Pseudo-analytic structures, quantum tori and non-commutative geometry

B. Zilber

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Categoricity and Stability are very powerful notions mastered by model-theoreticians. By definition a structure is *categorical* in a certain formalisation if the description of this structure in terms of the formal language determines the structure uniquely, up to isomorphism. This has been developed into the powerful Morley-Shelah classification theory. Various possible levels of *stability* of structures form a hierarchy, basically indicating how far a given structure is from categoricity.

On the one hand it is now commonly understood that these notions cover rather narrow class of mathematical structures, one way or another related to algebraic geometry. On the other hand the mere assumption that a structure under consideration is stable, or even better categorical, has very strong consequences. I would like to claim that this kind of assumption (in fact the assumption of *uniqueness* of a model of a description) has a very strong *predictive* power (see e.g. [Z10]). This seems to be the property of a mathematical theory direly needed for models of physical reality. At the same time model theory is extremely interested in finding and understanding 'new stable structures' most if not all of them mysteriously being associated with Hrushovski's construction [H] (see also [Z3] for a discussion of the construction).

We speculate here that there might be a two-way link between quantum physics and stability theory. The former can be a source of important examples for the latter and the latter can provide the former with unconventional mathematical tools and ideas.

In sections 4 we duscuss two types of structures, the first of which is a known example of a non-algebraic Zariski curve and the second one is one of the 'new structures with exponentiation', also known as "Poizat's bicolored field'. We show how the first class of examples can be considered as non-commutative algebraic varieties, more specifically we study a class of non-commutative algebraic curves $T_{\frac{1}{N}}$ "at root of unity". The second example we identify with the Connes' quantum torus T_h^2. Importantly, we show that T_h^2 can be seen as a limit of $T_{\frac{1}{N}}$ as $N \to \infty$.

One starts with a *naive* geometric *nonlinear* structure $\mathbf{M}$. By the main result on Zariski geometries [HZ] there is an algebraically closed field F living in $\mathbf{M}$ (which one can see immediately in the examples discussed). Thus, one gets Zariski-continuous coordinate functions $\mathbf{M} \to \mathrm{F}$, which in a way can be seen as measurements (observations) in $\mathbf{M}$ in terms of numbers of F. Naively one could hope that the usual commutative algebra $\mathrm{F}(\mathbf{M})$ of regular maps $f : \mathbf{M} \to \mathrm{F}$ is rich enough to represent the geometry of $\mathbf{M}$ by the spectrum of $\mathrm{F}(\mathbf{M})$.

This is not the case in the examples $T_{\frac{1}{N}}$ mentioned above. The spectrum of $\mathrm{F}(\mathbf{M})$ represents only the quotient $C = \mathbf{M}/E$, where E is a nontrivial equivalence relation given by the action of a finite group H. In fact, H is a subgroup of a group G of regular automorphisms of the structure, which does not split. For this reason the bundle $\mathbf{M} \to C$ is nontrivial.

In order to represent all the structural information by way of "observations" we introduce sections $C \to \mathbf{M}$, which we show to generate an infinite-dimensional vector space $\mathcal{H}(\mathbf{M}) = \mathcal{H}$.

Now the group G acts on the space $\mathcal{H}$. The sections are not regular functions (that is are not definable) but it turns out that $\mathcal{H}$ along with the action of G on it is determined uniquely up to isomorphism.

The F-algebra $\mathrm{A}(\mathbf{M})$ of the geometric transformations of $\mathcal{H}$ generated by G is precisely the noncommutative coordinate ring of $\mathbf{M}$ which, as we show, determines the geometry on $\mathbf{M}$: the points of $\mathbf{M}$ can be recovered as irreducible representations of $\mathrm{A}(\mathbf{M})$ of a certain *orientation* and Zariski relations on $\mathbf{M}$ are also represented by the algebra.

We then go on to analyse the construction which leads to the quantum torus T_h^2. In model theory this construction has been found by B.Poizat in an attempt to produce a difficult counterexample of a "bad field". In popular terms this is a structure based on an algebraically closed field F which behaves in many ways like an object of complex algebraic geometry, and at the same time has certain non-algebro-geometric properties. First of all, F contains a subgroup $\mathcal{G} \leq \mathrm{F}^*$ of dimension equal to half of that of F.

We have shown that Poizat's example can be reinterpreted in terms of the real-complex geometry, that is $\mathrm{F} = \mathbb{C}$ and $\mathcal{G}$ is a certain real curve (a tragectory) on the complex plane. Crucially, we could see that the only way a real trajectory on a complex plane can be compatible with stability is by assuming that the trajectory is *indeterministic,* in a typically quantum way. Moreover, we show that the quotient $\mathbb{C}^*/\mathcal{G}$ can be identified with the space of leaves of the Kronecker foliation and so with the quantum torus T_h^2 in the sense of A.Connes (see [Co]).

We show that the latter example is naturally related to the series of examples above. In fact, T_h^2 can be interpreted as a limit of $T_{\frac{1}{N}}$ as $N \to \infty$. We don't have a full clarity yet as to what properties can be passed to the limit structure, which is obviously a very important question. We hope that understanding the limit properties will help to clarify the meaning of many typically nonconvergent infinite expressions in the theory of quantum structures.

It is important also to stress the difference between the real and complex numbers as structures. From model theoretic point of view the algebraic structure $(\mathbb{C}, +, \cdot)$ of complex numbers is very good, technically its first-order theory is categorical in uncountable cardinalities, and so is stable and indeed Morley rank in this theory has the same meaning as the dimension in algebraic geometry. In contrast the field of reals $(\mathbb{R}, +, \cdot)$ is very unstable, though can be treated successfully in model theory by means of o-minimality. Our approach is based on categoricity and stability assumptions so the reals seem to be excluded from the context. In principle this poses a problem in regards to the analogy of *measurements* or *observations* over a structure **M** (see above). Rather interestingly our structure behind the quantum torus T_h^2 resolves this seeming contradiction. We explain in section 6 that the structure despite being superstable interpretes the reals albeit in non-elementary way. More precisely one can still produce a real number as a 'measurement' for a point in **M** but this has to come as a result of countably infinite process (an $L_{\omega_1,\omega}$-formula).

1 Analytic Zariski structures and real analytic structures

We think in terms of (proper) Zariski- and analytic-Zariski- ([HZ],[PZ] and [Z1]) structures and 'non-standard analysis' in analytic Zariski structures in the sense of [Z1] (though it is still to be developed in the analytic Zariski context). The (analytic) Zariski class has been introduced as an attempt to develop a formal theory of a 'good' analyticity, an analyticity relevant to algebraic geometry and including both structures based on classical transcendental analytic functions (exponentiation, Weierstrass functions e.c.) and 'new stable structures' produced by Hrushovski's construction as in [H],[Po] and others.. See [Z2] and [Z3] for discussions around this subject.

The class of (proper) Zariski structures can be easily illustrated by classical examples, first of all the F-points of an algebraic variety over an algebraically closed field F as well as compact complex spaces, both in their *natural* languages [Z1]. The class of analytic Zariski, except for its proper Zariski subclass, needs much more hard work to be provided with proven examples. In most interesting cases we apparently would need the Schanuel type conjectures (and probably a related Diophantine conjecture).

On the other end of our project there are examples of structures on the reals which can be treated in a fashion similar to Zariski and which are close to so called o-minimal structures. Of course, o-minimal analysis has a rich topological ingredient in it. But it becomes even more interesting when it is possible to combine o-minimality with stability, as e.g. Peterzil and Starchenko show in [PS], Zariski-type topology and geometry arise immediately.

In [Z8] an example which is quite close to the real one-dimensional noncommutative torus is shown to behave model-theoretically nicely, in fact there is a natural dimension theory in it, similar to o-minimality or d-minimality. Interestingly, this example can be naturally linked to a very similar ω-stable structure on the complexes. In [Z5] (and in section 5) more complicated structures are shown to have both real- and complex-analytic properties and at the same time have ω-stable theories. We will try to exhibit here that this is essentially an effect of what we call (maybe incorrectly) a quantization of a more simple d-minimal structure.

Finally in section 6 we show that the real and complex model theory can be really brought together providing a further analysis of the quantum torus combined with a 'bad' field of section 5. This turns out to be a superstable structure with the field of reals $L_{\omega_1,\omega}$-definable in it. This seems to be a phenomenon that might explain how in quantum mathematical physics the real analysis interplays with the complex algebraic and noncommutative geometry. This example, in my opinion, alludes to the mystery of mirror symmetry which puts in correspondence purely algebro-geometric structure of moduli of Calabi-Yau manifolds with the essentially real symplectic structure of Kähler classes.

2 Non-standard analysis in Zariski structures.

Following [HZ], [Z1] and [PZ] a (analytic) Zariski structure M is given with a coarse (Zariski) topology on M^n, each n, sometimes assumed compactifyable, i.e. there are definable completions $\overline{M^n}$ agreeing with natural embeddings. Often it poses a problem to find or prove that certain completions are indeed (atomic) compact. In particular, the compactified $\bar{M}$ has the property that, for any elementary extension $\bar{M} \prec {}^*\bar{M}$ there is a surjective homomorphism, which is the identity on $\bar{M}$,

$$\sigma : {}^*\bar{M} \to \bar{M}.$$

σ is said to be a specialisation, an example of a specialisation in case $M = \mathbb{R}$ would be the standard part mapping, with $\bar{R} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. We fix a universal specialisation σ and, given a point $a \in \overline{M^n}$, call an **infinitesimal neighborhood of** a the set

$$\mathcal{V}_a = \{\alpha \in {}^*\overline{M^n} : \sigma(\alpha) = a\}.$$

In fact, as is shown in the latest version of [Z1], we don't need to consider compactification to have infinitesimal calculus. A maximal partial mapping (universal specialisation)

$$\sigma : {}^*M \to M$$

preserving Zariski closed sets (relations) provides us with the same tools.

The structure of such an infinitesimal neighborhood of an M is very rich and we can study them *when the theory of M allows elimination of quantifiers.* This is the case with the proper Zariski structures [Z1], and for analytic Zariski we usually have elimination of quantifiers to the level of $\exists$-formulas.

In particular, one can recover the intersection theory of algebraic varieties (in any characteristic) as well as of compact manifolds in a uniform fashion from the local infinitesimal theory.

Let $\mathcal{V}$ and $\mathcal{V}_0$ be infinitesimal neighborhoods. We say that $\phi : \mathcal{V} \to \mathcal{V}_0$ is a **definable local function (germ)** if there is a definable relation F on M with

$$\text{graph } \phi = F \cap \mathcal{V} \times \mathcal{V}_0.$$

If F is (formal) analytic and regular at the point, ϕ is also said to be holomorphic.

Rigidity. *Under the latter assumptions and irreducibility of F the correspondence*

$$F \leftrightarrow \phi$$

between local functions and global irreducible analytic sets through the point is one-to-one.

In general M is a many sorted structure. If a is in sort S then we sometime write $\mathcal{V}_a(S)$ or just $\mathcal{V}(S)$ to point to the type of a non-standard neighborhood.

Suppose that in M there is a sort F which is an algebraically closed field (with $+$ and $\times$ in the language, of course).

Then for $\mathcal{V}$ in M and $\mathcal{V}_0 = \mathcal{V}_0(\mathrm{F})$ in F, infinitesimal neighborhoods, a definable local holomorphic function $\phi : \mathcal{V} \to \mathcal{V}_0$ is said to be a **local coordinate into** $\mathcal{V}_0$, or we rather should talk in general on $n = \dim \mathcal{V}$ independent functions $\phi_1, \ldots, \phi_n$ into $\mathcal{V}_1, \ldots, \mathcal{V}_n$ forming a system of local coordinates.

Notice that any two neighborhoods in F are biholomorphically isomorphic.

Obviously we can apply the algebraic (field) operations to local functions (though the result may change the target neighborhood). We restrict ourselves with neighborhoods of F proper, that is avoiding ∞, thus local coordinates form a usual commutative algebra over F.

3 Examples of Zariski structures

Examples

1. Algebraic varieties over an algebraically closed field in the natural language.

2. Compact complex spaces in the natural language (Zariski closed subset of $M^n \equiv$ analytic subset). [Z1]

3. Proper analytic varieties over complete algebraically closed non-Archimedean valued fields (in the sense of *rigid analytic geometry*). [Z1]

4. The solution spaces of systems of differential equations in *differentially closed fields*, provided 'the number of variables equals the number of equations' (A.Pillay, [Pi])

Analytic Zariski

5. (M, R^3) Hrushovski's *ab initio* triple relation is analytic Zariski (see [PZ]). The closest analog is $M = \mathbb{C}$ with the triple relation $R(x, y, z)$ given as $f(x, y, z) = 0$ with f a generic entire function.

6. $(\mathrm{F}, +, \cdot, R^3)$ free triple relation on a.c.field F (see [P1]). The algebraic geometry extended by R.

7. $(\mathbb{C}, +, \cdot, f(n))_{n\in\mathbb{Z}}$ $f(n)$ the Liouville function and its derivatives. Under assumptions that the "Schanuel property" of the Liouville function (A.Wilkie, [W]) can be extended to its derivatives, see [P2].

8. A big class of two-sorted structures $((U, p, \mathcal{A})$ with $\mathcal{A}$ a complex algebraic variety (typically semi-abelian variety) with the full Zariski structure, U the covering space of $\mathcal{A}$ in a rich language and $p : U \to \mathcal{A}$ the corresponding holomorphic covering. This requires rather involved arithmetic (M.Gavrilovich [G] and [Z10]).

9. $(\mathrm{F}, +, \cdot, x^a)$ powered algebraically closed field of characteristic 0. This is a two-sorted structure $(V, \mathrm{ex}, \mathrm{F})$ with $V = (V, +, a\cdot)$ a vector

space over $\mathbb{Q}(a)$, $\mathrm{F} = (F, +, \cdot)$ a field, and $\mathrm{ex} : V \to F^\times$ covering homomorphism satisfying the Schanuel property

$$\mathrm{tr.d.}(x_1, \ldots, x_n, \mathrm{ex}\, x_1, \ldots, \mathrm{ex}\, x_n) \geq \mathrm{lin.d.}(x_1, \ldots, x_n).$$

We can prove that the theory is superstable and the canonical model of the theory is analytic Zariski ([Z6],[Z9]).

10. $(\mathbb{C}, +, \cdot, P)$ Poizat's black points (see [Z5])

11. $(\mathbb{C}, +, \cdot, G)$ Poizat's green points (assuming the Schanuel conjecture, see [Z5]) This and the previous examples are conjectured to be analytic Zariski. Only some of the properties have been proved.

12. Same as example 9 but with the canonical $V = \mathbb{C}$, $\mathrm{F} = \mathbb{C}$ and $\mathrm{ex} = \exp$. We need here the Schanuel conjecture (restricted to a). Most of this is done in [Z9] using [Z6] and results of Khovanski and others.

13. $(\mathrm{F}, +, \cdot, \mathrm{ex})$ abstract field with pseudo-exponentiation. The structure (or rather its $L_{\omega_1,\omega}(Q)$ theory) is *excellent* (see [Z7]), and is analytic Zariski.

14. $(\mathbb{C}, +, \cdot, \exp)$ the classical exponentiation. We can prove that it is the same as 13 above, assuming Schanuel conjecture, CIT and existential closedness conjectures.

4 Non-algebraic Zariski geometries

4.0.1 Recall the following theorem C of [HZ].

Theorem *There exist irreducible pre-smooth Zariski structures (in particular of dimension* 1*) which are not interpretable in an algebraically closed field.*

The construction

Let M be an irreducible pre-smooth Zariski structure, $G \leq \mathrm{ZAut}\, M$ acting freely on M and for some $\tilde{G}$ with **finite** H :

$$1 \to H \to \tilde{G} \to^{p_0} G \to 1.$$

Consider a set $S \subseteq M$ of representatives of G-orbits: for each $a \in M$, $G \cdot a \cap S$ is a singleton.

Consider the formal set

$$M(\tilde{G}) = \tilde{M} = \tilde{G} \times S$$

and the projection map

$$p : (g, s) \mapsto p_0(g) \cdot s.$$

Consider also, for each $f \in \tilde{G}$ the function

$$f : (g, s) \mapsto (fg, s).$$

One can prove rather easily

Claim 1. *The structure*

$$(\tilde{M},\ \{f\}_{f\in\tilde{G}},\ p^{-1}(\textit{Zariski relations on } M))$$

is an irreducible pre-smooth Zariski structure, its isomorphism type is determined by M *and* $\tilde{G}$ *only and* $\dim \tilde{M} = \dim M$.

Claim 2. *Suppose H does not split, for every proper $G_0 < \tilde{G}$*

$$G_0 \cdot H \neq \tilde{G}.$$

Then, every equidimensional Zariski expansion $\tilde{M}'$ of $\tilde{M}$ is irreducible.

Claim 3. $\tilde{G} \leq \mathrm{ZAut}\, \tilde{M}$, that is $\tilde{G}$ is a subgroup of the group $\mathrm{ZAut}\, M$ of Zariski-continuous bijecions of M.

More technical is the following Lemma the proof of which uses the Claims and some analysis of groups of rational automorphisms of algebraic curves.

Lemma. *Suppose M is a rational or elliptic curve (over an algebraically closed field F of characteristic zero), H does not split, $\tilde{G}$ is nilpotent and for some big enough integer μ there is a non-abelian subgroup G_0*

$$|\tilde{G} : G_0| \geq \mu.$$

Then $\tilde{M}$ is not interpretable in an algebraically closed field.

In general it is harder to analyse the situation when $\dim M > 1$ since the group of birational automorphisms is not so immediately reducible to the group of biregular automorphisms of a smooth variety in higher dimensions. But nevertheless the same method can prove the useful fact that the construction produces examples essentially of non algebro-geometric nature.

Proposition (i) Suppose M is an abelian variety, H does not split and $\tilde{G}$ is nilpotent not abelian. Then $\tilde{M}$ can not be an algebraic variety with $p: \tilde{M} \to M$ a regular map.

(ii) Suppose M is the (semi-abelian) variety $(\mathrm{F}^{\times})^n$. Suppose also that $\tilde{G}$ is nilpotent and for some big enough integer $\mu = \mu(n)$ has no abelian subgroup G_0 of index bigger than μ. Then $\tilde{M}$ can not be an algebraic variety with $p: \tilde{M} \to M$ a regular map.

Proposition. *Suppose M is an F-variety and, in the construction of $\tilde{M}$, the group G is finite. Then $\tilde{M}$ is definable in any expansion of the field F by a total linear order.*

In particular, if M is a complex variety, $\tilde{M}$ is definable in the reals.

Proof Extend the ordering of F to a linear order of M and define

$$S := \{s \in M : \ s = \min G \cdot s\}.$$

The rest of the construction of $\tilde{M}$ is definable.□

Remark In other known examples of non-algebraic $\tilde{M}$ (with G infinite) $\tilde{M}$ is still definable in any expansion of the field F by a total linear order.

Problem (i) Classify Zariski structures definable in the reals.

(ii) Classify Zariski structures definable in the reals as a smooth real manifold.

(iii) Find new Zariski structures definable in $\mathbb{R}_{an}$ as a smooth real manifold.

We hope that a solution to these problems may connect the theory of Zariski geometries to the theory of symplectic manifolds.

4.1 A non-algebraic Zariski curve and its coordinate algebra

4.1.1 Let F be an algebraically closed field of characteristic 0 and N a positive integer. Consider the groups given by generators and defining relations,

$$G = \langle \mathbf{u}, \mathbf{v} : \ \mathbf{uv} = \mathbf{vu} \rangle,$$

$$\tilde{G} = \langle \mathbf{U}, \mathbf{V} : \ [\mathbf{U}, [\mathbf{U}, \mathbf{V}]] = [\mathbf{V}, [\mathbf{U}, \mathbf{V}]] = 1 = [\mathbf{U}, \mathbf{V}]^N \rangle.$$

Let $a, b \in \mathrm{F}^*$ multiplicatively independent.
G acts on $\mathrm{F}^\times$:

$$\mathbf{u} \cdot x = ax, \ \mathbf{v} \cdot x = bx.$$

Taking M to be $\mathrm{F}^\times$ this determines, by 4.0.1, a presmooth non-algebraic Zariski curve $\tilde{M}$ which from now on we denote T_N. Correspondingly, T_N or simply T is the universum of T_N.

Since $[\mathbf{U}, \mathbf{V}]$ is a central element, in every representation of $\tilde{G}$ one can replace $[\mathbf{U}, \mathbf{V}]$ by an $\epsilon \in \mathrm{F}$, a primitive root of unity of order N. So, the defining relation for $\tilde{G}$ becomes just

$$\mathbf{VU} = \epsilon \mathbf{UV},$$

or

$$\mathbf{VUV}^{-1}\mathbf{U}^{-1} = \epsilon.$$

The correspondent definition for the covering map $p : \tilde{M} \to M$ then gives us

$$p(\mathbf{U}t) = ap(t), \quad p(\mathbf{V}t) = bp(t). \tag{1}$$

4.1.2 Semi-definable functions.

By 4.0.1 T_N constructed above is not algebraic. In particular, the field $\mathrm{F}(\mathrm{T}_N)$ of definable (Zariski continuous) functions $\mathrm{T}_N \to \mathrm{F}$ represents T_N up to the equivalence relation $p(t) = p(t')$, that is it can not distinguish t and t' if $t' = H \cdot t$, where $H = \langle [\mathbf{U}, \mathbf{V}] \rangle$, the cenral subgroup of order N.

We want to introduce a wider class of functions, so that it represents T_N faithfully. We notice first that in correspondence with the local theory, in infinitesimal neighborhoods $\mathcal{V}_t$ of any $t \in T$ the map $p : T \to \mathrm{F}^*$ is a (local) bijection onto $\mathcal{V}_{p(t)}$. That is p is similar to the map $t \mapsto t^N$ and T_N, at least locally, can be identified with F^*.

We are going to extend the local bijection by the cost of losing definability.

Lemma *Given α, β such that $\alpha^N = a,\ \ \beta^N = b$, one can define bijections*

$$x_k : \mathrm{T}_N \to \mathrm{F}^* \quad k = 0, \ldots, N-1$$

so that for any $t \in T_N$ the following **functional equations** *are satisfied,*

$$x_k(t)^N = p(t) \tag{2}$$

$$x_k(\mathbf{U}t) = \alpha\epsilon^k x_k(t), \tag{3}$$

$$x_k(\mathbf{V}t) = \beta x_{k+1}(t), \ \textit{where } x_N = x_0, \tag{4}$$

$$\frac{x_{k+1}(t)}{x_k(t)} = \frac{x_k(t)}{x_{k-1}(t)}. \tag{5}$$

Proof First, notice that (3),(4) imply

$$x_k([\mathbf{U},\mathbf{V}]^{-1}t) = \epsilon x_k(t), \tag{6}$$

where $[\mathbf{U},\mathbf{V}]^{-1} = \mathbf{U}^{-1}\mathbf{V}^{-1}\mathbf{U}\mathbf{V}$.

To construct the x_k choose randomly an injection $\sqrt[N]{\ }\ :\ \mathrm{F}^\times \to \mathrm{F}^\times$ with the property

$$(\sqrt[N]{w})^N = w.$$

$$\text{For } t = \mathbf{U}^m\mathbf{V}^n[\mathbf{U},\mathbf{V}]^l \cdot s, \quad s \in S \text{ set } \ x_k(t) := \alpha^m\beta^n\epsilon^{mk-l}\sqrt[N]{s}.$$

This satisfies (2)-(5) and is bijective. □

4.1.3 Define the **angular function** on F^* as a function ang : $\mathrm{F}^\times \to \mathrm{F}[N]$, roots of unity of order N.

Set for $\lambda \in \mathrm{F}^*$,

$$\operatorname{ang}(\lambda) = \frac{x_1(t)}{x_0(t)}, \text{ if } \lambda = x_0(t).$$

This is well-defined since x_0 is a bijection. One can easily deduce from definitions

$$\operatorname{ang} \alpha\lambda = \epsilon \operatorname{ang} \lambda, \tag{7}$$

and

$$\operatorname{ang} \epsilon\lambda = \operatorname{ang} \lambda, \quad \operatorname{ang} \beta\lambda = \operatorname{ang} \lambda. \tag{8}$$

Now we consider the structure

$$\check{\mathrm{T}}_N := (\mathrm{F}, +, \cdot, \operatorname{ang}).$$

It is clear that F is partitioned into N 'sectors' using the angular function:

$$P_\delta = \{\mu \in \mathrm{F}^* : \operatorname{ang} \mu = \delta\}.$$

Proposition T_N is definable in $\check{\mathrm{T}}_N$ using parameters α and β. Moreover, $x_0, \ldots, x_{N-1}$ are definable in the structure as well.

Proof Define $T = \mathrm{F}^\times$ as a set, and for any $t \in \mathrm{F}^\times$ set

$$p(t) = t^N, \quad \mathbf{U}t = \alpha\, t, \quad \mathbf{V}t = \beta \operatorname{ang}(t)\, t.$$

This satisfies , so is, up to isomorphism, the same structure as T_N.
Set $x_k(t) := (\operatorname{ang} t)^k \cdot t$. This satisfies 4.1.2. □

Notice that the map p in the presence of the angular function can be definably inverted. That is there is a section

$$y : \mathrm{F}^\times \to T$$

such that $y(p(t)) = t$. In fact we can have N such functions, $y_k, \quad k = 0, \ldots, N-1$. Just set, for $w \in \mathrm{F}^\times$,

$$y_k(w) = \text{ the unique } t \in P_{\epsilon^k} \text{ such that } p(t) = w.$$

In other words, the above representation of T_N as the cover of $\mathrm{F}^\times$ by another copy of $\mathrm{F}^\times$ can be seen as the bundle of N-fibres on $\mathrm{F}^\times$ with structural group $\mathrm{F}[N]$, and the y_k's are sections of the bundle.

4.1.4 Consider the class $\mathcal{T}_N$ of structures satisfying (7) - (**??**). Suppose $\check{\mathrm{T}}_N$ is existentially closed in this class. What is the model-theoretic status of the theory of this structure?

Theorem (D.Evans) $\mathcal{T}_N$ has model completion and every complete extension of the theory of $\mathcal{T}_N$ is supersimple.

The fact that $\check{\mathrm{T}}_N$ is supersimple has certain methodological significance. There is a common, albeit informal, understanding that simple structures (theories) come basically from stable structures by introducing a 'random noise'. So, one may think of $\check{\mathrm{T}}_N$ as an algebraic curve with a random angular function.

Consider the **group of periods of** ang

$$\mathcal{G}(\mathrm{T}_N) = \mathcal{G} := \{w \in \mathrm{F}^* : \ \forall t \ \mathrm{ang}(w \cdot t) = \mathrm{ang}\, t\}.$$

$\mathcal{G}$ is obviously definable in $\check{\mathrm{T}}_N$ and by (8) $\mathcal{G}$ contains $\langle \beta, \epsilon \rangle$, the group generated by β and ϵ.

In case $\check{\mathrm{T}}_N$ *is existentially closed in* $\mathcal{T}_N$ *we have the equality*

$$\mathcal{G} = \langle \beta, \epsilon \rangle.$$

Problem *Study the structure of definable subsets on* $\check{\mathrm{T}}_N$. *Is there a good probabilistic measure theory on* $\check{\mathrm{T}}_N$?

4.1.5 The space of semi-definable functions.

Let R_N be the F-algebra of semi-definable functions on T_N generated by $x_0, \ldots, x_{N-1}, x_0^{-1}, \ldots, x_{N-1}^{-1}$.

Notice that R_N is determined as a commutative F-algebra uniquely up to isomorphism by its generators $x_0, \ldots, x_{N-1}$ satisfying the relations (2).

We may also regard R_N as an F-vector space.

We define linear operators $\mathbf{U}^*$ and $\mathbf{V}^*$ on the linear space $\mathbb{R}_N$:

$$\begin{array}{l} \mathbf{U}^* : \ \psi(t) \mapsto \psi(\mathbf{U}t), \\ \mathbf{V}^* : \ \psi(t) \mapsto \psi(\mathbf{V}t). \end{array} \tag{9}$$

Denote $\tilde{G}^*$ the group generated by the operators $\mathbf{U}^*$, $\mathbf{V}^*$ and their inverses.

R_N with the action of $\tilde{G}^*$ on it is determined uniquely up to isomorphism by the defining relation (2)-(6) and so is independent on the arbitrariness in the choices of $x_0, \ldots, x_{N-1}$.

Notice that

The correspondence $\mathbf{U} \mapsto \mathbf{U}^*, \quad \mathbf{V} \mapsto \mathbf{V}^*$ generates the anti-isomorphism $\tilde{G} \to \tilde{G}^*$ satisfying the property

$$(g_1 g_2)^* = g_2^* g_1^*, \text{ for any } g_1, g_2 \in \tilde{G}.$$

This can be easily seen if we define the pairing

$$R_N \times T \to \mathrm{F}, \quad (\psi, t) \mapsto \psi(t).$$

This allows to consider the adjoint action of any $g \in \tilde{G}$ on R_N setting $g^*\psi$ as the unique element of R_N such that

$$(g^*\psi, t) = (\psi, gt), \text{ for all } t \in T.$$

We can immediately identify that this definition extends (9). The desired formula follows.

The advantage of using $\mathbf{U}^*$ and $\mathbf{V}^*$ instead of $\mathbf{U}$ and $\mathbf{V}$ is that $\mathbf{U}^*$, $\mathbf{V}^*$ and their inverses generate an F-algebra of linear operators acting on R_N. The group $\tilde{G}^*$ is a subgroup of the group of units of the algebra.

Consider the action of the central element $[\mathbf{U}^*, \mathbf{V}^*]$. It is easy to prove

Lemma *The eigenvalues of the operator* $[\mathbf{U}^*, \mathbf{V}^*]$ *on the linear space* R_N *are* $1, \epsilon, \ldots, \epsilon^{N-1}$ *and the linear space can be represented as the direct sum*

$$R_N = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{N-1}$$

of the corresponding eigenspaces.

In fact $\mathcal{H}_m$ as a vector space is generated by monoms of order $d \equiv m \mod m$, where the order of the monomial $x_0^{m_0} \cdot \cdots \cdot x_{N-1}^{m_{N-1}}$ is $m_0 + \cdots + m_{N-1}$.

Denote $\mathcal{H} := \mathcal{H}_1$. The algebra of linear operators on the linear space $\mathcal{H}$ generated by $\mathbf{U}^*$ and $\mathbf{V}^*$ we denote $\mathrm{A}(\mathrm{T}_N)$ or simply A. Obviously, the only defining relation for the algebra is

$$\mathbf{U}^*\mathbf{V}^* = \epsilon \mathbf{V}^*\mathbf{U}^*.$$

4.1.6 Let $\mathrm{Max}(R_N)$ be the space of maximal ideals of the commutative algebra R_N.

The standard fact of commutative algebra:

$\mathrm{Max}(R_N)$ consists of ideals $I_{\bar{\mu}}$, $\bar{\mu} = \langle \mu_0, \dots, \mu_{N-1}\rangle$, $\mu_0^N = \cdots = \mu_{N-1}^N$,

$$I_{\bar{\mu}} = \langle (x_0 - \mu_0), \dots, (x_{N-1} - \mu_{N-1})\rangle.$$

Assuming F is endowed with an angular function $\mathrm{ang} : \mathrm{F}^\times \to \mathrm{F}[N]$ we call $\bar{\mu}$ as above **oriented positively** if $\mu_k = \mathrm{ang}(\mu_0)^k \cdot \mu_0$. Correspondingly, we call an ideal $I_{\bar{\mu}}$, oriented positively if $\bar{\mu}$ is.

$\mathrm{Max}^+(R_N)$ will denote the subspace of $\mathrm{Max}(R_N)$ consisting of positively oriented ideals I.

It is easy to see that $\bar{\mu}$ is positively oriented if and only if

$$\langle \mu_0, \dots, \mu_{N-1}\rangle = \langle x_0(t), \dots, x_{N-1}(t)\rangle,$$

for some $t \in T$.

Moreover:

(i) There is a bijective correspondence $\Xi : \ \mathrm{Max}^+(R_N) \to \mathrm{T}_N$ between the space of positively oriented maximal ideals and T_N.

(ii) The action (9) of $\tilde{G}^*$ on R_N induces an action on $\mathrm{Max}(R_N)$ and leaves $\mathrm{Max}^+(R_N)$ setwise invariant.

(iii) The action of $g^* \in \tilde{G}^*$ on $\mathrm{Max}(R_N)$(and so on T_N) can be identified as

$$g^* : I_{\langle x_0(t), \dots, x_{N-1}(t)\rangle} \mapsto I_{\langle x_0(g^{-1}t), \dots, x_{N-1}(g^{-1}t)\rangle}.$$

4.1.7 We may also treat T as the space of F-linear functionals $\mathcal{H} \to \mathrm{F}$ defined by the pairing of 4.1.5,

$$\mathcal{H}_T^* = \{F_t : \ \psi \mapsto (\psi, t), \quad t \in T\}.$$

Obviously, the kernel of a nonzero functional is a maximal ideal. Moreover,

$$\ker F_t = \{\phi \in \mathcal{H} : \ (\phi, t) = 0\} = I_{\langle x_0(t), \dots, x_{N-1}(t) \rangle}.$$

We also denote $\ker F_t := I^t$.

We call a linear functional F on $\mathcal{H}$ **positive** if $\ker F$ is a positive maximal ideal.

Notice that a functional $\psi \mapsto \psi(t)$ defined on $\mathcal{H}$ can be extended to the subalgebra $\mathrm{F}(p, p^{-1})$ of R_N generated by the function p, since $p = x_k^N$, $k = 0, \dots, N-1$. Obviously $\mathrm{F}(p, p^{-1})$ is just the coordinate algebra of of the usual torus and its maximal spectrum is F^*. We can hence say that the functionals F_t^p are in the bijective correspondence $F_t^p \mapsto p(t)$ with the points of F^*.

Proposition

(i) The correspondence

$$t \mapsto F_t$$

between T and the space $\mathcal{H}_+^*$ of positive linear functionals on $\mathcal{H}$ is bijective.

(ii) The correspondence transfers isomorphically the natural action of $\tilde{G}$ on T to a natural action of $\tilde{G}$ on $\mathcal{H}_+^*$.

(iii) Consider also the commutative algebra $\mathrm{F}(p, p^{-1})$ generated by the function $t \mapsto p(t)$ and, for each linear functional F_t its restriction F_t^p on $\mathrm{F}(p, p^{-1})$. Then, for any $t_1, t_2 \in T$,

$$F_{t_1}^p = F_{t_2}^p \text{ iff } p(t_1) = p(t_2) \text{ iff } F_{t_1} = \epsilon^j F_{t_2}, \text{ for some } j \in \{0, \dots, N-1\},$$

and the correspondence

$$F_t \mapsto F_t^p$$

is the one-to-N map from the space space $\mathcal{H}_+^*$ of all the positive functionals onto the space of linear functionals on $\mathrm{F}(p, p^{-1})$. This is in exact correspondence with the map $p : T \to \mathrm{F}^\times$.

In other words, the structure T_N is faithfully represented by the positive linear functionals on $\mathcal{H}$.

4.2 The limit of T_N and Connes' non-commutative torus

4.2.1 Let $\alpha, \beta \in \mathbb{C}^\times$, $\alpha\mathbb{R}+\beta\mathbb{R} = \mathbb{C}$. Set, for $w \in \mathbb{C}$, the α-β - *decomposition* to be the uniquely determined decomposition

$$w = w_a\alpha + w_b\beta, \quad w_a, w_b \in \mathbb{R}.$$

Let $i_a, i_b \in \mathbb{R}$ be the coordinates of the decomposition

$$= i_a\alpha + i_b\beta, \text{ here and below } {}^2 = -1.$$

We also choose a real number h and assume that 1, $2\pi i_a$ and $2\pi i_a\mathrm{h}$ are linearly independent over $\mathbb{Q}$.

We define an additive α-β-version of the angular function, which we call **band**

$$\mathrm{bd}_h : \mathbb{C} \to 2\pi\mathrm{h}\mathbb{Z}, \text{ fixed } \mathrm{h} \in \mathbb{R} \setminus \mathbb{Q}$$

as follows.

First we define the function $r \mapsto [r]_h$ from $\mathbb{R}$ to $\mathbb{Z}$, the **pseudo-integer part of** r with the properties, for all $r \in \mathbb{R}$,

$$[0]_h = 0, \quad [r+1]_h = [r]_h + 1, \tag{10}$$

$$[r + 2\pi i_a]_h = [r]_h, \tag{11}$$

$$[r + 2\pi i_a \mathrm{h}]_h = [r]_h \tag{12}$$

Example Consider a direct sum decomposition

$$\mathbb{R} = R' \dot{+} 2\pi i_a\mathbb{Q} \dot{+} 2\pi i_a\mathrm{h}\mathbb{Q}, \text{ some subgroup } \mathbb{Q} < R' < \mathbb{R},$$

and set, for all $r' \in R'$, $c \in \mathbb{Q}$,

$$[r' + c_1 \cdot 2\pi i_a + c_2 \cdot 2\pi i_a\mathrm{h}]_h := [r' + (c_1 - [c_1]) \cdot 2\pi i_a + (c_2 - [c_2]) \cdot 2\pi i_a\mathrm{h}],$$

$[\,\cdot\,]$ the usual integer part of a real number. This satisfies (10)-(12).

Set

$$\mathrm{bd}_h\, w := 2\pi\mathrm{h}\,[w_a]_h.$$

Set,

$$\tilde{\mathbf{U}}: \; w \mapsto \alpha + w,$$
$$\tilde{\mathbf{V}}: \; w \mapsto \beta + w + \mathrm{bd}_h\, w.$$

We have

$$\tilde{\mathbf{V}}\tilde{\mathbf{U}}\, w = \tilde{\mathbf{U}}\tilde{\mathbf{V}} w + 2\pi \mathrm{h}, \tag{13}$$

4.2.2 Define the additive subgroup of $\mathbb{C}$

$$\mathcal{A}_h = \beta\mathbb{R} + 2\pi\mathrm{h}\mathbb{Z} + 2\pi\mathbb{Z}.$$

It is easy to see that
$\mathcal{A}_h$ *is the subgroup of all* **periods** *of* bd_h, *that is*

$$\mathcal{A}_h = \{a \in \mathbb{C} : \; \mathrm{bd}_h(a+w) = \mathrm{bd}_h w\}.$$

$\mathcal{A}_h$ *is exactly the subgroup of shifts* $w \mapsto a + w$ *of* $\mathbb{C}$ *which are automorphisms of* $(\mathbb{C}, \tilde{\mathbf{U}}, \tilde{\mathbf{V}})$.
$\mathcal{A}_h$ *is definable in* $(\mathbb{C}, +, \mathrm{bd}_h)$.

4.2.3 We have a very interesting two-sorted structure here with both sorts living on $\mathbb{C}$. One sort is $(\mathbb{C}, +, \mathrm{bd}_h)$ where also $\mathcal{A}_h$, $\mathbf{V}$ and $\tilde{\mathbf{U}}$ are definable. We also might want to consider multiplication by some elements, such as h, which brings the structure of $\mathbb{Q}(\mathrm{h})$-module on this sort.

The other sort is $(\mathbb{C}, +, \cdot)$ with 0 removed. It is perhaps more appropriate to consider this sort in the language of Zariski closed relations, which is of course interdefinable with $(+, \cdot)$. We denote this sort $\mathbb{C}^*$.

An important component of the whole structure is the map $\exp : \mathbb{C} \to \mathbb{C}^*$ from the first sort onto the other. We would like to know the stability status of any version of the structure. So far we can prove the following.

Proposition 1 The theory of $((\mathbb{C}, +, \mathcal{A}_h), \exp, \mathbb{C}^\times)$ is superstable, provided the Schanuel conjecture is true.

Proof It is easy to see that the statement follows if the expansion of $\mathbb{C}^\times$ with the unary predicate for the subgroup $\mathcal{G}_h = \exp(\mathcal{A}_h) = \exp(2\pi\mathrm{h}\mathbb{Z} + \beta\mathbb{R})$ is superstable. A stronger theorem, stating ω-stability of the theory, for $\mathcal{G} = \exp(\beta\mathbb{R} + \delta\mathbb{Q})$, $\beta \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{R})$, $\delta \in \mathbb{R} \setminus 2\pi\mathbb{Q}$, was proved in [Z2].

The same proof describes the elementary theory of the structure and yields superstability for the present theory. See also [Z3]. □

Notation $\mathcal{G}_h$ will stand for the subgroup $\exp(\mathcal{A}_h)$ of $\mathbb{C}^\times$.

On the other hand $((\mathbb{C},+,\mathrm{bd}_h),\exp,\mathbb{C}^\times)$ defines the following unstable structure on the sort $\mathbb{C}^\times$.

Denote, for $t=\exp w$,

$$\mathrm{ang}_h t := \exp \mathrm{bd}_h w.$$

This is well-defined and we have analogues of (7)-(8), where $q=\exp 2\pi \mathrm{h}$,

$$\begin{array}{l} \mathrm{ang}_h qt = \mathrm{ang}_h t, \\ \mathrm{ang}_h e^\beta t = \mathrm{ang}_h t, \quad \mathrm{ang}_h e^\alpha t = q\cdot \mathrm{ang}_h t. \end{array}$$

Hence, defining

$$\mathbf{U}: t\mapsto e^\alpha\cdot t, \quad \mathbf{V}: t\mapsto e^\beta\cdot t\cdot \mathrm{ang}_h t,$$

we get

$$\mathbf{VU}=q\mathbf{UV}, \text{ on } \mathbb{C}^\times.$$

It is easy to see that also

$$\mathbf{U}\exp w = \exp \tilde{\mathbf{U}} w, \quad \mathbf{V}\exp w = \exp \tilde{\mathbf{V}} w.$$

We define

$$\check{\mathrm{T}}_h := (\mathbb{C},+,\cdot,\mathrm{ang}_h).$$

This is an obvious analogue of $\check{\mathrm{T}}_N$ defined in 4.1.3.

Note that the group $\Gamma_h=\exp 2\pi\mathrm{h}\mathbb{Z}=\mathrm{ang}_h(\mathbb{C}^\times)$ is definable in $\check{\mathrm{T}}_h$.

The full analogy with $\check{\mathrm{T}}_N$ of 4.1.3 requires also a definition of p_h. We define

$$p_h:\mathbb{C}^\times\to\mathbb{C}^\times/\Gamma_h,$$

the canonical homomorphism. This agrees with 4.1.3, moreover in the finite case $\mathbb{C}^\times/\langle\epsilon\rangle$ can be definably identified with $\mathbb{C}^\times$ in the full Zariski language, in particular the whole construction is a Zariski structure (obviously, of finite Morley rank).

We also define the maps $\mathbf{u}$ and $\mathbf{v}$ on $\mathbb{C}^\times/\Gamma_h$ by

$$\mathbf{u}\, p_h(t) := p_h(\mathbf{U}t), \quad \mathbf{v}\, p_h(t) := p_h(\mathcal{V}t),$$

that is

$$\mathbf{u}:\ t\cdot\Gamma_h \mapsto e^{\alpha}\cdot t\cdot\Gamma_h, \quad \mathbf{v}:\ t\cdot\Gamma_h \mapsto e^{\beta}\cdot t\cdot\Gamma_h.$$

This is obviously well-defined.

Proposition 2 The group of shifts $t \mapsto gt$ on $\mathbb{C}^\times$ commuting with ang_h (and so with $\mathbf{U}$ and $\mathbf{V}$) is $\mathcal{G}_h$. This group is definable in $\check{\mathrm{T}}_h$. The theory of the structure $(\mathbb{C}, +, \cdot, \mathcal{G}_h, \Gamma_h)$ is superstable.

Proof Essentially the same argument as for Proposition 1. The superstability of the weaker structure $(\mathbb{C}, +, \cdot, \Gamma_h)$ is well-known and follows from *the Lang property* of Γ_h. $\Box$

4.2.4 Now we redefine $\check{\mathrm{T}}_N$ in a way compatible both with 4.1.3 and 4.2.3.

Define, for each positive $N \in \mathbb{N}$ the map

$$e_{Nh}:\ \mathbb{C} \to \mathbb{C}^\times; \quad e_{Nh}(w) = \exp(N^{-1}\mathrm{h}^{-1}w).$$

It is convenient to distinguish the copies of $\mathbb{C}^\times$ which are images of e_{Nh} for different N as T_N.

Set, for $t = e_{Nh}(w) \in T_N$,

$$\mathbf{U}_N t := e_{Nh}(\tilde{\mathbf{U}}w), \quad \mathbf{V}_N t := e_{Nh}(\tilde{\mathbf{V}}w).$$

It follows,

$$\mathbf{U}_N t := e_{Nh}(\alpha)\cdot t, \qquad \mathbf{V}_N t := e_{Nh}(\beta)\cdot t\cdot \exp\frac{2\pi}{N}[w_a]_h.$$

Denote

$$\mathrm{ang}_N(t) := \exp\frac{2\pi}{N}[w_a]_h.$$

This is well-defined. Indeed, any other representation of t would be of the form $t = e_{Nh}(w + 2\pi \mathrm{h} Nk)$, $k \in \mathbb{Z}$. But $(w + 2\pi \mathrm{h} Nk)_a = w_a + 2\pi i_a \mathrm{h} Nk$, and $[w_a + \mathrm{h}Nk]_h = [w_a]_h$ by (12).

Define

$$\check{\mathrm{T}}_N = (\mathbb{C}, +, \cdot, \mathrm{ang}_N)$$

This is the same definition as 4.1.3 except here we specified our choice of the angular function.

Proposition The group of periods of ang_N, that is $g \in \mathbb{C}^\times$ such that $\mathrm{ang}_N(g \cdot t) = \mathrm{ang}_N t$ is equal to

$$\mathcal{G}_{N^{-1}h^{-1}, \alpha h^{-1}} \cdot \mathbb{C}[N] = \exp(2\pi N^{-1}\mathrm{h}^{-1} + \alpha \mathrm{h}^{-1}\mathbb{Z} + \beta\mathbb{R}) \cdot \mathbb{C}[N].$$

In particular, this group is definable in the above $\check{\mathrm{T}}_N$ and the theory of

$$(\mathbb{C}, +, \cdot, \mathcal{G}_{N^{-1}h^{-1}, \alpha h^{-1}})$$

is superstable.

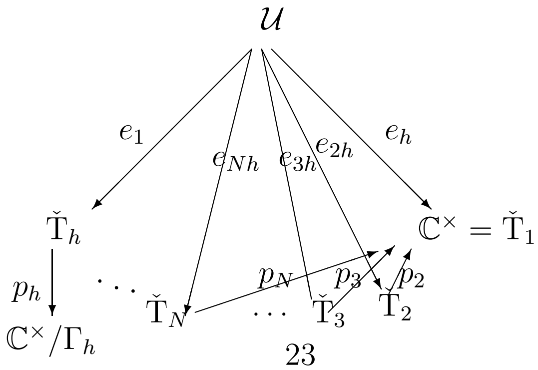
Problem Is the theory of $\check{\mathrm{T}}_N$ as given by the present construction, supersimple?

4.2.5 Denote

$$\mathcal{U} = (\mathbb{C}, +, \mathrm{bd}_h, \mathrm{h}\cdot).$$

By the construction in 4.2.3 and 4.2.4 $\check{\mathrm{T}}_N$ is definable in $(\mathcal{U}, \exp, \mathbb{C}^\times)$, for all $N \in \mathbb{N} \cup \{\mathrm{h}\}$.

The resulting picture is as follows, with the arrows showing definable surjections.



where $e_1(w) := \exp w$.

5 Quantum torus

Our aim here is to connect the construction of $\check{\mathrm{T}}_h$ to the well-known definition of the **noncommutative (quantum) torus** usually denoted T^2_h.

5.0.6 Following the pattern of 4.1.2 and 4.1.3 we introduce the $\mathbb{C}$-linear space $\mathcal{H}$ spanned by functions

$$x_k : \ \mathbb{C}^\times \to \mathbb{C}^\times, \quad k \in \mathbb{Z},$$

where $x_0 = x$ is the identity function and

$$x_k = \xi^k \cdot x, \quad \xi(t) = \mathrm{ang}_h t.$$

We have by 4.2.3,

$$x_k(\mathbf{U}t) = e^\alpha q^k \cdot x_k(t),$$
$$x_k(\mathbf{V}t) = e^\beta x_{k+1}(w),$$
$$\xi(\mathbf{U}t) = q \cdot \xi(t), \quad \xi(\mathbf{V}t) = \xi(t).$$

We can normalise $\mathbf{V}^*$ so that on $\mathcal{H}$ we have equivalent operators

$$\dot{\mathbf{U}} : \ \psi \mapsto \mathbf{U}^*\psi, \quad \mathbf{U}^*\psi(w) = \psi(\mathbf{U}w);$$
$$\dot{\mathbf{V}} : \ \psi \mapsto \xi \cdot \psi.$$

Using the identities above we get immediately the usual

$$\dot{\mathbf{U}}\dot{\mathbf{V}} = q\dot{\mathbf{V}}\dot{\mathbf{U}}.$$

5.0.7 The space $\mathcal{H}$ is an analogue of the space $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ of all Schwartz functions $\mathbb{R}^2 \to \mathbb{C}$ decaying at infinity along with all its derivatives faster than $\frac{1}{|x|^n}$, any n (see [C]), or $\mathcal{S}(\mathbb{Z}^2, \mathbb{C})$ the Hilbert space of Schwartz sequences, that is complex valued sequences $(c_{m,n})$ decaying faster than any polynomial of m, n.

In [C] with each leaf of the Kronecker foliation

$$L_a = \{\langle r, s\rangle \in \mathbb{R}^2 : \ s + \theta r = a\}$$

one associates the $\mathbb{C}[\dot{\mathbf{U}}, \dot{\mathbf{V}}, \dot{\mathbf{U}}^{-1}, \dot{\mathbf{V}}^{-1}]$-module $\mathcal{H}_a$ obtained by restricting functions of $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ to L_a and defining operators $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$. Namely, the operator

$\dot{\mathbf{U}}$ is defined by exactly the same formula as here and $\dot{\mathbf{V}}$ sends $\psi(r,s)$ (function of two real variables r and s) to $\exp(s)\cdot\psi(r,s)$ (notice that extra to these data there is a linear dependence between r and s). So, ξ is a good analogue of the function $\exp(s)$ taking values in the unit circle.

Notice that $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ are unitary operators if we see $\mathcal{H}_a$ as a Hilbert space. This makes the completion of $\mathbb{C}[\dot{\mathbf{U}},\dot{\mathbf{V}},\dot{\mathbf{U}}^{-1},\dot{\mathbf{V}}^{-1}]$ a C^*-algebra.

By A.Connes the quantum torus T^2_θ is the space of all the modules $\mathcal{H}_a$ on the correspondent L_a.

Remark Consider again the space of functions $\mathcal{H}$ and denote, for $a\in\mathbb{C}$, $\mathcal{H}_a$ the algebra obtained by restricting functions from $\mathcal{H}$ to the coset $a+\mathcal{A}_h$. It follows from Proposition 4.2.2(ii) that the action of $\dot{\mathbf{U}}$ and $\dot{V}$ on $\mathcal{H}$ induces a well-defined action on $\mathcal{H}_a$, so this is a $\mathbb{C}[\dot{\mathbf{U}},\dot{\mathbf{V}},\dot{\mathbf{U}}^{-1},\dot{\mathbf{V}}^{-1}]$-module for any $a\in\mathbb{C}$.

5.0.8 To understand further relations of Connes' construction to our T_h we prove the following.

Claim 1. There is a natural bijective correspondence

$$\phi:\ \mathbb{C}/\mathcal{A}_h\to\mathrm{T}^2_\theta,$$

for $\theta=\mathrm{h}$, where T^2_θ is seen as the space of leaves of the Kronecker foliation.

Indeed, we have the decomposition of $\mathbb{C}$ into two real lines

$$\mathbb{C}=\mathbb{R}+\alpha\mathbb{R},\quad \text{for any } z\in\mathbb{C}\ z=x+y\alpha,\ x,y\in\mathbb{R}.$$

Rescale the real coordinates

$$r:=\mathrm{h}^{-1}x,\quad s:=2\pi(2\pi i_a)^{-1}y$$

and consider the mapping onto the direct product of two unit circles

$$z\mapsto\langle x,y\rangle\mapsto\langle r,s\rangle\mapsto\langle\exp r,\exp s\rangle.$$

Under the map

$$2\pi\mathrm{h}\mathbb{Z}+2\pi i_a\alpha\mathbb{Z}\to\langle 2\pi\mathrm{h}\mathbb{Z},2\pi i_a\mathbb{Z}\rangle\to\langle 2\pi\mathbb{Z},2\pi\mathbb{Z}\rangle\to 1,$$

and since $2\pi - 2\pi i_a \alpha \in \beta\mathbb{R}$,

$$\beta\mathbb{R} \to \langle 2\pi, -2\pi i_a \rangle\mathbb{R} \to \langle 2\pi \mathrm{h}^{-1}, -2\pi \rangle\mathbb{R} \to L_0.$$

This establishes the bijection between the cosets of $\mathcal{A}_h$ and the leaves L_a of the foliation.

Claim 2. There is a bijective correspondence

$$\tilde{p}_h : \ \mathbb{C}/\mathcal{A}_h \to \mathbb{C}^\times/\mathcal{G}_h,$$

induced by p_h. Moreover, the action of $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ on $\mathbb{C}$ induces a well-defined action on $\mathbb{C}/\mathcal{A}_h$ and correspondingly the action on $\mathbb{C}^\times/\mathcal{G}_h$. The latter action coincides with the one induced by $\mathbf{u}$ and $\mathbf{v}$ on the cosets of $\mathcal{G}_h$.

This is the direct consequence of Proposition 4.2.2(iii) and the definition of p_h.

Corollary $\tilde{p}_h \circ \phi^{-1}$ identifies T^2_h with $\mathbb{C}^\times/\mathcal{G}_h$, with all the structure on the latter induced from $\check{T}_h$.

6 Recovering a real curve in a superstable structure

The structure $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$ of the complex field with the spiral (Kronecker foliation) playing the key role in sections 4.2 and 5 has another amazing model-theoretic property. It shows a way to reconcile the apparent contradiction of the attempt to interprete the Zariski geometries as observable worlds. The ultimate observations are made in trms of real numbers, yet the classical examples of Zariski structures come from algebraic geometry over an algebraically closed field. Moreover, the field of real numbers is inherently unstable in the sense of model theory.

6.0.9 Proposition Assume Schanuel's conjecture. Then the field $(\mathbb{R}, +, \cdot)$ of real numbers is $L_{\omega_1,\omega}$-definable in $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$.

Proof First we notice that the spiral

$$\mathcal{G}_h^0 = \exp(\beta\mathbb{R}) \subset \mathbb{C}$$

is an $L_{\omega_1,\omega}$-definable subset of our structure. In fact it is first-order type-definable.

Indeed, $\mathcal{G}_h/\mathcal{G}_h^0$ is the cyclic group $q^{\mathbb{Z}}$ and $\mathcal{G}_h^0$ is 'divisible' (if written additively). So, the subgroups $\mathcal{G}_h^n$ of n-powers of $\mathcal{G}_h$ is of index n in $\mathcal{G}_h$ and

$$\mathcal{G}_h^0 = \bigcap_{n\in\mathbb{N}} \mathcal{G}_h^n,$$

which is type-definable.

Now we refer to a general technical argument by D.Marker (unpublished):

Suppose $C \subset \mathbb{C}$ is a Jordan curve which is not dense in $\mathbb{C}$. Then in $(\mathbb{C}, +, \cdot, C)$ the reals $\mathbb{R} \subset \mathbb{C}$ are definable.

Applying this to $C = \mathcal{G}_h^0$ we get the desired conclusion.

This remarkable property of the superstable structure $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$ says that the reals are in effect present in our otherwise very *algebro-geometric* structure but to extract a real number one has to *go through an infinite (countable) process encoded in the correspondent $L_{\omega_1,\omega}$-formula.* In contrast the complex geometric structure is readily (first-order) available. Moreover, the subgroup $\mathcal{G}_h$ itself behaves in a typically global way like a complex analytic subvariety.

6.0.10 On the other hand the first-order theory of $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$ is tame and can be obtained by Hrushovski construction similarly to examples 5-14 of section 3. More precisely the Hrushovski construction adapted as in [Z7] produces a *canonical model*, call it $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$, of the first-order theory, see [Z12]. One of the defining fitures of the canonical model is the non-elementary stability and probably categoricity for uncountable cardinals. This immediately excludes the possibility of a real closed field being definable in $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$, in particular $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$ is not isomorphic to $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$. One of the abstract properties ditinguishing the two structures:

$(\mathbb{C}, +, \cdot, \mathcal{G}_h)$ is rigid while $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$ is at least ω-homogeneous.

Problem Study non-elementary properties of $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$.

(i) Prove that a natural non-elementary axiomatisation of the structure is uncountably categorical.

(ii) Prove that $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$ is analytic Zariski.

Very unusually we have here a case of a nice complete first-order theory which has two canonical models of cardinality continuum, $(\mathbb{C}, +, \cdot, \mathcal{G}_h)$ and $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$. In particular, both models must have a common elementary extension $(\tilde{\mathrm{F}}, +, \cdot, \mathcal{G}_h(\tilde{\mathrm{F}}))$.

In the first model various construction of real and complex analysis, such as integration, power series calculations and so on make sense. These constructions can be carried over to $(\tilde{\mathrm{F}}, +, \cdot, \mathcal{G}_h(\tilde{\mathrm{F}}))$ in the style of non-standard analysis. Now one can try to use a specialisation from $(\tilde{\mathrm{F}}, +, \cdot, \mathcal{G}_h(\tilde{\mathrm{F}}))$ to $(\mathrm{F}, +, \cdot, \mathcal{G}_h(\mathrm{F}))$ to give a new topological meaning to the same constructions.

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