Finite approximations of the Veneziano model (Draft)

Boris Zilber

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1 Introduction

1.1 This article is a step in a work program that aims, by meaans of model theory, to test the hypothesis: **universe is a huge-finite structure**.

A "huge-finite" structure is modeled as a pseudo-finite structure. In concrete terms we aim to solve a series of problems of the form:

Given a continuous structure $\mathbf{M}$ *(of physics) find an ultraproduct of (nice) finite structures* $\mathbf{M}_i$ *such that*

$$\mathrm{lm}\, \mathbf{M}_i = \mathbf{M}.$$

The success of such a program should imply that the continuous structures of physics can be replaced by huge-finite structures and thus the hypothesis is consistent with the established physical theory. Author's papers in the list of references report on some steps along the program.

The notion of limit (approximation) of structures has been studied in [Z26-1] where two forms of approximation, **local** and **global** have been introduced. We assume the reader to consult the paper for definitions and basic facts.

Global approximation corresponds to the situation when structure is seen globally along with its "boundaries", in contrast with local approximation which mimics a reconstruction of reality on the basis of observations with bounded capacity. It is useful to have in mind the basic examples of approximations by finite fields F_p (see [Z26-1]):

$$\mathsf{lm}^{\mathrm{loc}}\mathrm{F}_{\mathfrak{p}} = \mathbb{R}$$
$$\mathsf{lm}^{\mathrm{glob}}\mathrm{F}_{\mathfrak{p}} = \bar{\mathbb{C}}$$

Here $\mathrm{F}_{\mathfrak{p}}$ is the pseudo-finite field, $\mathrm{F}_{\mathfrak{p}} = {}^*\mathbb{Z}/\mathfrak{p}{}^*\mathbb{Z}$, $\mathfrak{p}$ is a non-standard prime. (It is useful and meaningful to assume that $\mathfrak{p}-1$ is highly divisible, that is divisble by any standard integer n).

Global approximation is given by a homomorphism (place) of structures, so the above can be written in the form:

$$\mathsf{lm}^{\mathrm{glob}} : \mathrm{F}_{\mathfrak{p}} \twoheadrightarrow \bar{\mathbb{C}}$$

Note, that compactified field $\bar{\mathbb{C}}$ can also be globally approximated by a sequence of residue rings $\mathbb{Z}/n\mathbb{Z}$ since there is a homomorphism

$${}^*\mathbb{Z}/\mathfrak{n}{}^*\mathbb{Z} \twoheadrightarrow \mathrm{F}_{\mathfrak{p}}.$$

1.2 The Veneziano model as a natural structure of physics. The Veneziano model is a structure introduced by G.Veneziano in 1967 with the aim of interpreting certain unexpected regularities in experimental data; see [Schwartz14]. In model-theoretic terms, the structure consists of the compactified complex plane $\bar{\mathbb{C}}$ equipped with a meromorphic binary function $\mathcal{B}(x, y)$, which was identified with the Euler beta function.

The ensuing physical theory relies on essential properties of $\mathcal{B}$ as a complex meromorphic function, including the location and structure of its poles, the residues at the poles, analytic continuation, and symmetries (dualities).

It is important to mention that the Veneziano model soon led to the notion of strings and to the beginning of string theory.

In this paper, we discuss the problem of globally approximating the Veneziano model by finite structures. Since the beta function is readily expressible in terms of the gamma function, it is enough to consider the approximation of the complex meromorphic function $\Gamma(z)$.

An appropriate model-theoretic setting requires treating the structure hosting Γ as a formal structure, akin to the pseudo-analytic formalism applied to exponentiation and other holomorphic functions. In particular, this approach is used in the paper [EP24] by S. Eterovi and A. Padgett, who initiated the study of Γ from a model-theoretic perspective.

Here, we construct a structure over a pseudo-finite field $\mathrm{F}_{\mathfrak{p}}$ equipped with functions $\Gamma_{\mathfrak{p}}$ and $\mathcal{B}_{\mathfrak{p}}$, and prove that it satisfies most of the identities of their classical counterparts, as well as reproducing the structure around the poles. In principle, the pseudo-finite structure appears sufficiently close to the classical one to be informally regarded as an approximation to the Veneziano model. Our aim, however, is to obtain a mathematically precise statement. We hope that it may be possible to prove the existence of a **global** approximation (as defined in [Z26-1]), assuming a suitable Schanuel-type conjecture for Γ.

As a matter of historical fact, soon after the publication of Veneziano's paper, it was realised that the structure could be interpreted in terms of an even richer structure that introduced new objects - strings; see [Schwartz14]. We mention this with the hope that this richer structure may also be shown to be approximable, in this case **locally**, by finite structures.

1.3 Gamma and Euler's beta functions.

Recall that

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

and the basic identity for Γ is

$$x \cdot \Gamma(x) = \Gamma(x+1) \tag{1}$$

This implies the identities for $\mathcal{B}$:

$$\frac{x}{x+y}\mathcal{B}(x,y) = \mathcal{B}(x+1,y); \quad \frac{y}{x+y}\mathcal{B}(x,y) = \mathcal{B}(x,y+1) \tag{2}$$

In fact, the Veneziano model treats $\mathcal{B}(x,y)$ as a diagram (corresponding to probable outcomes of experiments) where the marks over the arrows are calculated on the basis of (2).

$$\begin{tikzcd}
 & & \mathcal{B}(3,1)\ldots & \\
 & \mathcal{B}(2,1) \arrow[ur, "\frac{2}{3}"] \arrow[dr, "\frac{1}{3}"] & & \\
1=\mathcal{B}(1,1) \arrow[ur, "\frac{1}{2}"] \arrow[rr, "\frac{1}{6}"] & & \mathcal{B}(2,2)\ldots & \mathcal{B}(\pi,2)\ldots \\
 & \infty=\mathcal{B}(0,1) \arrow[ul, "0"] & & \\
 & & -1=\mathcal{B}(-1,1)\ldots \arrow[ul, "-\infty"] &
\end{tikzcd}$$

Here the value ∞ for $\mathcal{B}$ signify simple poles which are well-defined $\mathcal{B}$. It is also meaningful and important for applications to include residues of poles into the diagram.

For Γ we have a simpler diagram:

$$\begin{tikzcd}
 & & \Gamma(3)\ldots \\
 & \Gamma(2) \arrow[ur, "2"] & \\
1=\Gamma(1) \arrow[ur, "1"] & & \\
 & \infty=\Gamma(0) \arrow[ul, "0"] & \\
 & & -1=\Gamma(-1)\ldots \arrow[ul, "-\infty"]
\end{tikzcd}$$

1.4 Our formalism for Γ (and hence for $\mathcal{B}$) is a two-sorted structure $(\bar{\mathbb{C}}, \bar{\mathbb{C}}; \Gamma)$ with

$$\Gamma : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$$

the domain having the structure of the compactified field of complex numbers and the range - the compactified multiplicative semigroup.

Our aim supported by partial results is to show that there is a pseudo-finite structure that approximates globally $(\bar{\mathbb{C}}, \bar{\mathbb{C}}; \Gamma)$. Namely,

$$\Gamma_{\mathfrak{p}} : \; \mathrm{F}_{\mathfrak{p}} \to \mathrm{S}_{\mathfrak{p}}$$

where $\mathrm{S}_{\mathfrak{p}}$ is a pseudo-finite ring. We prove that our $\Gamma_{\mathfrak{p}}$ satisfies the analogue of the basic identity (1). In particular, the diagram for $\Gamma_{\mathfrak{p}}$ is cyclic of length $\mathfrak{p}$ and certain elements of $\mathrm{S}_{\mathfrak{p}}$ play role of poles. Also, $\Gamma_{\mathfrak{p}}$ satisfies a version of Gauss' multiplication formula and an analogue of Euler's reflection formula with $\sin \pi z$ replaced by an abstract periodic function $\mathbf{s}(z)$.

2 $\mathfrak{p}$-Gamma-function.

2.1 Let

$$\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})} = {}^*\mathbb{Z}/\mathfrak{p}^{\mathfrak{l}}\,{}^*\mathbb{Z}, \quad \mathfrak{l} \geq 1, \ \mathfrak{l} \in {}^*\mathbb{Z}$$

the set of residues modulo $\mathfrak{p}^{\mathfrak{l}}$ of non-standard integers.

We write elements of $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$ as $\mathbf{1}, \mathbf{2}, \ldots$, boldface numerals up to $\mathfrak{p}^{\mathfrak{l}}$, to distinguish form elements of $\mathrm{F}_{\mathfrak{p}} = \{1, 2, \ldots, \mathfrak{p}-1, \mathfrak{p}\}$. We use a canonical injection

$$z \mapsto \mathbf{z}. \tag{3}$$

We treat $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$ as the multiplicative semigroup and so (3) is the map from the field into the semigroup.

2.2 First we study the case when $\mathfrak{l} = 1$, $\mathrm{S}_{\mathfrak{p}} := \mathrm{S}_{\mathfrak{p}}^{(1)}$. Define

$$\Gamma_{\mathfrak{p}} : \mathrm{F}_{\mathfrak{p}} \to \mathrm{S}_{\mathfrak{p}},$$

the internally definable $\mathfrak{p}$-Γ-function

$$\Gamma_{\mathfrak{p}}(z) = \mathbf{1} \cdot \mathbf{2} \cdot \ldots \cdot (\mathbf{z} - 1).$$

Remark. ($\Gamma_{\mathfrak{p}}$ of a rational number).

Recall that we assume that $\mathfrak{p}-1$ is highly divisible. Then, for any $k, n \in \mathbb{N}$, $\frac{k(\mathfrak{p}-1)}{n}$ is a non-standard integer and in $\mathrm{F}_{\mathfrak{p}}$

$$\frac{k(\mathfrak{p}-1)}{n} = -k \cdot n^{-1} = -\frac{k}{n}.$$

Thus, expressions like

$$\Gamma_{\mathfrak{p}}(\pm\frac{k}{n})$$

is well-defined in the model.

2.3 A duplication formula (for $0 < z \leq \frac{\mathfrak{p}-1}{2}$)

$$\Gamma_{\mathfrak{p}}(2z) = \{1 \cdot 3 \cdot \ldots \cdot (2z-1)\} \cdot \{2 \cdot 4 \cdot \ldots \cdot (2z-2)\} =$$
$$\{1 \cdot 3 \cdot \ldots \cdot (2z-1)\} \cdot 2^{z-1}\{1 \cdot 2 \cdot \ldots \cdot (z-1)\} =$$
$$2^z\{\frac{\mathfrak{p}+1}{2} \cdot \frac{\mathfrak{p}+3}{2} \cdot \ldots \cdot \frac{\mathfrak{p}+2z-1}{2}\} \cdot 2^{z-1}\{1 \cdot 2 \cdot \ldots \cdot (z-1)\} =$$

$$2^{2z-1}\frac{(\frac{\mathfrak{p}-1}{2}+z)!}{(\frac{\mathfrak{p}-1}{2})!}\cdot(z-1)!=2^{2z-1}\Gamma_{\mathfrak{p}}(z)\frac{\Gamma_{\mathfrak{p}}(z+\frac{1}{2})}{\Gamma_{\mathfrak{p}}(\frac{1}{2})}$$

That is we proved:

$$\Gamma_{\mathfrak{p}}(2z)=2^{2z-1}\frac{(\frac{\mathfrak{p}-1}{2}+z)!}{(\frac{\mathfrak{p}-1}{2})!}\cdot(z-1)!=2^{2z-1}\Gamma_{\mathfrak{p}}(z)\frac{\Gamma_{\mathfrak{p}}(z+\frac{1}{2})}{\Gamma_{\mathfrak{p}}(\frac{1}{2})} \tag{4}$$

This is a correct formula for the classical Γ for all values of $z\in\mathbb{C}$.

The Legendre duplication formula

$$\Gamma_{\mathfrak{p}}(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{5}$$

is more precise in establishing in addition that

$$\Gamma(\frac{1}{2})=\sqrt{\pi}.$$

2.4 The multiplication formula.

More generally, for $0<z\leq\mathfrak{p}-\frac{\mathfrak{p}-1}{m}$,

$$\Gamma_{\mathfrak{p}}(mz)=$$

$$=\{1\cdot(m+1)\cdot\ldots\cdot((\mathbf{z-1})m+1)\}\cdot\{2\cdot(m+2)\cdot\ldots\cdot((\mathbf{z-1})m+2)\}\cdot\ldots\cdot\{m\cdot 2m\cdot\ldots\cdot((\mathbf{z-1})m)\}=$$

$$m^z\cdot\{\frac{(m-1)\mathfrak{p}+1}{m}\cdot\frac{(m-1)\mathfrak{p}+m+1}{m}\cdot\ldots\cdot\frac{(m-1)\mathfrak{p}+(\mathbf{z-1})m+1}{m}\}\cdot\ldots\cdot$$

$$m^z\cdot\{\frac{(m-2)\mathfrak{p}+2}{m}\cdot\frac{(m-2)\mathfrak{p}+m+2}{m}\cdot\ldots\cdot\frac{(m-2)\mathfrak{p}+(z-1)m+2}{m}\}\cdot\ldots\cdot\ldots$$

$$\ldots m^{z-1}\{1\cdot 2\cdot\ldots\cdot(\mathbf{z-1})\}=$$

$$m^{mz-1}\prod_{k=1}^{m-1}\frac{(\frac{(m-k)\mathfrak{p}+(z-1)m+k}{m})!}{(\frac{(m-k)\mathfrak{p}-m+k}{m})!}\cdot(\mathbf{z-1})!=$$

$$m^{mz-1}\prod_{k=1}^{m-1}\frac{(\mathbf{z}+\mathfrak{p}+\frac{k(\mathbf{1}-\mathfrak{p})}{m})!}{(\mathfrak{p}+\frac{k(\mathbf{1}-\mathfrak{p})}{m})!}\cdot(\mathbf{z-1})!=$$

$$=m^{mz-1}\Gamma_{\mathfrak{p}}(z)\prod_{k=1}^{m-1}\frac{\Gamma_{\mathfrak{p}}(z+\frac{k}{m})}{\Gamma_{\mathfrak{p}}(\frac{k}{m})}$$

Hence we proved:

$$\Gamma_{\mathfrak{p}}(mz) = m^{mz-1}\Gamma_{\mathfrak{p}}(z)\prod_{k=1}^{m-1}\Gamma_{\mathfrak{p}}(z+\frac{k}{m})\cdot\gamma_{\mathfrak{p}}(m)^{-1} \tag{6}$$

$$\gamma_{\mathfrak{p}}(m) := \prod_{k=1}^{m-1}\Gamma_{\mathfrak{p}}(\frac{k}{m}).$$

The canonical identity is

$$\Gamma(mz) = \frac{m^{mz-1/2}}{\sqrt{2\pi}^{m-1}}\Gamma(z)\prod_{k=1}^{m-1}\Gamma(z+\frac{k}{m}) \tag{7}$$

This uses a basic identity, the Euler's reflection formula relating $\Gamma(z)$ with $\sin \pi z$, and the multiplicative identity

$$\prod_{k=1}^{m-1}\sin(\pi\frac{k}{m}) = \frac{m}{2^{m-1}} \tag{8}$$

3 The case of infinite $\mathfrak{l}$

We assume below $\mathfrak{l}$ is an infinite non-standard integer.

3.1 Define $\Gamma_{\mathfrak{p}}^{(\mathfrak{l})} : \mathrm{F}_{\mathfrak{p}} \to \mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$,

$$\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z) := \mathbf{1}\cdot\mathbf{2}\cdot\ldots\cdot(\mathbf{z}-\mathbf{1}).$$

(with $\mathfrak{p} \in \mathrm{F}_{\mathfrak{p}}$).

We then have the basic identity, for $0 < z < \mathfrak{p}$:

$$\mathbf{z}\cdot\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z) = \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z+1) \tag{9}$$

3.2 Treating $\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(\mathfrak{p})$ as a pole.

Set

$$\eta := \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(\mathfrak{p}) = \mathbf{1}\cdot\mathbf{2}\cdot\ldots\cdot(\mathfrak{p}-1) \tag{10}$$

Let $\sim$ be the equivalence relation on $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$ generated by $\mathfrak{p}\eta \sim 1$, that is

$$a \sim b \text{ iff for some } m, n \in \mathbb{N}:\ a(\mathfrak{p}\eta)^m = b(\mathfrak{p}\eta)^n.$$

Let

$$\bar{\mathrm{S}}_{\mathfrak{p}} := \mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}/\sim$$

Set

$$\bar{\Gamma}_{\mathfrak{p}}(z) := \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z)/\sim$$

the image of $\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}$ in $\bar{\mathrm{S}}_{\mathfrak{p}}$.

3.3 Proposition.

- *The canonical map*

$$\mathbf{z} \mapsto \bar{\mathbf{z}}; \quad \mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})} \to \bar{\mathrm{S}}_{\mathfrak{p}}$$

is injective on the subset $\{\mathbf{z} \in \mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})} : \ z < \mathfrak{p}^k, \ k \in \mathbb{N}\}$.

- $\bar{\mathrm{S}}_{\mathfrak{p}}$ *is a semigroup.*
- *For all* $z \in \mathrm{F}_{\mathfrak{p}}$

$$\mathbf{z} \cdot \bar{\Gamma}_{\mathfrak{p}}(\mathbf{z}) = \bar{\Gamma}_{\mathfrak{p}}(\mathbf{z} + \mathbf{1}) \tag{11}$$

Proof. For invertible a, b, asssuming $a(\mathfrak{p}\eta)^m = b(\mathfrak{p}\eta)^n$ and comparing the $\mathfrak{p}$-valuations, one concludes $m = n$. Now use the ring structure on $\mathrm{S}_{\mathfrak{p}}$ to get $(a-b)(\mathfrak{p}!)^n = 0$. Hence $(a-b)$ is divisible by $\mathfrak{p}^{\mathfrak{l}-n}$ which is impossible for distinct $a, b < \mathfrak{p}^k$.

It is easy to check that $\sim$ is a congruence relation and thus the quotient is a semigroup.

For (11) we just need to extend (9) to $z = \mathfrak{p}$, that is to check that

$$\mathfrak{p} \cdot \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(\mathfrak{p}) \sim 1 = \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(1).$$

This is just a calculation by definitions.

□

3.4 Let Δ be a subgroup of $\bar{\mathrm{S}}_{\mathfrak{p}}$ generated by elements

$$\delta_z := \mathbf{1} \pm \mathfrak{p} \cdot \mathbf{x} \cdot \mathbf{y}^{-1}; \quad 0 < x, y < \mathfrak{p}.$$

We consider the factor-semigroup

$$\hat{\mathrm{S}}_{\mathfrak{p}} := \bar{\mathrm{S}}_{\mathfrak{p}}/\Delta$$

Lemma. *The canonical homomorphism*

$$\bar{\mathrm{S}}_{\mathfrak{p}} \twoheadrightarrow \hat{\mathrm{S}}_{\mathfrak{p}}$$

is injective on the subsets

$$R_\Gamma := \{\bar{\Gamma}_{\mathfrak{p}}(z) : z \in \mathrm{F}_{\mathfrak{p}}\} \textit{ and } \{\mathbf{z} : z \in \mathbb{Z}\}$$

Proof. It follows from (11) using induction in the pseudo-finite structure $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$, that for two distinct elements γ_1 of R_Γ

$$\gamma_1 \cdot \gamma_2^{-1} \neq 1 \bmod \mathfrak{p}.$$

Thus $\gamma_1 \cdot \gamma_2^{-1} \notin \Delta$.

The same argument applies for the second subset. □

3.5 Poles and residues in $\hat{\mathrm{S}}_{\mathfrak{p}}$.

We set $\eta = \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p})$ to represent a simple pole with residue equal 1.

Claim. For $n \in \mathbb{N}$

$$\bar{\Gamma}_{\mathfrak{p}}(-n) = \frac{(-1^n)}{n!} \cdot \eta \tag{12}$$

Proof. Note that by definitions $\bar{\Gamma}_{\mathfrak{p}}(-n) = \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p} - n)$ and by (11),

$$(\mathfrak{p} - n)\bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p} - n) = \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p} + 1 - n).$$

Apply the latter identity n times:

$$(\mathfrak{p} - 1) \cdot \ldots \cdot (\mathfrak{p} - n)\bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p} - n) = \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p}).$$

Now use the fact that

$$(\mathfrak{p} - k)/\Delta = -k/\Delta$$

for $k = 1, \ldots, n$. One gets

$$(-1^n)n! \cdot \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p} - n) = \bar{\Gamma}_{\mathfrak{p}}(\mathfrak{p})$$

and (12) follows. □

3.6 Corollary. $\bar{\Gamma}_{\mathfrak{p}}(-n)$ *is a simple pole with residue* $\frac{(-1)^n}{n!}$.

Remark. This is a true statement for the complex Γ.

3.7 The multiplication formula in $\hat{\mathrm{S}}_{\mathfrak{p}}$.

We analyse $\bar{\Gamma}_{\mathfrak{p}}(mz)$ as in 2.4 and consider, for $k = 0, 1, \ldots, m-1$ products

$$P_k(z) := k \cdot (m + k) \cdot (2m + k) \cdot \ldots \cdot ((z - 1)m + k) = \prod_{i=0}^{z-1}(m\mathbf{i} + \mathbf{k})$$

so that

$$\bar{\Gamma}_{\mathfrak{p}}(mz) = \prod_{k=0}^{m-1} P_k(z).$$

Now we assume that z is finite, that is $z \in \mathbb{N}$. Note that in $\hat{\mathrm{S}}_{\mathfrak{p}}$

$$(m\mathbf{i} + \mathbf{k}) = (m-k)\mathfrak{p} + (m\mathbf{i} + \mathbf{k}) = k(\mathbf{1} - \mathfrak{p}) + m(\mathbf{1} + \mathbf{i})$$

Note that $\mathfrak{p} - 1$ is divisible by m and thus the finite product $P_k(z)$ can be rewirtten

$$P_k(z) = m^z \cdot \prod_{i=0}^{z-1} (\mathbf{1} + \mathbf{i} - \frac{k(\mathfrak{p}-1)}{m})$$

Note that the integers $\mathbf{1} + \mathbf{i} - \frac{k(\mathfrak{p}-1)}{m})$ are positive since $(m-k)\mathfrak{p} + (m\mathbf{i} + \mathbf{k}) > 0$. Therefore the factors in the product are consecuitive positive numbers starting from $\frac{(m-k)\mathfrak{p}+\mathbf{k}}{m}$ and ending at $\frac{(m-k)\mathfrak{p}+\mathbf{k}}{m} + \mathbf{z} - \mathbf{1}$, both within $\{\mathbf{1}, \ldots, \mathfrak{p} - \mathbf{1}\}$. Thus we can write

$$\prod_{i=0}^{z-1} (\mathbf{1} + \mathbf{i} - \frac{k(\mathfrak{p}-1)}{m}) = \frac{\bar{\Gamma}_{\mathfrak{p}}(z + \frac{k}{m})}{\bar{\Gamma}_{\mathfrak{p}}(\frac{k}{m})}.$$

Hence

$$\begin{array}{l} \bar{\Gamma}_{\mathfrak{p}}(mz) = m^{mz-1} \bar{\Gamma}_{\mathfrak{p}}(z) \prod_{k=1}^{m-1} \frac{\bar{\Gamma}_{\mathfrak{p}}(z+\frac{k}{m})}{\bar{\gamma}_{\mathfrak{p}}(m)} \\ \text{where} \\ \quad \bar{\gamma}_{\mathfrak{p}}(m) = \prod_{k=1}^{m-1} \bar{\Gamma}_{\mathfrak{p}}(\frac{k}{m}) \end{array} \tag{13}$$

Thus we proved the analogue of Gauss' multiplication formula (see reference (7) and related commentaries above) restricted to $z \in \mathbb{N}$.

3.8 The reflection formula in $\hat{\mathrm{S}}_{\mathfrak{p}}$..

For $z \in \mathrm{F}_{\mathfrak{p}}^*$

$$\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z) \cdot \Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(\mathfrak{p} + 1 - z) = \mathbf{1} \cdot \mathbf{2} \cdot \ldots \cdot (\mathbf{z} - \mathbf{1}) \cdot \mathbf{1} \cdot \mathbf{2} \cdot \ldots \cdot (\mathfrak{p} - \mathbf{z}) =$$

$$\mathbf{1} \cdot \mathbf{2} \cdot \ldots \cdot (\mathbf{z} - \mathbf{1}) \cdot (\mathfrak{p} - \mathbf{1})! \cdot (\mathfrak{p} - \mathbf{1})^{-1} \cdot (\mathfrak{p} - \mathbf{2})^{-1} \cdot \ldots \cdot (\mathfrak{p} - (\mathbf{z} - \mathbf{1}))^{-1} =$$

$$=: \mathbf{s}(z)^{-1} \tag{14}$$

where we denoted

$$\mathbf{s}(z) = \eta^{-1} \cdot \prod_{k=1}^{z-1} (\mathfrak{p} - \mathbf{k})\mathbf{k}^{-1} = \eta^{-1} \cdot \prod_{k=1}^{z-1} (\mathfrak{p}\mathbf{k}^{-1} - \mathbf{1})$$

and consider (14) in $\hat{\mathrm{S}}_{\mathfrak{p}}$.

Note that

$$\mathbf{s}(\mathfrak{p}) = \mathbf{s}(0) = \eta^{-1} \tag{15}$$

The function is periodic, with half-period = 1

$$\mathbf{s}(z+1) = -\mathbf{s}(z) \tag{16}$$

since

$$\mathbf{s}(z+1) = (\mathfrak{p}\mathbf{z}^{-1} - \mathbf{1}) \cdot \mathbf{s}(z).$$

Clearly, (14) is an analogue of Euler's reflection formula

$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

where η takes the role of ∞ or a value at the pole of Γ.

3.9 Proposition. *There exists a multiplicative place (homomorphism into a compactified semigroup)*

$$\mathfrak{h} : \bar{\mathrm{S}}_{\mathfrak{p}} \twoheadrightarrow \bar{\mathbb{C}}$$

satisfying:

$$\mathfrak{h}(\mathfrak{p}\eta) = 1 \tag{17}$$

$$\mathfrak{h}(\mathfrak{p}) = 0 \tag{18}$$

$$\mathfrak{h}(\eta) = \infty \tag{19}$$

$$\mathfrak{h}(\mathbf{z}) = z, \text{ for } z \in \mathbb{Z} \tag{20}$$

$$\mathfrak{h}(\frac{\mathfrak{p} - 1}{n}) = -\frac{1}{n}, \text{ for } n \in \mathbb{N} \tag{21}$$

Proof. Let $v_{\mathfrak{p}}$ and v_η be valuations on $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$ and define

$$v^*(\mathbf{z}) = v_{\mathfrak{p}}(\mathbf{z}) - v_\eta(\mathbf{z}).$$

v^* is then also well-defined for $\bar{\mathbf{z}} \in \bar{\mathrm{S}}_{\mathfrak{p}}$.

Define

$$\begin{array}{l}\mathfrak{h}(\mathbf{z}) = 0 \text{ iff } v^*(\mathbf{z}) > 0 \\ \mathfrak{h}(\mathbf{z}) = \infty \text{ iff } v^*(\mathbf{z}) < 0\end{array}.$$

It remains to define $\mathfrak{h}$ on the semigroup of elements satisfying $v^*(\mathbf{z}) = 0$. By 3.3 this group is isomorphic to a subgroup H of the group of invertible elements of $\mathrm{S}_{\mathfrak{p}}^{(\mathfrak{l})}$ and the latter is torsion-free and of rank at least continuum. Such a group has always a surjective homomorphism $\mathfrak{h}$ onto $\mathbb{C}^\times$. Moreover, we can arrange that $\mathfrak{h}(\mathbf{z}) = z$ and $\mathfrak{h}(\mathfrak{p} - \mathbf{z}) = -z$ for all $z \in \mathbb{Z}$. Both (20) and (21) follow. $\Box$

In order to suggest a direction of a possible interesting research we formulate the following:

3.10 Conjecture. *Assuming a "Schanuel conjecture" for* Γ, *there is a homomorphism*

$$\mathfrak{h} : \left(\mathrm{F}_{\mathfrak{p}}, \bar{\mathrm{S}}_{\mathfrak{p}}\right) \twoheadrightarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}\right)$$

that is a place of fields in the first coordinate and a place of semigroups in the second, such that (17) - (21) hold and

$$\mathfrak{h}(\Gamma_{\mathfrak{p}}^{(\mathfrak{l})}(z)) = \Gamma(\mathfrak{h}(z)) \tag{22}$$

$$\mathfrak{h}(\mathbf{s}(z)) = \pi^{-1} \sin(\pi \mathfrak{h}(z)) \tag{23}$$

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