma_h)$ is well-known and follows from *the Lang property* of Γ_h. □

Problems 1. Fix the theory $\mathcal{T}_h^{\mathcal{G}}$ of structures of the form $(\mathrm{F}, +, \cdot, \mathrm{ang}, e_a)$, ($e_a$ a constant) saying that

$$(\mathrm{F}, +, \cdot, \mathrm{Aut}(\mathrm{ang}), \mathrm{ang}(\mathrm{F}^\times), e_a) \equiv (\mathbb{C}, +, \cdot, \mathcal{G}_h, \Gamma_h, e^\alpha)$$

(where Aut(ang) is the group of shifts of $\mathrm{F}^\times$ commuting with ang, and $\mathrm{ang}(\mathrm{F}^\times)$ is the image under ang)

and

$$\forall t \in \mathrm{F}^\times \ \mathrm{ang}\, g \cdot t = q \cdot \mathrm{ang}\, t \text{ iff } g^{-1} e_a \in \mathrm{Aut}(\mathrm{ang}).$$

Consider the class $\dot{\mathcal{T}}_h^{\mathcal{G}}$ of existentially closed models of $\mathcal{T}_h^{\mathcal{G}}$. What is the stability status of completions of $\dot{\mathcal{T}}_h^{\mathcal{G}}$. Are they supersimple?

2. Is $\check{\mathrm{T}}_h$ above based on the band function bd_h given in the Example in 3.1 existentially closed in $\mathcal{T}_h^{\mathcal{G}}$? Is it supersimple?

3.4 We notice here that in $((\mathbb{C}, +, \mathrm{bd}_h, 2\pi i_a\cdot, \ \mathrm{h}\cdot\ ), \exp, \mathbb{C}^\times)$ ($2\pi i_a\cdot$ and $\mathrm{h}\cdot$ are unary operations here) one can definably construct an inverse to the usual exponentiation $\exp : \mathbb{C} \to \mathbb{C}^\times$.

Define the function

$$\ln_0 : \ \mathbb{C}^\times \to \mathbb{C}$$

by setting, for $t = \exp w$,

$$\ln_0 t = w - \mathrm{h}^{-1}\mathrm{bd}_h(w/2\pi i_a).$$

It is immediate that

$$\exp(\ln_0 t) = t.$$

Claim $\ln_0 t$ is well-defined and is injective.

Indeed, if also $\exp w' = t$, $w' = w + 2\pi \mathrm{i} k$, some $k \in \mathbb{Z}$, then

$$\mathrm{bd}_h(w'/2\pi i_a) = \mathrm{bd}_h(\frac{w + 2\pi \mathrm{i} k}{2\pi i_a}) = \mathrm{bd}(\frac{w}{2\pi i_a}) + 2\pi \mathrm{i} \mathrm{h} k, \text{ by (23).}$$

Hence,

$$w' - \mathrm{h}^{-1}\mathrm{bd}_h(w'/2\pi i_a) = w - \mathrm{h}^{-1}\mathrm{bd}_h(w/2\pi i_a),$$

as required.

In more detail,

$$\ln_0 t = w - 2\pi \mathrm{i}[\frac{w_a}{2\pi i_a}]_h. \tag{25}$$

So,

$$\ln_0 t = \ln_0 t' \text{ iff } w - 2\pi\mathrm{i}[\frac{w_a}{2\pi i_a}]_h = w' - 2\pi\mathrm{i}[\frac{w'_a}{2\pi i_a}]_h,$$

$$\text{whence } w - w' \in 2\pi\mathrm{i}\mathbb{Z} \text{ and } t = t',$$

hence $\ln_0$ is injective.

Remark The logarithm constructed here resembles the *random logarithm* constructed (non-effectively) by T.Hyttinen [Hy].

3.5 Now we redefine $\check{\mathrm{T}}_N$ in a way compatible both with 2.3 and 3.3.

Define, for each positive $N \in \mathbb{N}$ the map

$$e_{Nh}: \mathbb{C} \to \mathbb{C}^\times; \quad e_{Nh}(w) = \exp(N^{-1}\mathrm{h}^{-1}w).$$

It is convenient to distinguish the copies of $\mathbb{C}^\times$ which are images of e_{Nh} for different N as T_N.

Set, for $t = e_{Nh}(w) \in T_N$,

$$\mathbf{U}_N t := e_{Nh}(\tilde{\mathbf{U}}w), \quad \mathbf{V}_N t := e_{Nh}(\tilde{\mathbf{V}}w).$$

It follows,

$$\mathbf{U}_N t := e_{Nh}(\alpha) \cdot t, \quad \mathbf{V}_N t := e_{Nh}(\beta) \cdot t \cdot \exp\frac{2\pi\mathrm{i}}{N}[w_a]_h.$$

Denote

$$\mathrm{ang}_N(t) := \exp\frac{2\pi\mathrm{i}}{N}[w_a]_h.$$

This is well-defined. Indeed, any other representation of t would be of the form $t = e_{Nh}(w + 2\pi\mathrm{i}\mathrm{h}Nk)$, $k \in \mathbb{Z}$. But $(w + 2\pi\mathrm{i}\mathrm{h}Nk)_a = w_a + 2\pi i_a \mathrm{h}Nk$, and $[w_a + \mathrm{h}Nk]_h = [w_a]_h$ by (19).

Similarly one checks that ang_N satisfies (7)-(9) with $\epsilon = \exp\frac{2\pi\mathrm{i}}{N}$ and corresponding parameters for α, β. So we get, by 2.3

$$\mathbf{V}_N\mathbf{U}_N\, t = \epsilon\, \mathbf{U}_N\mathbf{V}_N\, t. \tag{26}$$

Define

$$\check{\mathrm{T}}_N = (\mathbb{C}, +, \cdot, \mathrm{ang}_N)$$

This is the same definition as 2.3 except here we specified our choice of the angular function.

Proposition The group of periods of ang_N, that is $g \in \mathbb{C}^\times$ such that $\mathrm{ang}_N(g \cdot t) = \mathrm{ang}_N t$ is equal to

$$\mathcal{G}_{N^{-1}h^{-1},\alpha h^{-1}} \cdot \mathbb{C}[N] = \exp(2\pi \mathrm{i} N^{-1}\mathrm{h}^{-1} + \alpha \mathrm{h}^{-1}\mathbb{Z} + \beta\mathbb{R}) \cdot \mathbb{C}[N].$$

In particular, this group is definable in the above $\check{\mathrm{T}}_N$ and the theory of

$$(\mathbb{C}, +, \cdot, \mathcal{G}_{N^{-1}h^{-1},\alpha h^{-1}})$$

is superstable.

Proof By calculation: for $t = \exp N^{-1}\mathrm{h}^{-1}w$ and $g = \exp N^{-1}\mathrm{h}^{-1}u$, by definition,

$$\mathrm{ang}_N(gt) = \exp \frac{2\pi\mathrm{i}}{N}[w_a + u_a]_h,$$

so g is a period if and only if

$$\forall r \in \mathbb{R} \;\; [r + u_a]_h \equiv [r]_h \bmod N\mathbb{Z},$$

iff $u_a \in 2\pi i_a\mathbb{Z} + 2\pi i_a \mathrm{h}\mathbb{Z} + N\mathbb{Z}$ iff

$$g \in \exp(2\pi i_a \mathrm{h}^{-1}N^{-1} + 2\pi i_a \alpha N^{-1}\mathbb{Z} + \alpha\mathrm{h}^{-1}\mathbb{Z} + \beta\mathbb{R}) = \exp(2\pi\mathrm{i}N^{-1}\mathrm{h}^{-1}\mathbb{Z} + 2\pi\mathrm{i}N^{-1}\mathbb{Z} + \alpha\mathrm{h}^{-1}\mathbb{Z} + \beta\mathbb{R}).$$

The superstability follows by the same argument as in 3.3. □

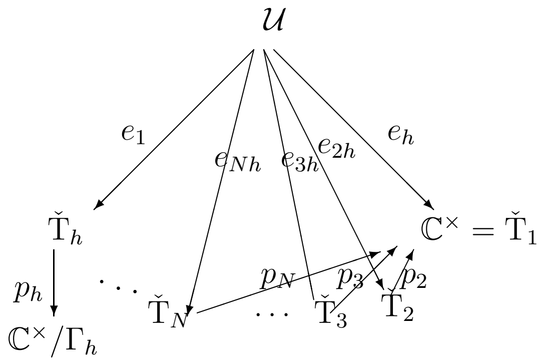
Problem Is the theory of $\check{\mathrm{T}}_N$ as given by the present construction, supersimple?

3.6 Denote

$$\mathcal{U} = (\mathbb{C}, +, \mathrm{bd}_h, \mathrm{h}\cdot\,).$$

By the construction in 3.3 and 3.5 $\check{\mathrm{T}}_N$ is definable in $(\mathcal{U}, \exp, \mathbb{C}^\times)$, for all $N \in \mathbb{N} \cup \{\mathrm{h}\}$.

The resulting picture is as follows, with the arrows showing definable surjections.



where $e_1(w) := \exp w$.

4 Quantum torus

Our aim here is to connect the construction of $\check{\mathrm{T}}_h$ to the well-known definition of the **noncommutative (quantum) torus** usually denoted T_h^2.

4.1 Following the pattern of 2.2 and 2.3 we introduce the algebra $\mathcal{H}$ generated by functions

$$x_k : \mathbb{C}^\times \to \mathbb{C}^\times, \quad k \in \mathbb{Z},$$

where $x_0 = x$ is the identity function and

$$x_k = \xi^k \cdot x, \quad \xi(t) = \mathrm{ang}_h t.$$

We have by 3.3,

$$x_k(\mathbf{U}t) = e^\alpha q^k \cdot x_k(t),$$
$$x_k(\mathbf{V}t) = e^\beta x_{k+1}(w),$$
$$\xi(\mathbf{U}t) = q \cdot \xi(t), \quad \xi(\mathbf{V}t) = \xi(t).$$

As in **??** we normalise the operators $\mathbf{U}^*$ and $\mathbf{V}^*$ on functions by defining operators on $\mathcal{H}$,

$$\begin{array}{l}\dot{\mathbf{U}} : \psi \mapsto \mathbf{U}^*\psi, \quad \mathbf{U}^*\psi(w) = \psi(\mathbf{U}w);\\ \dot{\mathbf{V}} : \psi \mapsto \xi \cdot \psi.\end{array}$$

Using the identities above we get immediately the usual

$$\dot{\mathbf{U}}\dot{\mathbf{V}} = q\dot{\mathbf{V}}\dot{\mathbf{U}}.$$

4.2 We can introduce an isomorphic space with operators in an alternative but closely connected way.

Let z and ζ be the functions $\mathbb{C} \to \mathbb{C}^\times$ given by

$$z(w) = \exp w, \quad \zeta(w) = \exp \mathrm{bd}_h w.$$

Denote $\dot{\mathcal{H}}$ the commutative F-algebra generated by z and ζ, and denote $z_k = \zeta^k z$.

We have, using identities for bd_h,

$$z(\tilde{\mathbf{U}}w) = e^\alpha \cdot z(w), \quad \zeta(\tilde{\mathbf{U}}w) = q \cdot \zeta(w),$$

$$z(\tilde{\mathbf{V}}w) = e^\beta \zeta(w) z(w), \quad \zeta(\mathbf{V}w) = \zeta(w).$$

Again, we define operators on $\dot{\mathcal{H}}$:

$$\dot{\mathbf{U}}: \ \psi \mapsto \tilde{\mathbf{U}}^*\psi,$$
$$\dot{\mathbf{V}}: \ \psi \mapsto \zeta \cdot \psi.$$

The space $\dot{\mathcal{H}}$ is an analogue of the space $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ of all Schwartz functions $\mathbb{R}^2 \to \mathbb{C}$ decaying at infinity along with all its derivatives faster than $\frac{1}{|x|^n}$, any n (see [C]), or $\mathcal{S}(\mathbb{Z}^2, \mathbb{C})$ the Hilbert space of Schwartz sequences, that is complex valued sequences $(c_{m,n})$ decaying faster than any polynomial of m, n.

In [C] with each leaf of the Kronecker foliation

$$L_a = \{\langle r, s\rangle \in \mathbb{R}^2 : \ s + \theta r = a\}$$

one associates the $\mathbb{C}[\dot{\mathbf{U}}, \dot{\mathbf{V}}, \dot{\mathbf{U}}^{-1}, \dot{\mathbf{V}}^{-1}]$-module $\mathcal{H}_a$ obtained by restricting functions of $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ to L_a and defining operators $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$. Namely, the operator $\dot{\mathbf{U}}$ is defined by exactly the same formula as here and $\dot{\mathbf{V}}$ sends $\psi(r, s)$ (function of two real variables r and s) to $\exp(\mathrm{i}s) \cdot \psi(r, s)$ (notice that extra to these data there is a linear dependence between r and s). So, ξ is a good analogue of the function $\exp(\mathrm{i}s)$ taking values in the unit circle.

Notice that $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ are unitary operators if we see $\mathcal{H}_a$ as a Hilbert space. This makes the completion of $\mathbb{C}[\dot{\mathbf{U}}, \dot{\mathbf{V}}, \dot{\mathbf{U}}^{-1}, \dot{\mathbf{V}}^{-1}]$ a $\mathbb{C}^*$-algebra.

By A.Connes the quantum torus T^2_θ is the space of all the modules $\mathcal{H}_a$ on the correspondent L_a.

Remark Consider again the algebra of functions $\dot{\mathcal{H}}$ and denote, for $a \in \mathbb{C}$, $\dot{\mathcal{H}}_a$ the algebra obtained by restricting functions from $\dot{\mathcal{H}}$ to the coset $a + \mathcal{A}_h$. It follows from Proposition 3.2(ii) that the action of $\dot{\mathbf{U}}$ and $\dot{V}$ on $\dot{\mathcal{H}}$ induces a well-defined action on $\dot{\mathcal{H}}_a$, so this is a $\mathbb{C}[\dot{\mathbf{U}}, \dot{\mathbf{V}}, \dot{\mathbf{U}}^{-1}, \dot{\mathbf{V}}^{-1}]$-module for any $a \in \mathbb{C}$.

4.3 To understand further relations of Connes' construction to our T_h we prove the following.

Claim 1. There is a natural bijective correspondence

$$\phi : \ \mathbb{C}/\mathcal{A}_h \to \mathrm{T}^2_\theta,$$

for $\theta = \mathrm{h}$, where T^2_θ is seen as the space of leaves of the Kronecker foliation.

Indeed, we have the decomposition of $\mathbb{C}$ into two real lines

$$\mathbb{C} = \mathrm{i}\mathbb{R} + \alpha\mathbb{R}, \quad \text{for any } z \in \mathbb{C}\ z = x\mathrm{i} + y\alpha,\ x, y \in \mathbb{R}.$$

Rescale the real coordinates

$$r := \mathrm{h}^{-1}x, \quad s := 2\pi(2\pi i_a)^{-1}y$$

and consider the mapping onto the direct product of two unit circles

$$z \mapsto \langle x, y\rangle \mapsto \langle r, s\rangle \mapsto \langle \exp \mathrm{i}r, \exp \mathrm{i}s\rangle.$$

Under the map

$$2\pi\mathrm{ih}\mathbb{Z} + 2\pi i_a\alpha\mathbb{Z} \to \langle 2\pi\mathrm{h}\mathbb{Z}, 2\pi i_a\mathbb{Z}\rangle \to \langle 2\pi\mathbb{Z}, 2\pi\mathbb{Z}\rangle \to 1,$$

and since $2\pi\mathrm{i} - 2\pi i_a\alpha \in \beta\mathbb{R}$,

$$\beta\mathbb{R} \to \langle 2\pi, -2\pi i_a\rangle\mathbb{R} \to \langle 2\pi\mathrm{h}^{-1}, -2\pi\rangle\mathbb{R} \to L_0.$$

This establishes the bijection between the cosets of $\mathcal{A}_h$ and the leaves L_a of the foliation.

Claim 2. There is a bijective correspondence

$$\tilde{p}_h :\ \mathbb{C}/\mathcal{A}_h \to \mathbb{C}^\times/\mathcal{G}_h,$$

induced by p_h. Moreover, the action of $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ on $\mathbb{C}$ induces a well-defined action on $\mathbb{C}/\mathcal{A}_h$ and correspondingly the action on $\mathbb{C}^\times/\mathcal{G}_h$. The latter action coincides with the one induced by $\mathbf{u}$ and $\mathbf{v}$ on the cosets of $\mathcal{G}_h$.

This is the direct consequence of Proposition 3.2(iii) and the definition of p_h.

Corollary $\tilde{p}_h \circ \phi^{-1}$ identifies T^2_h with $\mathbb{C}^\times/\mathcal{G}_h$, with all the structure on the latter induced from $\check{T}_h$.

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