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CHARACTERISTIC ELEMENTS FOR p-TORSION IWASAWA MODULES

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Abstract

Let G be a compact $p-$adic analytic group with no elements of order p. We provide a formula for the characteristic element [3] of any finitely generated $p-$torsion module M over the Iwasawa algebra Λ_G of G in terms of twisted $\mu-$invariants of M, which are defined using the Euler characteristics of M and its twists. A version of the Artin formalism is proved for these characteristic elements. We characterize those groups having the property that every finitely generated pseudo-null $p-$torsion module has trivial characteristic element as the $p-$nilpotent groups. It is also shown that these are precisely the groups which have the property that every finitely generated $p-$torsion module has integral Euler characteristic. Under a slightly weaker condition on G we decompose the completed group algebra Ω_G of G with coefficients in $\mathbb{F}_p$ into blocks and show that each block is prime; this generalizes a result of Ardakov and Brown [1]. We obtain a generalization of a result of Osima [12], characterizing the groups G which have the property that every block of Ω_G is local. Finally, we compute the ranks of the K_0 group of Ω_G and of its classical ring of quotients $Q(\Omega_G)$ whenever the latter is semisimple.

1. Introduction

1.1. Iwasawa algebras. In recent years there has been increasing interest in noncommutative Iwasawa algebras. These are the completed group algebras

$$\Lambda_G \quad := \quad \varprojlim \mathbb{Z}_p[G/N],$$

where $\mathbb{Z}_p$ denotes the ring of $p-$adic integers, G is a compact $p-$adic analytic group, and the inverse limit is taken over the open normal subgroups of G. Closely related is the epimorphic image Ω_G of Λ_G,

$$\Omega_G \quad := \quad \varprojlim \mathbb{F}_p[G/N],$$

where $\mathbb{F}_p$ is the field of p elements. In the paper [3], Coates *et al* develop the notion of a characteristic element for a certain class of finitely generated Λ_G-modules, when G has no elements of order p. We briefly recall how this is done.

1.2. The localisation sequence. An additional hypothesis on G is involved: it is assumed in [3] that G has a closed normal subgroup H such that G/H is isomorphic to $\mathbb{Z}_p$. Let $\mathfrak{M}_H(G)$ denote the category of all finitely generated Λ_G-modules M such that $M/M(p)$ is finitely generated over Λ_H; here $M(p)$ denotes the $p-$primary part of M. In fact, $\mathfrak{M}_H(G)$ consists of precisely the S^*-torsion modules for a certain Ore subset S^* of Λ_G depending on H.

Because G is assumed to have no elements of order p, Λ_G has finite global dimension. As a result, associated with the Ore set S^* we have an exact sequence of $K-$groups

$$K_1(\Lambda_G) \to K_1((\Lambda_G)_{S^*}) \stackrel{\partial_G}{\to} K_0(\mathfrak{M}_H(G)) \to K_0(\Lambda_G) \to K_0((\Lambda_G)_{S^*}) \to 0.$$

The connecting homomorphism ∂_G is shown to be surjective in [3, Proposition 3.4], enabling us to define a *characteristic element* of a module $M \in \mathfrak{M}_H(G)$ to be any $\xi_M \in K_1((\Lambda_G)_{S^*})$ such that $\partial_G(\xi_M) = [M] \in K_0(\mathfrak{M}_H(G))$.

1.3. In this paper, we will be concerned with a smaller class of Iwasawa modules, namely the $p-$torsion ones. Let $\mathcal{D}$ denote the category of all finitely generated $p-$torsion Λ_G-modules. One can parallel the above construction for the central Ore set $T = \{1, p, p^2, \ldots\}$ of Λ_G and obtain an analogous exact sequence of $K-$groups

$$K_1(\Lambda_G) \to K_1((\Lambda_G)_T) \stackrel{\partial_G}{\to} K_0(\mathcal{D}) \to K_0(\Lambda_G) \to K_0((\Lambda_G)_T) \to 0.$$

Again, it can be shown (see Corollary 5.2) that ∂_G is surjective, so we may define a *characteristic element* of M to be any $\xi_M \in K_1((\Lambda_G)_T)$ such that

$$\partial_G(\xi_M) = [M] \in K_0(\mathcal{D}).$$

1.4. Recall [3, §3] that S^* is defined to be $\cup_{n=0}^{\infty} Sp^n$, where

$$S = \{x \in \Lambda_G : \Lambda_G/x\Lambda_G \quad \text{is finitely generated over} \quad \Lambda_H\}.$$

Hence T is always contained in S^*, so there exists a natural commutative diagram of $K-$groups

$$\begin{array}{ccc} K_1((\Lambda_G)_T) & \xrightarrow{\partial_G} & K_0(\mathcal{D}) \\ \downarrow & & \downarrow \\ K_1((\Lambda_G)_{S^*}) & \xrightarrow{\partial_G} & K_0(\mathfrak{M}_H(G)) \end{array}$$

which shows that our characteristic elements are compatible with those considered in [3]. Moreover, any S^*-torsion module M fits into a short exact sequence $0 \to M(p) \to M \to M/M(p) \to 0$ where $M(p)$ is $p-$torsion and $M/M(p)$ is $p-$torsion free and $S-$torsion. This shows that it is sufficient to consider characteristic elements for $p-$torsion modules and those for $S-$torsion modules separately.

1.5. Twisted $\mu-$invariants. Now let G be an arbitrary compact $p-$adic analytic group. Then Λ_G has finitely many simple modules up to isomorphism, $V_1, \ldots, V_s$ say, and each one is a finite dimensional $\mathbb{F}_p-$vector space. Assuming G has no elements of order p, every finitely generated $p-$torsion Λ_G-module M has finite *Euler characteristic*, defined by

$$\chi(G, M) = \prod_{n \geqslant 0} |\operatorname{Tor}_n^{\Lambda_G}(M, \mathbb{Z}_p)|^{(-1)^n}.$$

We define the *$i-$th twisted $\mu-$invariant* of M for $i = 1, \ldots, s$ by the formula

$$\mu_i(M) = \frac{\log_p \chi(G, (\operatorname{gr}_p M) \otimes_{\mathbb{F}_p} V_i^*)}{\dim_{\mathbb{F}_p} \operatorname{End}_{\Omega_G}(V_i)}.$$

Here V_i^* is the dual module to V_i and $\operatorname{gr}_p M$ is the graded module of M with respect to the $p-$adic filtration on M; this is a finitely generated Ω_G-module. It turns out that $\mu_i(M)$ is always an *integer*; moreover, we are able to give an explicit description of the characteristic element of M in terms these twisted $\mu-$invariants:

Theorem. *Let $\theta : (\Lambda_G)_T^\times \to K_1((\Lambda_G)_T)$ be the canonical homomorphism and let M be a finitely generated $p-$torsion Λ_G-module. Then*

$$\xi_M = \theta\left(\prod_{i=1}^{s} f_i^{\mu_i(M)}\right),$$

where $f_i = 1 + (p-1)e_i$ and e_i is an idempotent in Λ_G such that V_i is the unique simple quotient module of $e_i\Lambda_G$.

See (5.6) for more details.

1.6. $\mu-$invariants by Venjakob and Howson. By [1, Theorem C], Ω_G is a domain if and only if G is a pro-p group of finite rank with no elements of order p. If these equivalent conditions hold, then the *rank* of a finitely generated Ω_G-module is defined in the usual way, using the fact that Ω_G is a Noetherian domain.

Venjakob [18, Definition 3.32] defines the $\mu-$invariant of a finitely generated Λ_G-module M to be the Ω_G-rank of $\operatorname{gr}_p M(p)$, the graded module of the $p-$torsion part of M. See also the paper [8] by Howson for more precursors to this notion.

Because G is pro-p, Λ_G has a unique simple module, namely the trivial module $V_1 = \mathbb{F}_p$. It now follows immediately from Lemma 8.3 that when M is $p-$torsion, $\mu(M)$ coincides with the first twisted $\mu-$invariant $\mu_1(M)$ of M defined in (1.5) - this motivates our terminology.

Note that in this case we may take $e_1 = 1$ in Theorem 1.5. Then the formula for the characteristic element of our $p-$torsion module M simplifies down to

$$\xi_M = \theta\left(p^{\mu(M)}\right). \tag{1}$$

1.7. Artin formalism for characteristic elements. Let H be an open normal subgroup of G. It is convenient to have a connection between the characteristic element of a Λ_G-module M and the characteristic element of the restriction $\operatorname{Res}^G_H M$ of M to Λ_H. Such a connection is commonly known as the *Artin formalism*, and it usually involves twists of M at certain Artin representations of G; recall that a continuous representation $\rho : G \to \mathrm{GL}_n(\mathbb{Z}_p)$ of G is said to be an *Artin representation* if the kernel of ρ is an open subgroup of G. [3, Theorem 3.10] establishes an Artin formalism for Euler characteristics.

Let Δ denote the finite group G/H and let $\mathcal{V}(\Delta)$ be the set of all absolutely irreducible representations of G over $\overline{\mathbb{Q}_p}$. Then there exists a finite extension L of $\mathbb{Q}_p$ such that each $\rho \in \mathcal{V}(\Delta)$ can be realized over L. Let $\mathcal{O}_L$ be the ring of integers of L. For each $\rho \in \mathcal{V}(\Delta)$ we can then find a finitely generated $\mathcal{O}_L-$module E_ρ of $\mathcal{O}_L-$rank n_ρ, say, such that the image of ρ is contained in $\operatorname{Aut}(E_\rho)$; in this way, E_ρ becomes a Λ_G-module. Let $\mathrm{tw}_\rho(M)$ denote the Λ_G-module $M \otimes_{\mathbb{Z}_p} E_\rho$ equipped with the diagonal action of G - this is $p-$torsion whenever M is. Our version of the Artin formalism is given by the following result:

Theorem. *Let $\lambda_{G,H} : K_1((\Lambda_H)_T) \to K_1((\Lambda_G)_T)$ be the natural map and let M be a finitely generated $p-$torsion Λ_G-module. Then*

$$\lambda_{G,H}(\xi_{\operatorname{Res}^G_H M})^{|L:\mathbb{Q}_p|} = \prod_{\rho \in \mathcal{V}(\Delta)} \xi^{n_\rho}_{\mathrm{tw}_\rho(M)}.$$

See (6.8) for more details. Note that if we "evaluate this at 0", or equivalently, take the image of this equation under the canonical map

$$K_1((\Lambda_G)_T) \to K_1(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times,$$

we obtain [3, Theorem 3.10] for $p-$torsion modules, as shown in Corollary 6.8.

1.8. Pseudo-null modules. Recall [18, Theorem 3.26] that if G has no elements of order p, then Λ_G is an Auslander regular ring. We will not give the full technical definition of Auslander regularity here; see [4] for an excellent introduction to the subject. If R is a ring, then a finitely generated $R-$module M is said to be *pseudo-null* if its *grade* $j_R(M)$ satisfies $j_R(M) \geqslant 2$.

Let M be a finitely generated $p-$torsion Λ_G-module. It is shown in [18, Remark 3.33] that if G is a pro-p group of finite rank with no elements of order p then $\mu(M) = 0$ if and only if M is pseudo-null. One would hope that a suitable generalization of this would be true for compact $p-$adic analytic groups which are not necessarily pro-p. In view of (1), one might hope that

$$\xi_M = 1 \quad \text{if and only if} \quad M \quad \text{is pseudo-null.} \tag{2}$$

Whilst Corollary 8.3 shows that $\xi_M = 1$ certainly implies that M is pseudo-null, the converse is false in general, as is shown in Example 9.6.

1.9. Integrality. We say that a finitely generated Λ_G-module M has *integral Euler characteristic* if $\chi(G, M) \in \mathbb{Z}$. Another nice property of Euler characteristics in the case when G is pro-p without elements of order p is *integrality*: every finitely generated $p-$torsion Λ_G-module has integral Euler characteristic.

Again, Example 9.6 shows that this property fails for more general groups G. On the positive side, we are able to characterize those G for which integrality holds. Intriguingly, these groups coincide with those for which (2) holds:

Theorem. *Let G be a compact $p-$adic analytic group with no elements of order p. Then the following are equivalent:*

(a) $\xi_M = 1$ for all finitely generated $p-$torsion pseudo-null Λ_G-modules M,
(b) $\chi(G, M) \in \mathbb{Z}$ for all finitely generated $p-$torsion Λ_G-modules M,
(c) G is $p-$nilpotent.

See (11.5) for a proof. The definition of *$p-$nilpotent* groups is given in 11.2; we simply note here that if G has no elements of order p, then G is $p-$nilpotent if and only if it is a semidirect product of a finite $p'-$group with a pro-p group of finite rank.

1.10. Blocks of Ω_G. To establish Theorem 1.9, we need the concept of *blocks* (2.2), which is a standard tool in the modular representation theory of finite groups. Let Δ^+ denote the largest finite normal subgroup of G.

Theorem. *Suppose that $p \nmid |\Delta^+|$. Then the each block of Ω_G is a prime ring.*

See (9.2) for more details. This result can be thought of as a generalization of [1, Theorem A], which states that Ω_G is prime if and only if $\Delta^+ = 1$. Note that the condition $p \nmid |\Delta^+|$ is equivalent to Ω_G being semiprime by [1, Theorem B].

1.11. Local blocks. In the modular representation theory of finite groups one is also sometimes interested in those blocks which have exactly one simple module (when viewing the block as a ring in its own right). Such blocks are called *primary* or *local.* It is a well-known result of Osima [12] that for a finite group G, every block of the group algebra $\mathbb{F}_p G$ is local if and only if G is $p-$nilpotent. We establish a generalization of this to compact $p-$adic analytic groups in (11.4):

Theorem. *Let G be a compact $p-$adic analytic group. Then every block of Ω_G is local if and and only if G is $p-$nilpotent.*

1.12. Ranks of K_0-groups of certain algebras. Let kG be the completed group algebra of G with coefficients in k, a finite field extension of $\mathbb{F}_p$. We are able to explicitly compute the number of simple $kG-$modules, or equivalently, the rank of $K_0(kG)$. By Proposition 7.2, kG has an Artinian ring of quotients, $Q(kG)$. When $p \nmid |\Delta^+|$, we also compute $\operatorname{rk} K_0(Q(kG))$; this number turns out to be equal to the number of blocks of kG by Proposition 9.4.

Theorem. *Let G be a compact $p-$adic analytic group. Fix an open normal pro-p subgroup N of G. Then*

(a) The rank of $K_0(kG)$ equals the number of $G \times \mathcal{G}_k-$orbits on $(G/N)_{\mathrm{reg}}$.
(b) If $p \nmid |\Delta^+|$, the rank of $K_0(Q(kG))$ equals the number of $G \times \mathcal{G}_k-$orbits on Δ^+.

See (12.7) for the relevant notation and details.

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1.14. Conventions. All rings are assumed to be associative and to have a unit element. All modules are assumed to be *right* modules, unless explicitly stated otherwise. When we speak of a ring-theoretic property like "Noetherian" or "regular", we implicitly assume that both the right and left handed property holds. For a ring R, $R^\times$ denotes the group of units of R. The reader should be aware of the slightly nonstandard notation adopted in (3.2).

2. Generalities

2.1. Idempotents. Let A be a ring. An element $e \in A$ is an *idempotent* if $e^2 = e$. Two idempotents e_1, e_2 in A are said to be *orthogonal* if $e_1e_2 = e_2e_1 = 0$; thus e and $1-e$ are always orthogonal whenever e is an idempotent.

The nonzero (central) idempotent e is *(centrally) primitive* if it is not possible to find two nonzero orthogonal (central) idempotents $e_1, e_2 \in A$ with $e = e_1 + e_2$.

2.2. Blocks. Let A be a ring and let $A = B_1 \oplus \cdots \oplus B_r$ be a decomposition of A into indecomposable two-sided ideals $B_i \neq 0$; such a decomposition exists whenever A is Noetherian.

The B_i's are known as the *blocks* of A. Each one is generated as an ideal by a central idempotent e_i of A, corresponding to the decomposition $1 = e_1 + \cdots + e_r$. Note that each e_i is centrally primitive, but need not be primitive.

Note also that each $B_i = e_i A$ is itself a ring, with the multiplication and addition inherited from A, but with identity element e_i. Thus block decomposition expresses A as a direct sum of algebras. We will write $b(A)$ for the number of blocks of A; note that this is also the number of terms in the decomposition of 1 into a sum of orthogonal centrally primitive idempotents.

2.3. Grothendieck groups. Recall that a category $\mathcal{A}$ is *small* if the collection of objects in $\mathcal{A}$ forms a set. Let $\mathcal{A}$ be a small abelian category. A full additive subcategory $\mathcal{B}$ of $\mathcal{A}$ is *admissible* if whenever $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$ such that M and M'' belong to $\mathcal{B}$, then M' also belongs to $\mathcal{B}$ [10, 12.4.2].

The *Grothendieck group* $K_0(\mathcal{B})$ of $\mathcal{B}$ is the abelian group with generators $[M]$ where M runs over all the objects of $\mathcal{B}$ and relations $[M] = [M'] + [M'']$ for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ [10, 12.4.3].

If A is a ring, then $\mathcal{P}(A)$, the category of all finitely generated projective modules is an admissible subcategory of $\mathcal{M}(A)$, the category of all finitely generated $A-$modules. The *Grothendieck groups* of A are defined as follows:

- $K_0(A) := K_0(\mathcal{P}(A))$, and
- $\mathcal{G}_0(A) := K_0(\mathcal{M}(A))$.

2.4. Semisimple rings. We record some information about K_0-groups of semisimple rings. The following result is well-known:

Lemma. *Let A be a semisimple ring and let $V_1, \ldots, V_s$ be a complete list of representatives for the isomorphism classes of simple $A-$modules.*

(a) If P, Q are finitely generated $A-$modules, then

$$P \cong Q \quad \textit{if and only if} \quad [P] = [Q] \quad \textit{in} \quad K_0(A)$$

(b) $K_0(A) = \bigoplus_{i=1}^{s} \mathbb{Z}[V_i]$ is free of rank s.
(c) $b(A) = \operatorname{rk} K_0(A)$.

2.5. Semilocal rings. Let A be a ring. We will always write $J(A)$ for the Jacobson radical of A and $\overline{A}$ for $A/J(A)$. We say that A is

- *semilocal* if $\overline{A}$ is Artinian,
- *local* if $\overline{A}$ is simple Artinian, and
- *scalar local* if $\overline{A}$ is a division ring.

We say that A is a *complete semilocal ring* if A is semilocal and complete with respect to the $J(A)-$adic filtration.

2.6. Idempotent lifting. Let A be a complete semilocal ring and let $V_1, \ldots, V_s$ be the simple $\overline{A}-$modules as in (2.4). Since $\overline{A}$ is semisimple, we can find primitive orthogonal idempotents $a_1, \ldots, a_s \in \overline{A}$ such that $V_i \cong a_i\overline{A}$ as an $\overline{A}-$module. Because A is $J(A)-$adically complete, we can lift the a_i to a set of primitive orthogonal idempotents $e_1, \ldots, e_s$ of A by [6, Volume I, Theorem 6.7]: $a_i = \overline{e_i}$ for each i.

Let $P_i = e_iA$, $i = 1, \ldots, s$. Since e_i is an idempotent in A, P_i is a projective $A-$module for each i. Write

$$\overline{P} = \frac{P}{PJ(A)} \cong P \otimes_A \overline{A}$$

for any finitely generated projective $A-$module P.

Lemma. *Let $\varphi : K_0(A) \to K_0(\overline{A})$ be the natural map given by $\varphi([P]) = [\overline{P}]$ for finitely generated projectives P.*

(a) φ is an isomorphism
(b) If P, Q are finitely generated projective $A-$modules, then

$$P \cong Q \quad \textit{if and only if} \quad [P] = [Q] \quad \textit{in} \quad K_0(A)$$

(c) $\varphi([P_i]) = [V_i]$
(d) $K_0(A) = \bigoplus_{i=1}^s \mathbb{Z}[P_i]$.

Proof. See [6, Volume I, Proposition 16.7] and its proof. □

Whenever P is a finitely generated projective with $V = \overline{P}$, we will say that P is a *projective cover* of V. Note that for any semisimple module V, a projective cover exists and is unique up to isomorphism by [6, Volume I, §6C].

2.7. Proposition. Let A be a complete semilocal ring. Then $b(A) \leqslant \operatorname{rk} K_0(A)$ with equality if and only if each block of A is local.

Proof. Note that if B is a semilocal ring, then B is local if and only if $\operatorname{rk} K_0(\overline{B}) = 1$.

Now, if $A = B_1 \oplus \cdots \oplus B_r$ is a decomposition of A into blocks, then each B_i is complete and semilocal. Moreover, $\overline{A} = \overline{B_1} \oplus \cdots \oplus \overline{B_r}$ is a decomposition of the semisimple ring $\overline{A}$ into a direct sum of two-sided ideals, so

$$K_0(\overline{A}) \cong K_0(\overline{B_1}) \oplus \cdots \oplus K_0(\overline{B_r}).$$

Hence by Lemma 2.6,

$$\operatorname{rk} K_0(A) = \operatorname{rk} K_0(\overline{A}) = \sum_{i=1}^r \operatorname{rk} K_0(\overline{B_i}) \geqslant r = b(A),$$

with equality if and only if $\operatorname{rk} K_0(B_i) = \operatorname{rk} K_0(\overline{B_i}) = 1$ for each i. □

2.8. Whitehead groups. If R is a ring, let $\mathrm{GL}_n(R)$ denote the group of all invertible matrices with coefficients in R. Note that $\mathrm{GL}_n(R)$ can also be thought of as the automorphism group of the free module R^n. There is an obvious inclusion of $\mathrm{GL}_n(R)$ into $\mathrm{GL}_{n+1}(R)$, given by

$$(X) \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We then define the *infinite general linear group* $\mathrm{GL}(R)$ to be the direct limit of all the $\mathrm{GL}_n(R)$'s with respect to these inclusions. The *Whitehead group* $K_1(R)$ of R is defined to be the abelianization of $\mathrm{GL}(R)$ [6, Volume II, §40]:

$$K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]}.$$

Since $GL_1(R) \cong R^\times$ is the group of units of R, there is a natural map

$$\theta : R^\times \to K_1(R).$$

It is shown in [6, Volume II, Theorem 40.31] that θ is a surjection whenever R is semilocal.

2.9. Localisation sequence of $K-$theory. Let R be a ring and let S be an Ore set in R consisting of regular elements. Then the localisation R_S exists by [10, Theorem 2.1.12].

The canonical map $\varphi : R \to R_S$ gives rise to an exact sequence of $K-$groups associated with the rings R and R_S as in [16, Theorem 15.5]:

$$K_1(R) \to K_1(R_S) \to K_0(R, \varphi) \to K_0(R) \to K_0(R_S).$$

Here $K_0(R, \varphi)$ is the *relative K_0-group* [16, p. 214].

Suppose in addition that the ring R is Noetherian and regular. Recall [10, 7.7.1] that a ring R is said to be *regular* if every finitely generated $R-$module has finite projective dimension. Of course, any ring of finite global dimension is regular.

Lemma. *$K_0(R, \varphi)$ can be identified with the group $K_0(\mathcal{C})$, where $\mathcal{C}$ is the category of all finitely generated $S-$torsion $R-$modules.*

Proof. Venjakob [17, (4.3)] shows precisely this, but in less generality. The whole result follows from [19]. □

In view of [10, Theorem 12.4.9] the above sequence becomes

$$K_1(R) \to K_1(R_S) \xrightarrow{\partial} K_0(\mathcal{C}) \xrightarrow{\alpha} K_0(R) \xrightarrow{\beta} K_0(R_S) \to 0. \quad (3)$$

Also, since R is a regular Noetherian ring, there is an isomorphism

$$\gamma : \mathcal{G}_0(R) \to K_0(R),$$

see [10, Theorem 12.4.8].

Below are partial descriptions of the maps β, γ, α and ∂ that we will need:

- $\beta([M]) = [M \otimes_R R_S]$ for all $M \in \mathcal{M}(R)$,
- $\gamma([M]) = \sum_{j=0}^{n}(-1)^j[X_j]$ if $0 \to X_n \to \cdots \to X_0 \to M \to 0$ is a finite projective resolution of $M \in \mathcal{M}(R)$,
- $\alpha([M]) = \gamma([M])$ for all $M \in \mathcal{C}$, and
- $\partial(\theta(x)) = [R/xR] \in \mathcal{C}$ for all $x \in R \cap R_S^\times$.

Here $\theta : R_S^\times \to K_1(R_S)$ is the natural map appearing in (2.8).

3. Iwasawa algebras

3.1. Notation. Let K be a finite field extension of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of K; this is a finite extension of $\mathbb{Z}_p$ and a complete local discrete valuation ring. We fix a uniformizer π of $\mathcal{O}$ and write $k = \mathcal{O}/\pi\mathcal{O}$ for the residue field of $\mathcal{O}$; this is a finite field of characteristic p. This notation will remain in force throughout the paper.

3.2. Completed group algebras. Let $\mathcal{O}[[G]]$ be the completed group algebra of the compact $p-$adic analytic group G with coefficients in $\mathcal{O}$:

$$\mathcal{O}[[G]] = \varprojlim \mathcal{O}[G/N]$$

where N runs over all the open normal subgroups of G. Similarly,

$$k[[G]] = \varprojlim k[G/N]$$

is the completed group algebra of G with coefficients in k. Note that these are just the usual group algebras when G is finite. Since we will not be considering the usual group algebra kG or $\mathcal{O}G$ if G is infinite, we will write kG for $k[[G]]$ and $\mathcal{O}G$ for $\mathcal{O}[[G]]$ throughout this paper. When k' is a finite extension of k, it is easily checked that

$$k'G \cong kG \otimes_k k'.$$

Also note that π is a central regular element of $\mathcal{O}G$ and that $\mathcal{O}G/\pi\mathcal{O}G \cong kG$.

3.3. Properties of kG and $\mathcal{O}G$. Let $R = k$ or $\mathcal{O}$. We collect some well-known results about RG below.

Proposition. *Let N be an open normal pro-p subgroup of G and let I be the kernel of the natural map $RG \to R[G/N]$. Then*

(a) I is contained in the Jacobson radical of RG,
(b) RG is complete with respect to the $I-$adic filtration,
(c) RG is a complete semilocal ring,
(d) RG is Noetherian,
(e) The global homological dimension of RG is finite if and only if G has no elements of order p.

Proof. See [11, Proposition 5.2.16] for part (c). Part (e) follows from [2, Theorem 4.1] and [14, Corollaire 1]. □

Corollary. *The natural map $K_0(RG) \to K_0(R[G/N])$ is an isomorphism for any open normal pro-p subgroup N of G.*

Proof. This follows from Lemma 2.6 and part (a) of the Proposition. □

We wish to apply the long exact sequence (3) of K-theory to the ring RG. In our setup (2.9), this requires RG to be Noetherian and regular. We will therefore be frequently assuming that G has no elements of order p.

4. Projective $kG-$modules and Euler characteristics

4.1. Let $V_1, \ldots, V_s$ be a complete list of representatives for the isomorphism classes of simple $kG-$modules; note that each V_i is finite dimensional over k because G is virtually pro-p. As in (2.6), choose a projective $kG-$cover P_i for V_i; thus $P_1, \ldots, P_s$ are the indecomposable projective $kG-$modules. It follows from Lemma 2.6 that any finitely generated projective $kG-$module X can be written as follows:

$$X \cong \bigoplus_{j=1}^{s} P_j^{\langle X, P_j \rangle}$$

for some well-defined $\langle X, P_j \rangle \in \mathbb{N}$.

Proposition. *Let X, Y be finitely generated projective $kG-$modules.*

(a) $X \cong Y$ if and only if $\langle X, P_i \rangle = \langle Y, P_i \rangle$ for all $i = 1, \ldots, s$

(b) $\langle X, P_i \rangle = \dim_k \operatorname{Hom}_{kG}(X, V_i)/\dim_k \operatorname{End}_{kG}(V_i)$ for each i.

Proof. Part (a) follows from Lemma 2.6. For part (b), consider

$$\operatorname{Hom}_{kG}(X, V_i) \cong \bigoplus_{j=1}^{s} \operatorname{Hom}_{kG}(P_j, V_i)^{\langle X, P_j \rangle}.$$

Note that the vector spaces involved here are finite dimensional over k, because each V_i is finite dimensional. Now since P_j is the projective cover of V_j and because V_i is semisimple, $\operatorname{Hom}_{kG}(P_j, V_i) \cong \operatorname{Hom}_{kG}(V_j, V_i)$. Hence by Schur's Lemma, we have

$$\dim_k \operatorname{Hom}_{kG}(P_j, V_i) = \delta_{ij} \dim_k \operatorname{End}_{kG}(V_i).$$

and the result follows. □

4.2. Twists and duality. Let V be an $kG-$module which is finite dimensional over k and let M be a finitely generated $kG-$module. Then the tensor product $M \otimes_k V$ is naturally an $kG-$module equipped with the diagonal action:

$$(m \otimes v).g = mg \otimes vg \quad \text{for all} \quad m \in M, v \in V, g \in G.$$

The vector space dual $V^* = \operatorname{Hom}_k(V, k)$ is also an $kG-$module in the usual way:

$$(f.g)(v) = f(vg^{-1}) \quad \text{for all} \quad f \in V^*, v \in V, g \in G.$$

4.3. Induction and restriction. Let H be an open subgroup of G. Then H has finite index in G, so whenever M is a finitely generated $kG-$module, M is also finitely generated over kH. We thus have induction and restriction functors

$$\operatorname{Ind}_H^G : \mathcal{M}(kH) \to \mathcal{M}(kG) \quad \text{and} \quad \operatorname{Res}_H^G : \mathcal{M}(kG) \to \mathcal{M}(kH).$$

Twists and induced modules are connected via the following very useful result.

Lemma. *Let $X \in \mathcal{M}(kH)$ and let $Y \in \mathcal{M}(kG)$. Suppose that Y is finite dimensional over k. Then there is an isomorphism of $kG-$modules*

$$\operatorname{Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y) \cong (\operatorname{Ind}_H^G X) \otimes_k Y.$$

Proof. There exists a $kH-$balanced map $(X \otimes_k \operatorname{Res}_H^G Y) \times kG \to (X \otimes_{kH} kG) \otimes_k Y$ which sends $(x \otimes y, g)$ to $(x \otimes g) \otimes yg$ for all $x \in X, y \in Y, g \in G$. This gives rise to a $kG-$module homomorphism

$$\varphi : \operatorname{Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y) \to (\operatorname{Ind}_H^G X) \otimes_k Y$$

such that $\varphi((x \otimes y) \otimes g) = x \otimes g \otimes yg$ for all $x \in X, y \in Y, g \in G$. There also exists a $k-$linear map

$$\psi : (\operatorname{Ind}_H^G X) \otimes_k Y \to \operatorname{Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y)$$

such that $\psi((x \otimes g) \otimes y) = (x \otimes yg^{-1}) \otimes g$ for all $x \in X, g \in G, y \in Y$. Then ψ is a $k-$linear inverse for φ, so φ is the required isomorphism. □

Lemma 4.3 is of course well known for finite groups, see for example [6, Volume I, Proposition 10.5] and [15, §3.3, Example 5].

4.4. Lemma. Let $X, Y, Z \in \mathcal{M}(kG)$ and suppose that Y, Z are finite dimensional over k. Then

(a) $X \otimes_k Y$ is a finitely generated $kG-$module.
(b) If X is projective, then so is $X \otimes_k Y$.
(c) There is a natural isomorphism of $k-$vector spaces

$$\operatorname{Hom}_{kG}(X \otimes_k Y^*, Z) \cong \operatorname{Hom}_{kG}(X, Y \otimes_k Z).$$

Proof. Since Y is finite dimensional over k and k is finite, we can find an open subgroup H of G which acts trivially on Y. Thus $\operatorname{Res}_H^G Y \cong k^n$ where k denotes the trivial $kH-$module and $n = \dim_k Y$. Now,

$$\operatorname{Res}_H^G(X \otimes_k Y) \cong (\operatorname{Res}_H^G X) \otimes_k k^n \cong (\operatorname{Res}_H^G X)^n.$$

But X is finitely generated over kG and H has finite index in G, so X is finitely generated over kH. Hence $X \otimes_k Y$ is finitely generated over kH and therefore also finitely generated over kG as required for part (a).

By Lemma 4.3, we have

$$kG \otimes_k Y \cong (\mathrm{Ind}_H^G kH) \otimes_k Y \cong \mathrm{Ind}_H^G(kH \otimes_k k^n) \cong \mathrm{Ind}_H^G(kH^n) \cong kG^n$$

as $kG-$modules, so the twist of a free $kG-$module with a finite dimensional module is again a free $kG-$module. Because tensor products commute with finite direct sums, part (b) follows.

Note that the vector spaces occurring in (c) are finite dimensional over k because Y and Z are finite dimensional. If $\theta \in Y^*$, let $\theta \otimes \mathrm{id}$ denote the $k-$linear map $Y \otimes_k Z \to Z$ which sends $y \otimes z$ to $\theta(y)z$.

Now, pick a basis $\{y_1, \ldots, y_n\}$ for Y and let $\{\theta_1, \ldots, \theta_n\}$ be the dual basis for Y^*. Define

$$\begin{aligned} \Theta &: \mathrm{Hom}_{kG}(X, Y \otimes_k Z) \to \mathrm{Hom}_{kG}(X \otimes_k Y^*, Z) \quad \text{and} \\ \Phi &: \mathrm{Hom}_{kG}(X \otimes_k Y^*, Z) \to \mathrm{Hom}_{kG}(X, Y \otimes_k Z) \end{aligned}$$

by setting

$$\begin{aligned} \Theta(f)(x \otimes \theta) &= (\theta \otimes \mathrm{id})f(x) \quad \text{and} \\ \Phi(g)(x) &= \textstyle\sum_{i=1}^n y_i \otimes g(x \otimes \theta_i) \end{aligned}$$

for all $f \in \mathrm{Hom}_{kG}(X, Y \otimes_k Z), x \in X, \theta \in Y^*, g \in \mathrm{Hom}_{kG}(X \otimes_k Y^*, Z)$. The reader can verify that Θ and Φ are mutually inverse $k-$linear maps. Part (c) follows. ☐

4.5. Euler characteristics. Let M be a finitely generated $kG-$module. Then

$$H_n(G, M) := \mathrm{Tor}_n^{kG}(M, k)$$

is a finite dimensional $k-$vector space for all $n \geqslant 0$.

Definition. *The* Euler characteristic *of M is defined to be*

$$\chi(G, M) = \prod_{n \geqslant 0} |H_n(G, M)|^{(-1)^n},$$

if this exists. M is said to have integral Euler characteristic *if $\chi(G, M) \in \mathbb{Z}$.*

Lemma. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated $kG-$modules. Suppose $\chi(G, A), \chi(G, B)$ and $\chi(G, C)$ all exist. Then*

$$\chi(G, B) = \chi(G, A)\chi(G, C).$$

Proof. This follows from the long exact sequence of homology. ☐

4.6. Let P_1 be the projective cover (2.6) of the trivial $kG-$module V_1.

Lemma. *Let X be a finitely generated projective $kG-$module. Then*

$$\chi(G, X) = q^{\langle X, P_1 \rangle}.$$

Proof. Let I_G be the augmentation ideal of kG, so that kG/I_G is the trivial module V_1. Then $X \otimes_{kG} k \cong X/XI_G =: X_G$, the coinvariants of X. Now

$$\mathrm{Hom}_{kG}(X, V_1) \cong \mathrm{Hom}_{kG}(X_G, V_1) \cong X_G^*$$

as $k-$vector spaces, so $\dim_k \mathrm{Hom}_{kG}(X, V_1) = \dim_k X_G$. Because $\mathrm{End}_{kG}(V_1) \cong k$, $\langle X, P_1 \rangle = \dim_k X_G$ by Proposition 4.1(b).

Since X is projective, $H_n(G, X) = 0$ whenever $n > 0$. Hence

$$\chi(G, X) = |H_0(G, X)| = |X \otimes_{kG} k| = |X_G| = q^{\dim_k X_G} = q^{\langle X, P_1 \rangle}$$

as required. □

4.7. Proposition. Let X be a finitely generated projective $kG-$module. Then

$$\langle X, P_i \rangle = \frac{\log_q \chi(G, X \otimes_k V_i^*)}{\dim_k \mathrm{End}_{kG}(V_i)}$$

for all $i = 1, \ldots, s$.

Proof. Since $V_i \otimes_k V_1 \cong V_i$ for all i, Lemma 4.4(c) gives

$$\mathrm{Hom}_{kG}(X, V_i) \cong \mathrm{Hom}_{kG}(X, V_i \otimes_k V_1) \cong \mathrm{Hom}_{kG}(X \otimes_k V_i^*, V_1).$$

By Lemma 4.4(a) and (b), $X \otimes V_i^*$ is finitely generated projective. Applying Proposition 4.1(b) we obtain

$$\langle X \otimes V_i^*, P_1 \rangle \dim_k \mathrm{End}_{kG}(V_1) = \dim_k \mathrm{Hom}_{kG}(X \otimes_k V_i^*, V_1),$$

and also

$$\langle X, P_i \rangle \dim_k \mathrm{End}_{kG}(V_i) = \dim_k \mathrm{Hom}_{kG}(X, V_i).$$

But $\mathrm{End}_{kG}(V_1) \cong k$, so

$$\langle X, P_i \rangle = \frac{\langle X \otimes_k V_i^*, P_1 \rangle}{\dim_k \mathrm{End}_{kG}(V_i)}.$$

The result now follows from Lemma 4.6. □

Corollary. *Let X, Y be finitely generated projective $kG-$modules. Then X is isomorphic to Y if and only if*

$$\chi(G, X \otimes_k V_i) = \chi(G, Y \otimes_k V_i) \quad \textit{for all} \quad i = 1, \ldots s.$$

Proof. By Proposition 4.1(a), X is isomorphic to Y if and only if $\langle X, P_i \rangle = \langle Y, P_i \rangle$ for all i. Because $V \mapsto V^*$ is an involution on the set of simple $kG-$modules, the result follows from Proposition 4.7. □

4.8. The image of γ. Assuming that G has no elements of order p, we have the following description of the map $\gamma : \mathcal{G}_0(kG) \to K_0(kG)$ appearing in (2.9).

Proposition. *Suppose G has no elements of order p. Then for any $M \in \mathcal{M}(kG)$,*

$$\gamma([M]) = \sum_{i=1}^{s} \left(\frac{\log_q \chi(G, M \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)} \right) [P_i].$$

Proof. By Proposition 3.3(e), kG has finite global dimension, so we can choose a finite projective resolution

$$0 \to X_n \to \cdots \to X_0 \to M \to 0$$

for M. Then the definition of γ given in (2.9) gives

$$\gamma([M]) = \sum_{j=0}^{n} (-1)^j [X_j] \in K_0(kG).$$

By Proposition 4.7, we have

$$[X_j] = \sum_{i=1}^{s} \langle X_j, P_i \rangle [P_i] = \sum_{i=1}^{s} \left(\frac{\log_q \chi(G, X_j \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)} \right) [P_i].$$

Since V_i^* is a flat $k-$module,

$$0 \to X_n \otimes_k V_i^* \to \cdots \to X_0 \otimes V_i^* \to M \otimes_k V_i^* \to 0$$

is an exact sequence of finitely generated $kG-$modules, by Lemma 4.4(a). Because $\log_q \chi(G, -)$ is additive on short exact sequences by Lemma 4.5, we obtain

$$\log_q \chi(G, M \otimes_k V_i^*) = \sum_{j=0}^{n} (-1)^j \log_q \chi(G, X_j \otimes_k V_i^*)$$

for each $i = 1, \ldots, s$, and the result follows. □

5. Characteristic elements

5.1. The localisation sequence. We are primarily interested in finitely generated $p-$torsion $\mathcal{O}G-$modules. Let $T = \{1, \pi, \pi^2, \ldots\}$; this is clearly a multiplicatively closed subset of $\mathcal{O}G$ consisting of central regular elements. Since we can write $p \in \mathcal{O}$ as some power of π times a unit in $\mathcal{O}$, we see that a finitely generated $\mathcal{O}G-$module is $p-$torsion if and only if it is $\pi-$torsion, or equivalently, $T-$torsion.

Until the end of this section G has no elements of order p.

Let $\mathcal{D}$ denote the category of all finitely generated $T-$torsion $\mathcal{O}G-$modules. By Proposition 3.3, $\mathcal{O}G$ is Noetherian and has finite global dimension since G has no elements of order p. Thus we obtain an exact sequence of $K-$groups from (2.9):

$$K_1(\mathcal{O}G) \xrightarrow{\tau} K_1(\mathcal{O}G_T) \xrightarrow{\partial_G} K_0(\mathcal{D}) \xrightarrow{\alpha} K_0(\mathcal{O}G) \xrightarrow{\beta} K_0(\mathcal{O}G_T) \to 0. \tag{4}$$

5.2. The following result is essentially [3, Proposition 3.4]. We give the proof for the convenience of the reader.

Lemma. *The map $\alpha : K_0(\mathcal{D}) \to K_0(\mathcal{O}G)$ appearing in (4) is zero.*

Proof. Fix an open normal pro-p subgroup N of G. We have a natural commuting diagram of rings

$$\begin{array}{ccc} \mathcal{O}G & \xrightarrow{\lambda_1} & \mathcal{O}G_T \\ {\scriptstyle\lambda_2}\downarrow & & \downarrow{\scriptstyle\lambda_3} \\ \mathcal{O}[G/N] & \xrightarrow[\lambda_4]{} & K[G/N] \end{array}$$

which induces by the functoriality of K_0 a commuting diagram of K_0-groups:

$$\begin{array}{ccc} K_0(\mathcal{O}G) & \xrightarrow{K_0(\lambda_1)} & K_0(\mathcal{O}G_T) \\ {\scriptstyle K_0(\lambda_2)}\downarrow & & \downarrow{\scriptstyle K_0(\lambda_3)} \\ K_0(\mathcal{O}[G/N]) & \xrightarrow[K_0(\lambda_4)]{} & K_0(K[G/N]). \end{array}$$

Now, $K_0(\lambda_1)$ is the map β appearing in (4) and is therefore surjective; moreover $K_0(\lambda_2)$ is an isomorphism by Corollary 3.3.

It is well known from the representation theory of finite groups that $K_0(\lambda_4)$ is injective [15, Chapter 16, Corollary 2 to Theorem 34]. Now an elementary diagram chase shows that $K_0(\lambda_1) = \beta$ is an isomorphism. Because the sequence (4) is exact at $K_0(\mathcal{O}G)$, α is zero as required. $\square$

From the exactness of (4), we also obtain

Corollary. *The connecting homomorphism ∂_G appearing in (4) is surjective.*

5.3. Characteristic elements for $T-$torsion modules. Following [3, (33)] we make the following definition:

Definition. *A* characteristic element *for a $T-$torsion module M is any element $\xi_M \in K_1(\mathcal{O}G_T)$ such that $\partial_G(\xi_M) = [M] \in \mathcal{D}$.*

Because $\partial_G : K_1(\mathcal{O}G_T) \to K_0(\mathcal{D})$ is surjective by Corollary 5.2, such a ξ_M always exists. By the exactness of (4), ξ_M is only defined modulo the image of $K_1(\mathcal{O}G)$ in $K_1(\mathcal{O}G_T)$. We will provide an explicit formula for ξ_M in terms of the natural map $\theta : (\mathcal{O}G_T)^\times \to K_1(\mathcal{O}G_T)$ in Proposition 5.6.

5.4. Euler characteristics for $T-$torsion $\mathcal{O}G-$modules. Let M be a finitely generated $\mathcal{O}G-$module. Then

$$H_n(G,M) := \mathrm{Tor}_n^{\mathcal{O}G}(M,\mathcal{O})$$

is a finitely generated $\mathcal{O}-$module for all $n \geqslant 0$. If M is $T-$torsion, M is killed by some power of π, so each $H_n(G,M)$ is also killed by some power of π and is hence finite.

Definition. *The* Euler characteristic *of M is defined to be*

$$\chi(G,M) = \prod_{n\geqslant 0} |H_n(G,M)|^{(-1)^n}.$$

M is said to have integral Euler characteristic *if $\chi(G,M) \in \mathbb{Z}$.*

It is easy to see that this definition extends the one given in (4.5). Moreover, as G has no elements of order p, $\chi(G,M)$ is always exists.

5.5. Dévissage. Since we can view each $M \in \mathcal{M}(kG)$ as a finitely generated $\mathcal{O}G-$module killed by π, we see that $\mathcal{M}(kG)$ is a full subcategory of the abelian category $\mathcal{D}$ which satisfies the conditions of [10, Theorem 12.4.7]:

- $\mathcal{M}(kG)$ is an admissible subcategory of $\mathcal{D}$,
- if $0 \to M' \to M \to M'' \to 0$ is exact in $\mathcal{D}$ and $M \in \mathcal{M}(kG)$ then $M', M'' \in \mathcal{M}(kG)$,
- each $M \in \mathcal{D}$ has a finite filtration $M \supseteq M\pi \supseteq M\pi^2 \supseteq \cdots \supseteq M\pi^{n+1} = 0$ for some n with $M\pi^i/M\pi^{i+1} \in \mathcal{M}(kG)$ for all $0 \leqslant i \leqslant n$.

Because these conditions are satisfied, [10, Theorem 12.4.7] guarantees that the natural map $\mathcal{G}_0(kG) \to K_0(\mathcal{D})$ induced from the inclusion of $\mathcal{M}(kG)$ in $\mathcal{D}$ is an isomorphism. The inverse is given by $\psi : K_0(\mathcal{D}) \to \mathcal{G}_0(kG)$, where

$$\psi([M]) = [\mathrm{gr}_\pi M] \in \mathcal{G}_0(kG) \quad \text{where} \quad \mathrm{gr}_\pi M := \bigoplus_{i=0}^{\infty} \frac{M\pi^i}{M\pi^{i+1}} \in \mathcal{M}(kG). \tag{5}$$

5.6. A formula for the characteristic element. We retain the notation of (4.1). By Lemma 2.6 we can choose a projective $\mathcal{O}G-$cover $e_i\mathcal{O}G$ for V_i. Here $e_1, \ldots, e_s$ are a collection of pairwise orthogonal idempotents in $\mathcal{O}G$ obtained from idempotent lifting. Then

$$P_i := \frac{e_i\mathcal{O}G}{\pi e_i\mathcal{O}G}$$

is a projective $kG-$cover for V_i.

Define $f_i = 1 + (\pi - 1)e_i \in \mathcal{O}G$ for each $i = 1, \ldots, s$ and note that

$$f_i(\pi + (1-\pi)e_i) = \pi. \tag{6}$$

Because π is a unit in $\mathcal{O}G_T$, we see that $f_1, \ldots, f_s$ all lie in $(\mathcal{O}G_T)^\times \cap \mathcal{O}G$.

Proposition. *Let $\theta : (\mathcal{O}G_T)^\times \to K_1(\mathcal{O}G_T)$ be the canonical homomorphism appearing in (2.8) and let $M \in \mathcal{D}$. Then*

$$\xi_M = \theta\left(\prod_{i=1}^s f_i^{\mu_i(M)}\right), \quad \textit{where} \quad \mu_i(M) = \frac{\log_q \chi(G, (\operatorname{gr}_\pi M) \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)}.$$

Proof. Since π is a multiple of f_i by (6), we see that $f_i\mathcal{O}G = f_i\mathcal{O}G + \pi\mathcal{O}G = (1-e_i)\mathcal{O}G + \pi\mathcal{O}G$. Hence

$$\frac{\mathcal{O}G}{f_i\mathcal{O}G} = \frac{\mathcal{O}G}{(1-e_i)\mathcal{O}G + \pi\mathcal{O}G} \cong \frac{e_i\mathcal{O}G}{\pi e_i \mathcal{O}G} = P_i$$

so $P_i \cong \mathcal{O}G/f_i\mathcal{O}G$ as $\mathcal{O}G-$modules for all $i = 1, \ldots, s$. Next, we have an isomorphism $\gamma : \mathcal{G}_0(kG) \to K_0(kG) = \oplus_{i=1}^s \mathbb{Z}[P_i]$ and an isomorphism $\psi : K_0(\mathcal{D}) \to \mathcal{G}_0(kG)$ given in (5). By Proposition 4.8, the composition $\gamma\psi : K_0(\mathcal{D}) \to K_0(kG)$ is given for $M \in \mathcal{D}$ by

$$\gamma\psi([M]) = \sum_{i=1}^s \mu_i(M)[P_i] \quad \text{where} \quad \mu_i(M) = \frac{\log_q \chi(G, (\operatorname{gr}_\pi M) \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)};$$

moreover $\gamma\psi([P_i]) = [P_i]$ for all $i = 1, \ldots, s$. From the definition of ∂_G given in (2.9), we have

$$\partial_G\theta\left(\prod_{i=1}^s f_i^{\mu_i(M)}\right) = \sum_{i=1}^s \mu_i(M)\left[\frac{\mathcal{O}G}{f_i\mathcal{O}G}\right] = \sum_{i=1}^s \mu_i(M)[P_i] \in K_0(\mathcal{D}),$$

so $\gamma\psi([M]) = \gamma\psi\delta_G\theta(\prod_{i=1}^s f_i^{\mu_i(M)})$. Since $\gamma\psi$ is an isomorphism, the result follows. □

5.7. Evaluation at zero. Let $\epsilon : \mathcal{O}G_T \to K$ denote the augmentation map. This gives rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}G_T^\times & \xrightarrow{\theta} & K_1(\mathcal{O}G_T) \\ \epsilon\downarrow & & \downarrow K_1(\epsilon) \\ K^\times & \xrightarrow{\cong} & K_1(K). \end{array}$$

Identifying $K_1(K)$ with $K^\times$ allows us to write $K_1(\epsilon)\circ\theta = \epsilon$. Compare the following result with [17, Proposition 8.6].

Lemma. *For any $M \in \mathcal{D}_G$, let $\xi_M(0) := K_1(\epsilon)(\xi_M) \in K^\times$. Then*

$$\chi(G, M) = N_{K/\mathbb{Q}_p}(\xi_M(0)).$$

Proof. Since K is a field, precisely one of the idempotents $e_1, \ldots, e_s$ appearing in (5.6) gets sent to 1 under ϵ, and it is clear that this is the idempotent corresponding to the trivial representation, e_1. Thus $\epsilon(e_i) = \delta_{i1}$, so

$$\epsilon(f_i) = 1 + (\pi - 1)\delta_{i1} = \pi\delta_{i1},$$

for all $i = 1, \dots, s$. Hence, Proposition 5.6 gives

$$\xi_M(0) = K_1(\epsilon)(\xi_M) = \prod_{i=1}^{s} \epsilon(f_i)^{\mu_i(M)} = \pi^{\mu_1(M)}.$$

But $\dim_k V_1 = 1$ and $(\mathrm{gr}_\pi M) \otimes_k V_1^* \cong \mathrm{gr}_\pi M$, so $\mu_1(M) = \log_q \chi(G, \mathrm{gr}_\pi M) = \log_q \chi(G, M)$. Since $N_{K/\mathbb{Q}_p}(\pi) = q$, the result follows. $\square$

6. Artin Formalism

6.1. We continue with the notation the previous sections, but do not make further assumptions on G for the time being. Let Δ be a finite quotient of G. Then $\mathcal{G}_0(k\Delta)$ is a commutative ring with multiplication given by

$$[X].[Y] = [X \otimes_k Y] \quad \text{for all} \quad X, Y \in \mathcal{M}(k\Delta).$$

Moreover, $\mathcal{G}_0(kG)$ becomes a $\mathcal{G}_0(k\Delta)-$module by Lemma 4.4(a) if we set

$$[M].[X] = [M \otimes_k X] \quad \text{for all} \quad M \in \mathcal{M}(kG), X \in \mathcal{M}(k\Delta).$$

6.2. The decomposition map. Let $V \in \mathcal{M}(K\Delta)$. A $\Delta-$*lattice* in V is defined to be a finitely generated $\mathcal{O}\Delta-$submodule E of V such that $V \cong E \otimes_{\mathcal{O}} K$. The *decomposition map* $d : \mathcal{G}_0(K\Delta) \to \mathcal{G}_0(k\Delta)$ is given by

$$d[V] = [\overline{E}], \quad \text{where} \quad \overline{E} = E/E\pi$$

for any choice of $\Delta-$lattice E in V. It is shown in [15, Chapter 15, Theorem 32] that $d[V]$ is independent of the choice of E and that d is in fact a *ring* homomorphism. In this way, $\mathcal{G}_0(kG)$ becomes a $\mathcal{G}_0(K\Delta)-$module:

$$[M].[V] = [M].d[V] \quad \text{for all} \quad M \in \mathcal{M}(kG), V \in \mathcal{M}(K\Delta).$$

6.3. Dévissage and twists. As in (5.1), let $\mathcal{D}_G$ be the category of all finitely generated $\pi-$torsion $\mathcal{O}G-$modules. In view of (5.5), we have an isomorphism of abelian groups $\psi : K_0(\mathcal{D}_G) \to \mathcal{G}_0(kG)$, so $K_0(\mathcal{D}_G)$ becomes a $\mathcal{G}_0(K\Delta)-$module via

$$[M].[V] = \psi^{-1}(\psi[M].d[V]) \quad \text{for all} \quad M \in \mathcal{D}_G, V \in \mathcal{M}(K\Delta).$$

Let E be a $\Delta-$lattice in V and let $M \in \mathcal{M}(\mathcal{O}G)$. The *twist* $M \otimes_{\mathcal{O}} E$ becomes an $\mathcal{O}G-$module with the diagonal action of G. An argument analogous to the proof of Lemma 4.4(a) shows that this module is finitely generated over $\mathcal{O}G$; moreover it is $\pi-$torsion whenever M is.

Lemma. *Let $M \in \mathcal{D}_G$ and let $V \in \mathcal{M}(K\Delta)$. Then for any $\Delta-$lattice E in V,*

$$[M].[V] = [M \otimes_{\mathcal{O}} E].$$

Proof. Because E is a flat $\mathcal{O}-$module, it is easy to verify that

$$\operatorname{gr}_\pi(M \otimes_{\mathcal{O}} E) \cong (\operatorname{gr}_\pi M) \otimes_k \overline{E}$$

as $kG-$modules. Hence

$$\psi[M].d[V] = [(\operatorname{gr}_\pi M) \otimes_k \overline{E}] = [\operatorname{gr}_\pi(M \otimes_{\mathcal{O}} E)] = \psi[M \otimes_{\mathcal{O}} E]$$

and the result follows. □

Let $\rho : \Delta \to \mathrm{GL}(V)$ be the group homomorphism associated to a finitely generated $K\Delta-$module V and let $M \in \mathcal{D}_G$. Because Δ is finite, the image of ρ is contained in $\mathrm{Aut}(E)$ for some $\Delta-$lattice E in V. We define the *twist* of M at ρ to be

$$\mathrm{tw}_\rho(M) = M \otimes_{\mathcal{O}} E \in \mathcal{D}_G.$$

Lemma 6.3 can now be rephrased as follows:

$$[M].[V] = [\mathrm{tw}_\rho(M)]$$

for any $M \in \mathcal{D}_G$ and any $V \in \mathcal{M}(K\Delta)$.

6.4. Induction and restriction. Let H be the kernel of the surjection $G \twoheadrightarrow \Delta$; this is an open normal subgroup of G. Then we have the induction functor

$$\mathrm{Ind}_H^G : \mathcal{M}(\mathcal{O}H) \to \mathcal{M}(\mathcal{O}G)$$

which sends M to $M \otimes_{\mathcal{O}H} \mathcal{O}G$. It is easy to see that $\mathrm{Ind}_H^G(M) \in \mathcal{D}_G$ whenever $M \in \mathcal{D}_H$. We also have the restriction functor

$$\mathrm{Res}_H^G : \mathcal{M}(\mathcal{O}G) \to \mathcal{M}(\mathcal{O}H)$$

which sends $\mathcal{D}_G$ to $\mathcal{D}_H$.

Lemma. *Let $M \in \mathcal{M}(\mathcal{O}G)$. Then the induced module of the restriction of M to $\mathcal{O}H$ is isomorphic as an $\mathcal{O}G-$module to the twist of M by $\mathcal{O}\Delta$:*

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G M) \cong M \otimes_{\mathcal{O}} \mathcal{O}\Delta.$$

Proof. By an appropriate modification of the proof of Lemma 4.3, we have an isomorphism $\mathrm{Ind}_H^G(\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Res}_H^G M) \cong \mathrm{Ind}_H^G(\mathcal{O}) \otimes_{\mathcal{O}} M$ of $\mathcal{O}G-$modules. Since $\mathrm{Ind}_H^G(\mathcal{O}) \cong \mathcal{O}\Delta$ and $\mathcal{O} \otimes_{\mathcal{O}} N \cong N$ for any $\mathcal{O}H-$module N, the result follows. □

6.5. Artin formalism expressed in $K_0(\mathcal{D}_G)$. We can find a finite field extension L of K such that the division rings appearing in the Wedderburn decomposition of the group algebra $L\Delta$ are all isomorphic to L. Such an L can always be obtained by adjoining sufficiently many roots of unity to K [15, Corollary to Theorem 24] and is called a *splitting field* for Δ.

Thus we have an isomorphism of $L\Delta-$modules

$$L\Delta \cong \bigoplus_{\rho \in \mathcal{V}(\Delta)} W_\rho^{n_\rho}. \tag{7}$$

Here $\mathcal{V}(\Delta)$ is the set of all absolutely irreducible representations of Δ over $\overline{\mathbb{Q}_p}$, W_ρ is the $L\Delta-$module corresponding to the representation ρ and $n_\rho = \dim_L W_\rho$.

Proposition. *Let $M \in \mathcal{D}_G$. Then*

$$|L:K| \cdot [\mathrm{Ind}_H^G(\mathrm{Res}_H^G M)] = \sum_{\rho \in \mathcal{V}(\Delta)} n_\rho [\mathrm{tw}_\rho(M)] \in K_0(\mathcal{D}_G).$$

Proof. Viewing (7) as an isomorphism of $K\Delta-$modules, we obtain

$$|L:K| \cdot [K\Delta] = \sum_{\rho \in \mathcal{V}(\Delta)} n_\rho [W_\rho] \in \mathcal{G}_0(K\Delta).$$

Now by (6.3), $K_0(\mathcal{D}_G)$ is a $\mathcal{G}_0(K\Delta)-$module, so we may apply this equation to $[M] \in K_0(\mathcal{D}_G)$ to get

$$|L:K| \cdot [M \otimes_{\mathcal{O}} \mathcal{O}\Delta] = \sum_{\rho \in \mathcal{V}(\Delta)} n_\rho [\mathrm{tw}_\rho(M)] \in K_0(\mathcal{D}_G).$$

Here we have chosen $\mathcal{O}\Delta$ to be the $\Delta-$lattice in $K\Delta$ and applied Lemma 6.3. The result now follows from Lemma 6.4. □

6.6. Until the end of this section, G has no elements of order p. The following result is essentially [3, Lemma 4.6]. We include a proof for the convenience of the reader.

Lemma. *The natural map $\theta : \mathcal{O}G_T^\times \to K_1(\mathcal{O}G_T)$ is surjective.*

Proof. Let $x \in K_1(\mathcal{O}G_T)$. By Proposition 5.6, each generator $[P_i]$ of $K_0(kG)$ is in the image of $\partial_G \circ \theta_G$, so $\partial_G \circ \theta$ is surjective. Hence there exists $y \in \mathcal{O}G_T^\times$ such that $\partial_G(x - \theta(y)) = 0$. Because (4) is exact at $K_1(\mathcal{O}G_T)$, $x = \theta(y) + \tau(z)$ for some $z \in K_1(\mathcal{O}G)$; here $\tau : K_1(\mathcal{O}G) \to K_1(\mathcal{O}G_T)$ is the natural map.

Now because $\mathcal{O}G$ is semilocal by Proposition 3.3(c), the natural map $\theta_1 : \mathcal{O}G^\times \to K_1(\mathcal{O}G)$ is surjective (2.8), so we can find $w \in \mathcal{O}G^\times$ such that $z = \theta_1(w)$. Moreover, $\theta(w) = \tau(\theta_1(w))$ by functoriality, whence

$$x = \theta(y) + \tau(z) = \theta(yw)$$

and θ is surjective as required. □

6.7. Lemma. There exists a commuting diagram of groups

$$\begin{array}{ccccc}
\mathcal{O}G_T^\times & \xrightarrow{\theta_G} & K_1(\mathcal{O}G_T) & \xrightarrow{\partial_G} & K_0(\mathcal{D}_G) \\
\iota\uparrow & & \lambda_{G,H}\uparrow & & \uparrow K_0(\mathrm{Ind}_H^G) \\
\mathcal{O}H_T^\times & \xrightarrow[\theta_H]{} & K_1(\mathcal{O}H_T) & \xrightarrow[\partial_H]{} & K_0(\mathcal{D}_H)
\end{array}$$

where $\iota : \mathcal{O}H \hookrightarrow \mathcal{O}G$ is the natural inclusion and $\lambda_{G,H} = K_1(\iota)$.

Proof. Any element $x \in \mathcal{O}H_T^\times$ can be written as $x = rs^{-1}$ with $r, s \in \mathcal{O}H$. Then both r and s lie in $\mathcal{O}H_T^\times \cap \mathcal{O}H$, so it is sufficient to check that the diagram commutes for all elements $x \in \mathcal{O}H_T^\times \cap \mathcal{O}H$.

The first square commutes by functoriality. If $x \in \mathcal{O}H_T^\times \cap \mathcal{O}H$ then

$$\partial_G \theta_G \iota(x) = \left[\frac{\mathcal{O}G}{x\mathcal{O}G}\right] = \left[\operatorname{Ind}_H^G\left(\frac{\mathcal{O}H}{x\mathcal{O}H}\right)\right] = K_0(\operatorname{Ind}_H^G)\partial_H\theta_H(x)$$

by (2.9). Since θ_H is surjective by Lemma 6.6, the second square also commutes as required. □

6.8. Artin formalism for characteristic elements. Recall that characteristic elements for modules in $\mathcal{D}_G$ are only defined modulo the image of $K_1(\mathcal{O}G)$ inside $K_1(\mathcal{O}G_T)$.

Theorem. *Keeping the notation of (6.5), let $M \in \mathcal{D}_G$. Then*

$$\lambda_{G,H}(\xi_{\operatorname{Res}_H^G M})^{|L:K|} = \prod_{\rho\in\mathcal{V}(\Delta)} \xi_{\operatorname{tw}_\rho(M)}^{n_\rho} \qquad \operatorname{mod} \tau(K_1(\mathcal{O}G)).$$

Proof. Apply Lemma 6.7 and Proposition 6.5. □

Evaluating at zero as in (5.7) gives [3, Theorem 3.10] for $p-$torsion modules.

Corollary. *For any $M \in \mathcal{D}_G$, we have*

$$\chi(H, \operatorname{Res}_H^G M)^{|L:K|} = \prod_{\rho\in\mathcal{V}(\Delta)} \chi(G, \operatorname{tw}_\rho(M))^{n_\rho}.$$

Proof. Let $\epsilon_G : \mathcal{O}G \to K$ and $\epsilon_H : \mathcal{O}H \to K$ be the augmentation maps. Then $\epsilon_G \circ \iota = \epsilon_H$, so $K_1(\epsilon_G) \circ \lambda_{G,H} = K_1(\epsilon_H)$ by functoriality. Now the result follows from Lemma 5.7. □

We now turn towards the question "When are Euler characteristics integral?". First, we must establish some preliminary results about torsion $kG-$modules.

7. Torsion $kG-$modules

7.1. Uniform pro-p groups. By a celebrated result of Lazard, any compact $p-$adic analytic group G always contains an open normal *uniform* pro-p subgroup N [7, Corollary 8.34]. Uniform pro$-p$ groups are defined at [7, Definition 4.1].

For any such N, there is a natural decomposition of kG as a crossed product of kN with the finite group G/N:

$$kG \quad \cong \quad kN * (G/N). \tag{8}$$

The following Lemma is well-known when $k = \mathbb{F}_p$, see [7, Corollary 7.25].

Lemma. *If N is uniform, then kN is a domain.*

Proof. Let J be the Jacobson radical of kN. Then $J = w_N \otimes_{\mathbb{F}_p} k$ where w_N is the augmentation ideal of $\mathbb{F}_p N$. Using [7, Theorem 7.24], we see that the graded ring $\operatorname{gr} kN$ of kN with respect to the $J-$adic filtration is isomorphic to $k[X_1, \ldots, X_d]$. Since kN is complete with respect to the $J-$adic filtration and $\operatorname{gr} kN$ is a domain, kN itself must be a domain by [7, Proposition 7.27]. □

7.2. Torsion modules. Recall [10, 2.1.14] that a ring R is said to have a *classical quotient ring* $Q(R)$ if the localisation of R at the set $S = \mathcal{C}_R(0)$ of regular elements of R exists. This is equivalent to S being an Ore set by [10, Theorem 2.1.12].

Proposition. *Let G be a compact $p-$adic analytic group. Then kG has an Artinian quotient ring $Q(kG)$.*

Proof. Choose an open normal uniform subgroup N of G as in (7.1) and let $T = kN\backslash\{0\}$. As kN is a Noetherian domain by Lemma 7.1, T is an Ore set in kN by [10, Theorem 2.1.15] and the localisation kN_T is a division ring.

Because N is normal, T is invariant under conjugation by G. In view of the crossed product decomposition (8), [13, Lemma 37.7] implies that T is actually an Ore set in kG, and that

$$kG_T \cong kN_T * (G/N).$$

Because kN_T is a division ring and G/N is finite, we see that kG_T is Artinian. Now, kG is a free $kN-$module so every element of T is regular in kG. Hence $T \subseteq S$ and kG embeds into the Artinian ring kG_T. Now every element of $S = \mathcal{C}_{kG}(0)$ is regular in kG_T and hence is a unit in kG_T by [10, Proposition 3.1.1]. This shows that kG_T is a quotient ring of kG with respect to S in the sense of [10, 2.1.3], so kG_S exists and $kG_S \cong kG_T$ is Artinian, as required. □

We will say that a $kG-$module M is *torsion* if it is torsion with respect to the canonical Ore set $S = \mathcal{C}_{kG}(0)$. Thus, M is torsion if and only if for all $m \in M$ there exists $s \in S$ such that $ms = 0$.

Corollary. *Let G be a compact $p-$adic analytic group with an open subgroup H and let M be a $kG-$module. Then M is torsion as a $kH-$module if and only if M is torsion as a $kG-$module.*

Proof. We can choose an open normal uniform subgroup N of G contained in H. The proof of the Proposition shows that M is $\mathcal{C}_{kG}(0)-$torsion if and only if M is $\mathcal{C}_{kN}(0)-$torsion if and only if M is $\mathcal{C}_{kH}(0)-$torsion, as required. □

7.3. Twists of torsion modules.

Proposition. *Let V be a $kG-$module which is finite dimensional over k and let M be a torsion $kG-$module. Then the twist $M \otimes_k V$ is also torsion.*

Proof. Since V is finite dimensional, we can find an open normal subgroup H of G which acts trivially on V. Then $M \otimes_k V$ is isomorphic to a finite direct sum of copies of M, viewed as a $kH-$module. Hence

$$(m \otimes v).t = mt \otimes v \quad \text{for all} \quad m \in M, v \in V, t \in kH.$$

Because M is $kG-$torsion, it is $kH-$torsion by Corollary 7.2. Hence $M \otimes_k V$ is $kH-$torsion and therefore also $kG-$torsion, again by Corollary 7.2. □

7.4. Pseudo-null $p-$torsion modules. An obvious extension of the argument used by Venjakob in [18, Theorem 3.26] together with the computation of $\operatorname{gr} kN$ when N is uniform performed in Lemma 7.1 shows that $\mathcal{O}G$ is an Auslander-Gorenstein ring.

Lemma. *Let M be a finitely generated $\pi-$torsion $\mathcal{O}G-$module. Then M is pseudo-null if and only if $\operatorname{gr}_\pi M$ is $kG-$torsion.*

Proof. Choose an open normal uniform subgroup N of G. Then by [1, Lemma 5.4],

$$j_{\mathcal{O}G}(M) = j_{\mathcal{O}N}(\operatorname{Res}^G_N M),$$

so M is pseudo-null if and only if $\operatorname{Res}^G_N M$ is. By Corollary 7.2, we may assume without loss of generality that $G = N$ is uniform. Furthermore, by dévissage, we may assume that M is actually a $kG-$module, so $\operatorname{gr}_\pi M = M$.

Now, M is $kG-$torsion if and only if $j_{kG}(M) \geqslant 1$ by [5, Lemma 1.4]. But

$$j_{\mathcal{O}G}(M) = j_{kG}(M) + 1$$

by the formula preceeding Theorem 3.30 in [18] and the result follows. □

8. Integrality of Euler characteristics

8.1. Throughout this section G has no elements of order p.

By Proposition 7.2, kG has a classical Artinian ring of quotients $Q(kG) = kG_S$, which can be obtained by localising kG at the Ore set of all regular elements $S = \mathcal{C}_{kG}(0)$ of kG. Let $\mathcal{C}$ be category of of all finitely generated $S-$torsion $kG-$modules. Because G has no elements of order p, kG is a Noetherian ring of finite global dimension by Proposition 3.3. We hence obtain the localisation sequence (3) of K_0-groups from (2.9):

$$K_0(\mathcal{C}) \xrightarrow{\alpha} K_0(kG) \xrightarrow{\beta} K_0(Q(kG)) \to 0. \tag{9}$$

We also have an isomorphism

$$\gamma : \mathcal{G}_0(kG) \to K_0(kG)$$

because kG has finite global dimension.

8.2. Euler characteristics of torsion $kG-$modules. We can now give a characterisation of those groups G which have the property that every finitely generated torsion $kG-$module has trivial Euler characteristic.

Theorem. *Keeping the notation of (8.1), the following are equivalent:*

(a) $\chi(G, M) = 1$ *for all* $M \in \mathcal{C}$,
(b) $\alpha = 0$,
(c) $\xi_M = 1$ *for all* $M \in \mathcal{C}$,
(d) β *is injective,*
(e) $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$.

Proof. Since $M \otimes_k V_1^* \cong M$, $M \in \mathcal{C}$ if and only if $M \otimes_k V_i^* \in \mathcal{C}$ for all $i = 1, \ldots, r$ by Proposition 7.3. Now by Proposition 4.8, the map α in (9) is given by

$$\alpha([M]) = \gamma([M]) = \sum_{i=1}^{s} \left(\frac{\log_q \chi(G, M \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)} \right) [P_i],$$

and the equivalence of (a) and (b) follows.

For any $\mathcal{O}G-$module $M \in \mathcal{D}$, ξ_M is completely determined by the element $[M] \in K_0(\mathcal{D}) \cong K_0(kG)$. So if $M \in \mathcal{C}$, $\xi_M = 1$ if and only if $\alpha([M]) = 0$, as required for the equivalence of (b) and (c).

Next, (b) and (d) are equivalent because the sequence (9) is exact at $K_0(kG)$. Now $\beta : K_0(kG) \to K_0(Q(kG))$ is surjective and $K_0(kG)$ is a torsionfree abelian group of finite rank. Hence β is injective if and only the $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$ as required for the equivalence of (d) and (e). □

8.3. Reduced rank. Let R be a Noetherian ring and let $S = \mathcal{C}_R(0)$ be the set of all regular elements of R. Suppose that the classical quotient ring $Q(R) = R_S$ exists and is Artinian.

Let M be a finitely generated $R-$module. Then M_S is a finitely generated module for the Artinian ring R_S and as such must have finite composition length $\rho(M)$, say. The *reduced rank* of M is defined to be $\rho(M)$. It is easy to see that this definition coincides with the slightly more general one given in [10, 4.1.2]. We list some fairly obvious properties of this invariant:

- $\rho(M)$ is a nonnegative integer,
- ρ is additive on short exact sequences,
- $\rho(M) = 0$ if and only if M is $S-$torsion.

Note that kG has an Artinian quotient ring by Proposition 7.2. In our setup, we have the following formula for $\rho(M)$:

Lemma. *Keeping the notation of (8.1), let M be a finitely generated $kG-$module. Then*

$$\rho(M) = \sum_{i=1}^{s} \mu_i(M)\rho(P_i) = \sum_{i=1}^{s} \left(\frac{\rho(P_i)}{\dim_k \operatorname{End}_{kG}(V_i)} \right) \log_q \chi(G, M \otimes_k V_i^*).$$

Proof. Because ρ is additive on short exact sequences, it factors through $\mathcal{G}_0(kG)$. Now apply Proposition 4.8. □

Corollary. *Let M be a finitely generated $\pi-$torsion $\mathcal{O}G-$module with $\xi_M = 1$. Then M is pseudo-null.*

Proof. By Proposition 5.6, $\mu_i(M) = 0$ for all $i = 1, \ldots, s$. By the Lemma, $\rho(\operatorname{gr}_\pi M) = 0$, so $\operatorname{gr}_\pi M$ is $kG-$torsion. Hence M is pseudo-null by Lemma 7.4. □

8.4. Integrality of Euler characteristics. Our main result is the following:

Theorem. *Suppose G is a compact $p-$adic analytic group with no elements of order p. Then every finitely generated $kG-$module has integral Euler characteristic if and only if $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$.*

Proof. ($\Rightarrow$) Let $M \in \mathcal{C}$. In view of Theorem 8.2, it is sufficient to show that $\chi(G, M) = 1$. Now as M is torsion, the reduced rank $\rho(M)$ of M is zero. On the other hand, Lemma 8.3 gives

$$\prod_{i=1}^{s} \chi(G, M \otimes_k V_i^*)^{r_i} = 1 \quad \text{where} \quad r_i = \frac{\rho(P_i)}{\dim_k \operatorname{End}_{kG}(V_i)}.$$

Note that $r_i \geqslant 0$ for all i. Since we are assuming that $\chi(G, N) \in \mathbb{Z}$ for all $N \in \mathcal{M}(kG)$, we see that $\chi(G, M \otimes_k V_i^*) = 1$ whenever $\rho(P_i) \neq 0$. But each P_i is a submodule of the $S-$torsionfree module kG and as such is torsionfree. It follows that $(P_i)_S \neq 0$, so $\rho(P_i) > 0$ for all $i = 1, \ldots, s$ and $\chi(G, M \otimes_k V_1) = \chi(G, M) = 1$ as required.

($\Leftarrow$). This will be given in (9.5), after we have obtained more information about blocks of kG. □

9. Blocks of kG

9.1. Recall [1, 1.3] the important characteristic subgroup Δ^+ of G, defined by

$$\Delta^+ = \Delta^+(G) = \{x \in G : |G : C_G(x)| < \infty \quad \text{and} \quad o(x) < \infty\}.$$

Thus Δ^+ consists of all elements of finite order whose $G-$conjugacy class is finite. Since G is a compact $p-$adic analytic group, it can be shown that Δ^+ is in fact the largest finite normal subgroup of G.

9.2. Suppose $p \nmid |\Delta^+|$, so that the group algebra $k\Delta^+$ is semisimple. Since Δ^+ is normal in G, G acts by conjugation on the centrally primitive idempotents of $k\Delta^+$. Whenever $\mathcal{C}$ is a $G-$orbit on these idempotents, $\widehat{\mathcal{C}} = \sum_{e \in \mathcal{C}} e$ is a central idempotent of kG. Let $f_1, \ldots, f_r$ be the central idempotents of kG obtained in this way; it is easy to see that they are pairwise orthogonal and that $1 = f_1 + \ldots + f_r$.

We then have a decomposition of kG into a direct sum of ideals:

$$kG = f_1 kG \oplus \cdots \oplus f_r kG. \tag{10}$$

The main result of this section can be thought of as a suitable generalization and refinement of [1, Theorem A], which says that $\mathbb{F}_p G$ is prime if and only if $\Delta^+ = 1$.

Theorem. *The ring $f_i kG$ is prime for every $i = 1, \ldots, r$.*

The proof is given in (10.6). First, we derive some important consequences.

Corollary. *Let G be a compact $p-$adic analytic group such that $p \nmid |\Delta^+|$. Then the number of blocks of kG equals the number of $G-$conjugacy classes of blocks of $k\Delta^+$.*

Proof. A prime ring is cannot be nontrivially decomposed into a direct sum of ideals, so (10) is actually a decomposition of kG into the required number of blocks. □

9.3. Semiprimeness of kG. Recall [1, Theorem B] that when $k = \mathbb{F}_p$, $kG = \mathbb{F}_p G$ is semiprime if and only if $p \nmid |\Delta^+|$. We obtain a generalization of this result, as another consequence of Theorem 9.2.

Proposition. *Let G be a compact $p-$adic analytic group. Then kG is semiprime if and only if $p \nmid |\Delta^+|$.*

Proof. Suppose $p \nmid |\Delta^+|$. Then by Theorem 9.2 and (10), kG is a direct sum of prime rings and is therefore semiprime. On the other hand, if $p \mid |\Delta^+|$, the Jacobson radical of $k\Delta^+$ generates a nonzero two-sided nilpotent ideal of kG. □

9.4. Local blocks. Our proof of Theorem 8.4 depends on the next result.

Proposition. *If G be a compact $p-$adic analytic group such that $p \nmid |\Delta^+|$, then $\operatorname{rk} K_0(Q(kG)) = b(kG)$. Moreover, $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$ if and only if every block of kG is local.*

Proof. Since kG is semiprime by Proposition 9.3, $Q(kG)$ is semisimple Artinian. Hence $\operatorname{rk} K_0(Q(kG)) = b(Q(kG))$ by Lemma 2.4(c). By Theorem 9.2, kG is a direct sum of $r = b(kG)$ prime rings, so $Q(kG)$ is a direct sum of r simple Artinian rings. Hence $b(Q(kG)) = b(kG)$ as required. The last part now follows directly from Proposition 2.7 and Proposition 3.3(c). □

9.5. Proof of Theorem 8.4($\Leftarrow$). Since $\operatorname{rk} K_0(Q(kG)) = \operatorname{rk} K_0(kG)$, we see that every block $f_i kG$ of kG is local by Proposition 9.4. By reordering the indecomposable projectives $P_1, \ldots, P_s$ if necessary, we may write $f_i kG \cong P_i^{m_i}$ for some integers $m_i \geqslant 1$. Thus $K_0(f_i kG) = \mathbb{Z}[P_i]$ for all i.

Next, as β is an isomorphism, β restricts to an isomorphism of $K_0(f_i kG)$ and $K_0(Q(f_i kG))$. Since $Q(f_i kG)$ is simple Artinian, we see that $\beta([P_i])$ must be a generator of $K_0(Q(f_i kG))$; in other words, each localisation $(P_i)_S$ is a simple $Q(kG)-$module. Moreover, $(P_1)_S, \ldots, (P_r)_S$ is then a complete list of representatives for the isomorphism classes of simple $Q(kG)-$modules.

Now let M be a finitely generated $kG-$module. Then the localisation M_S is a finitely generated module for the semisimple ring $Q(kG)$, so we may write

$$M_S = (P_1)_S^{a_1} \oplus \cdots \oplus (P_r)_S^{a_r} \cong (P_1^{a_1} \oplus \cdots \oplus P_r^{a_r})_S$$

for some integers $a_1, \ldots, a_r \geqslant 0$. Let $N = P_1^{a_1} \oplus \cdots \oplus P_r^{a_r}$, a finitely generated projective $kG-$module. By Lemma 4.6, $\chi(G, N) = q^{a_1} \in \mathbb{Z}$ so N has integral Euler characteristic.

Now $[M] - [N] \in \ker(\beta) = \operatorname{Im}(\alpha)$ and $\chi(G, X) = 1$ for all $X \in \mathcal{C}$ by Theorem 8.2. It follows that $\chi(G, M) = \chi(G, N) \in \mathbb{Z}$ as required. $\square$

9.6. An explicit example. Let p be an odd prime and let

$$G = \mathbb{Z}_p \rtimes C_2 = \overline{\langle x, y : y^{-1}xy = x^{-1}, y^2 = 1\rangle}$$

be the pro-p completion of the infinite dihedral group. This is a compact $p-$adic analytic group of dimension 1. Let $N = \overline{\langle x \rangle} \cong \mathbb{Z}_p$, an open normal subgroup of G.

Since G/N is cyclic of order 2 and p is odd, we see that $kG/J(kG) = k[G/N]$ is a direct product of two copies of k. Also $\Delta^+(G) = 1$ because otherwise G would be isomorphic to the direct product of N and G/N. Thus $\operatorname{rk} K_0(kG) = 2$ but $\operatorname{rk} K_0(Q(kG)) = 1$ since kG is prime by Theorem 9.2.

Let $e = \frac{1}{2}(1 + y)$ and $f = 1 - e$, a pair of orthogonal idempotents in kG. Then $P_1 = e.kG$ is the projective cover of the trivial simple $kG-$module V_1 and $P_2 = f.kG$ is the projective cover of the other simple $kG-$module, V_2 say. Moreover, $kG = P_1 \oplus P_2$ is a decomposition of kG into a direct sum of two indecomposable projectives.

Viewing kG as a $kN-$module, we see that P_1 and P_2 must both be finitely generated projective $kN-$modules of rank 1. Since $kN \cong k[[t]]$ is a scalar local Noetherian domain, this forces P_1 and P_2 to be uniform $kG-$modules; recall [10, 2.2.5] that a module U is said to be *uniform* if any two nonzero submodules X, Y of U have nonzero intersection.

Recall also that two uniform right ideals U and V of a semiprime Noetherian ring R are said to be *subisomorphic* if U contains an isomorphic copy of V, or equivalently, if V contains an isomorphic copy of U [10, 3.3.4]. By [10, Lemma 3.3.4(ii)], any two uniform right ideals U, V of a *prime* Noetherian ring R are necessarily subisomorphic.

Hence we can find an embedding $\varphi : P_1 \hookrightarrow P_2$ of $kG-$modules, leading to a short exact sequence

$$0 \to P_1 \xrightarrow{\varphi} P_2 \to \mathrm{coker}(\varphi) \to 0.$$

Taking Euler characteristics and applying Lemma 4.6, we see that

$$\chi(G, \mathrm{coker}(\varphi)) = \chi(G, P_2)/\chi(G, P_1) = q^{-1}.$$

Thus $\mathrm{coker}(\varphi)$ does *not* have integral Euler characteristic in this case.

With a bit of care, the injection φ can be chosen to have cokernel precisely V_2: set $\varphi(e) = f\alpha$ where $\alpha \in kN$ is such that $f\alpha kG = f.J(kG)$. We omit the computations which show that such an α exists. So $\chi(G, V_2) = q^{-1}$.

10. Proof of Theorem 9.2

10.1. A special case. A very special case of Theorem 9.2 is not too difficult to deal with:

Proposition. *Let N be a uniform pro-p group, F a finite group with $p \nmid |F|$ and $H = N \times F$. Suppose e is a centrally primitive idempotent of kF.*

(a) There exists a finite extension k' of k and an integer $t \geqslant 1$ such that $e.kH$ is isomorphic to a full $t \times t$ matrix ring with coefficients in $k'N$:

$$e.kH \cong M_t(k'N).$$

(b) The ring $e.kH$ is prime.

Proof. (a) Since $p \nmid |F|$, kF is semisimple so the block $e.kF$ is a simple finite dimensional $k-$algebra. Since k is finite, Wedderburn's theorem on the structure of finite division algebras implies that $e.kF \cong M_t(k')$ for some finite field extension k' of k. Now, because N commutes with F, we can think of kG as a group algebra of F with coefficients in kN: $kH = kN[F]$. We can also write this as a tensor product of $k-$algebras

$$kH \cong kN \otimes_k kF$$

where the multiplication on the right hand side is given by $(a \otimes b)(c \otimes d) = ac \otimes bd$. Hence,

$$e.kH \cong kN \otimes_k e.kF \cong kN \otimes_k M_t(k') \cong M_t(kN \otimes_k k') \cong M_t(k'N)$$

as required.

(b) Now, $k'N$ is a domain by Lemma 7.1 and is therefore prime. Since primeness is preserved by Morita equivalence [10, Proposition 5.10(iii)] and a ring A is always Morita equivalent to the matrix ring $M_t(A)$ [10, Proposition 5.6], we see that $e.kH \cong M_t(k'N)$ is prime. □

10.2. We will need a general Lemma.

Lemma. *Let A, B be $k-$algebras and let T be an Ore set in A. Then $T \otimes 1$ is an Ore set in $A \otimes_k B$ and*

$$(A \otimes_k B)_{T\otimes 1} \cong A_T \otimes_k B.$$

Proof. This is a straightforward application of [10, Lemma 2.1.8]. □

When N is a uniform pro-p group, write D_N for the division ring of fractions of $\mathbb{F}_p N$ which exists by [10, Theorem 2.1.15] and Lemma 7.1.

Proposition. *Let H and e be as in Proposition 10.1. Let $R = e.kH$ and let $S = e.\mathbb{F}_p N \backslash \{0\}$. Then:*

(a) S is an Ore set in R,
(b) $R_S \cong D_N \otimes_{\mathbb{F}_p} M_t(k')$ as $\mathbb{F}_p-$algebras,
(c) R_S is a simple ring.

Proof. From the proof of Proposition 10.1(a), we know that

$$R = e.kH \cong kN \otimes_k e.kF.$$

But $kN \cong \mathbb{F}_p N \otimes_{\mathbb{F}_p} k$ by (3.2) and $e.kF \cong M_t(k')$ for some finite field extension k' of k so we have an isomorphism

$$\theta : R \to \mathbb{F}_p N \otimes_{\mathbb{F}_p} M_t(k')$$

of $\mathbb{F}_p-$algebras. Now, $T = \mathbb{F}_p N \backslash \{0\}$ is an Ore set in $\mathbb{F}_p N$ so $T \otimes 1$ is an Ore set in $\mathbb{F}_p N \otimes_{\mathbb{F}_p} M_t(k')$ by the first part of the Lemma. It is easy to see that $\theta^{-1}(T \otimes 1) = S$, so S is an Ore set in R and

$$R_S \cong (\mathbb{F}_p N \otimes_{\mathbb{F}_p} M_t(k'))_{T\otimes 1} \cong D_N \otimes_{\mathbb{F}_p} M_t(k')$$

by the second part of the Lemma. This deals with parts (a) and (b).

Now, by Proposition 10.1(b), R is prime so R_S is also prime. But $R_S \cong D_N \otimes_{\mathbb{F}_p} M_t(k')$ is a finite module over the division subring $\theta^{-1}(D_N \otimes 1)$, so R_S is Artinian. Since any prime Artinian ring is simple, R_S is simple as required for part (c). □

10.3. Recall from [1, 2.2] the important subgroup $E_G(N)$ associated to any open normal uniform subgroup N of a compact $p-$adic analytic group G:

$$E_G(N) = \{x \in G : [N, x] \subseteq N^{p^\epsilon}\}.$$

Here, as in [1, 2.1],

$$\epsilon = \begin{cases} 2 & \text{if} \quad p = 2 \\ 1 & \text{otherwise.} \end{cases}$$

$E_G(N)$ is the kernel of the conjugation action of G on the finite set N/N^{p^ϵ} and as such is an open normal subgroup of G containing N.

10.4. Another special case. The following proposition reduces to [1, Proposition 2.2] in the case when $\Delta^+ = 1$. The proof is also broadly similar.

Proposition. *Let G be a compact $p-$adic analytic group with $p \nmid |\Delta^+|$. Suppose N is an open normal uniform subgroup of G such that $E_G(N) = N\Delta^+$ and suppose that the centrally primitive idempotent e of $k\Delta^+$ is central in kG. Then $e.kG$ is prime.*

Proof. Let $H = N\Delta^+$. Since N is torsionfree [7, Theorem 4.5], H is actually isomorphic to the direct product of N and Δ^+. Since H is normal in G we can write kG as a crossed product of kH with the finite group $\overline{G} = G/H$:

$$kG = kH * \overline{G}.$$

Since $e \in k\Delta^+ \subseteq kH$, we can also write $e.kG$ as a crossed product:

$$e.kG = R * \overline{G}$$

where $R = e.kH$ is the ring appearing in Proposition 10.1. Let $S = e\mathbb{F}_pN\backslash\{0\}$. Because kH is a free $\mathbb{F}_pN-$module and $\mathbb{F}_pN$ is a domain, we see that S consists of regular elements in R. Also, it is $\overline{G}-$stable and an Ore set in R by Proposition 10.2(a). Hence S is actually an Ore set of $e.kG$ consisting of regular elements by [13, Lemma 37.7], so by Proposition 10.2(b), we have

$$(e.kG)_S \cong R_S * \overline{G} \cong (D_N \otimes_{\mathbb{F}_p} M_t(k')) * \overline{G}.$$

We will now show that every nontrivial element of $\overline{G}$ induces an outer automorphism of the ring R_S.

The sets $e\mathbb{F}_pN$ and $e.k\Delta^+$ are stable under the conjugation action of G. Let $g \in G$ and let β_g and γ_g denote the automorphisms of D_N and $M_t(k') \cong e.k\Delta^+$ induced by conjugation by g on $e\mathbb{F}_pN$ and $e.k\Delta^+$, respectively.

Since $R_S \cong D_N \otimes_{\mathbb{F}_p} M_t(k')$, we see that the action of g on R_S is given by the automorphism $\alpha_g := \beta_g \otimes \gamma_g$. Suppose that α_g is an inner automorphism of R_S. Now, by the Skolem-Noether Theorem, $\gamma_g \in \mathrm{Aut}(M_t(k'))$ is inner, so $\beta_g \otimes 1 = \alpha_g(1 \otimes \gamma_g^{-1})$ is an inner automorphism of R_S which stabilizes $D_N \otimes 1$ and fixes $1 \otimes M_t(k')$. Let $\beta_g \otimes 1$ be given by conjugation by $x \in R_S$. Then x commutes with every matrix unit in $M_t(k')$ and therefore must lie in the subring $D_N \otimes 1$. Hence, by [1, Proposition 2.1], $[N, g] \subseteq N^{p^\epsilon}$ and $g \in E_G(N) = N\Delta^+$.

Hence every element $1 \neq \bar{g} \in \overline{G}$ induces an outer automorphism on R_S, which is simple by Lemma 10.2(c). Hence $R_S * \overline{G}$ is simple by [10, Theorem 7.8.12], so $e.kG = R * \overline{G}$ is prime, as required. $\square$

10.5. Now let G be an arbitrary compact $p-$adic analytic group such that $p \nmid |\Delta^+|$. By [7, Corollary 8.34], we can find an open normal uniform subgroup N of G.

Let e be a centrally primitive idempotent of $k\Delta^+$ and let f be the corresponding central idempotent in kG; thus f is the sum of the $G-$conjugates of e. We have a crossed product decomposition

$$f.kG = f.kH * \overline{G}$$

where $\overline{G} = G/H$ and $H = N\Delta^+ \cong N \times \Delta^+$, as in (10.1).

Suppose that we are given a crossed product $T * \overline{G}$. Recall [13, §14.4] that the coefficient ring T is said to be an $\overline{G}-$*prime* if whenever A, B are $\overline{G}-$stable ideals of T with $AB = 0$, then either $A = 0$ or $B = 0$.

Lemma. *The coefficient ring $f.kH$ appearing in the crossed product*

$$f.kG = f.kH * \overline{G}$$

is $\overline{G}-$prime.

Proof. Write $f = e_1 + \ldots + e_m$ as a sum of centrally primitive idempotents of $k\Delta^+$ and let $R = e.kH$ where $e = e_1$, say. Suppose A, B are nonzero $\overline{G}-$stable ideals of $f.kH$. Then $A \cap e_i f.kH \neq 0$ for some i. Since $\overline{G}$ acts transitively on the e_j's by construction and since A is $\overline{G}-$stable, we see that $A \cap R \neq 0$. Similarly $B \cap R \neq 0$. But R is prime by Proposition 10.1(b), so $(A \cap R)(B \cap R) \neq 0$. Hence $AB \neq 0$ and the result follows. □

10.6. Recall the following useful fact from [1, 2.2(3)]:

(11) *if H is a subgroup of G of finite index, then* $\Delta^+(H) \leqslant \Delta^+(G)$.

It follows immediately that $\Delta^+(H) = \Delta^+(G)$ for any open subgroup H of G containing $\Delta^+(G)$.

Proof of Theorem 9.2. Keeping the notation of (10.5), we have to show that the crossed product

$$f.kG = f.kH * \overline{G}$$

is prime. We know from Lemma 10.5 that $f.kH$ is $G-$prime. Let

$$Q = (f - e)kH = e_2.kH \oplus \cdots \oplus e_m.kH,$$

this is a minimal prime ideal of $f.kH$ by Proposition 10.1(b).

Let $G_Q/H = \mathrm{Stab}_{\overline{G}}(Q)$; since H centralizes the idempotent e, it is easy to see that $G_Q = \mathrm{Stab}_G(e)$. By [13, Corollary 14.8], $f.kH * \overline{G}$ is prime if and only if $(f.kH/Q) * \overline{G_Q} \cong e.kH * \overline{G_Q} \cong e.kG_Q$ is prime.

Note that $\Delta^+(G_Q) = \Delta^+$ by (11) because G_Q is an open subgroup of G containing Δ^+. We can therefore replace G by G_Q and assume that $f = e$ is central in G. In this case we have a crossed product decomposition

$$e.kG = e.kH * \overline{G}$$

where the coefficient ring $e.kH$ is prime by Proposition 10.1(b).

Now let l be a prime (possibly equal to p) and let K_l/H be a Sylow $l-$subgroup of $\overline{G} = G/H$. Then

$$e.kK_l = e.kH * \overline{K_l}$$

is a sub-crossed product and it is sufficient to show that $e.kK_l$ is prime for any prime l by [13, Theorem 17.5]. Also, note that $\Delta^+(K_l) = \Delta^+$ for any l, by (11).

Suppose first that $l \neq p$. If L is a Sylow $l-$subgroup of $E_{K_l}(N)$, then the conjugation action of L on N gives rise to an injection

$$L/C_L(N) \hookrightarrow \Gamma = \{\varphi \in \mathrm{Aut}(N) : [N, \varphi] \subseteq N^{p^\epsilon}\}.$$

But Γ is a pro-p group by [7, Theorem 5.2]; since L is an $l-$group and $l \neq p$, we see that $L = C_L(N)$, so $[N, L] = 1$. Hence every element of L has open centralizer in G, so $L \subseteq \Delta^+ \subseteq H$. Since K_l/H is an $l-$group by assumption, we have shown that $E_{K_l}(N) = H$ and the result follows from Proposition 10.4.

Finally, suppose that $l = p$ and let $K = K_p$, so that K/H is a $p-$group. Let P be a maximal open normal uniform subgroup of K containing N. We claim that $E_K(P) = P\Delta^+$; clearly $P\Delta^+ \leqslant E_K(P)$. Let $\overline{} : K \twoheadrightarrow K/\Delta^+$ denote the natural surjection.

Because $\Delta^+ \leqslant K$, it is easy to verify that $\overline{E_K(P)} = E_{\overline{K}}(\overline{P})$. Also, P is a maximal open normal uniform subgroup of K if and only if $\overline{P}$ is a maximal open normal uniform subgroup of $\overline{K}$, so for this part of the proof we may assume that $\Delta^+ = 1$. Thus, K is a pro-p group of finite rank with $\Delta^+(K) = 1$ and we have to show that $E_K(P) = P$.

If $E_K(P) > P$ then $E_K(P)/P$ is a nontrivial normal subgroup of the finite $p-$group K/P and as such meets the centre of K/P nontrivially. Let xP be a nontrivial element in this intersection and consider $L = \langle P, x\rangle$, an open normal subgroup of K properly containing L. Since L is itself uniform by [1, Lemma 2.3], this contradicts the maximality of P. Hence $E_K(P) = P$ in this special case, and $E_K(P) = P\Delta^+$ in general, as required.

The result now follows from Proposition 10.4, with P replacing N. □

11. For which groups G is every block of kG local?

11.1. Theorem 8.2 and Theorem 8.4 stimulate interest in those compact $p-$adic analytic groups G with the property that $\mathrm{rk}\, K_0(kG) = \mathrm{rk}\, K_0(Q(kG))$.

If G is such that $p \nmid |\Delta^+|$, these two numbers are equal if and only if every block of kG is local by Proposition 9.4.

11.2. $p-$**nilpotent groups.** Recall that a finite group G is said to be $p-$*nilpotent* if a Sylow $p-$subgroup of G has a normal complement. It is well known that any subgroup and any quotient of a $p-$nilpotent group is again $p-$nilpotent.

Following [1, 1.5] we will denote the largest finite normal $p'-$subgroup of G by $\Delta^+_{p'}(G)$. We will say that a compact $p-$adic analytic group G is $p-$*nilpotent* if $G/\Delta^+_{p'}(G)$ is pro-p; it is clear that this extends the usual notion of $p-$nilpotence.

Write $\Delta^+ = \Delta^+(G)$ as in (9.1). If G is such that $p \nmid |\Delta^+|$, then $\Delta^+_{p'}(G) = \Delta^+$, so in this case G is $p-$nilpotent if and only if it is a semidirect product of Δ^+ with a Sylow pro$-p$ subgroup of G.

11.3. We collect together some useful inequalities.

Lemma. *Let N be an open normal pro-p subgroup of G. Then*

$$b(kG) \leqslant b(k[G/N]) \leqslant \operatorname{rk}(K_0(k[G/N])) = \operatorname{rk}(K_0(kG)).$$

Proof. In view of Proposition 2.7 and Corollary 3.3, it is sufficient to prove the first inequality. Suppose e is an idempotent of kG contained in the Jacobson radical $J(kG)$. Then $1-e$ is invertible, but $e(1-e) = 0$ so $e = 0$. By Proposition 3.3(a), the kernel of the natural map $\pi : kG \to k[G/N]$ is contained in $J(kG)$ so the image of any nonzero idempotent in kG is nonzero in $k[G/N]$. Now, if $1 = e_1 + \cdots + e_r$ is a decomposition of $1 \in kG$ into a sum of r nonzero orthogonal centrally primitive idempotents, then $1 = \pi(e_1) + \cdots + \pi(e_r)$ is a decomposition of $1 \in k[G/N]$ into r nonzero orthogonal central idempotents, so $r = b(kG) \leqslant b(k[G/N])$ as required. ☐

11.4. Our main result in this section is the following:

Theorem. *The following are equivalent for a compact $p-$adic analytic group G:*

(a) every block of kG is local,
(b) every block of $k[G/N]$ is local, for every open normal pro-p subgroup N,
(c) G/N is $p-$nilpotent for every open normal pro-p subgroup N,
(d) G is $p-$nilpotent.

Proof. We will use Proposition 2.7 in what follows without further mention.

$(a) \Rightarrow (b)$. This follows from Lemma 11.3.

$(b) \Rightarrow (a)$. From Lemma 11.3 we have

$$b(kG) \leqslant b(k[G/N]) = \operatorname{rk}(K_0(k[G/N])) = \operatorname{rk}(K_0(kG)) = r, \quad \text{say,}$$

for every open normal pro-p subgroup N of G. Let $N_1 \leqslant N_2$ be two such subgroups and let $\pi : k[G/N_1] \twoheadrightarrow k[G/N_2]$ be the canonical projection. If $1 =$

$e_1+\cdots+e_r$ is a decomposition of 1 into nonzero orthogonal centrally primitive idempotents in $k[G/N_1]$, then $1 = \pi(e_1) + \cdots + \pi(e_r)$ is a decomposition of 1 into nonzero orthogonal central idempotents in $k[G/N_2]$. Because $r = b(k[G/N_1]) = b(k[G/N_2])$, we see that each $\pi(e_i)$ must be centrally primitive. This shows that we can "lift" primitive central idempotents modulo smaller and smaller open normal subgroups N. Using the definition of kG as the inverse limit of the various $k[G/N]$, we obtain r nonzero orthogonal central idempotents of kG. Hence $b(kG) \geqslant r$ and (a) follows.

$(b) \Leftrightarrow (c)$. This follows directly from [12, Theorem 1]. A more modern treatment of $(b) \Rightarrow (c)$ can be found at [9, Theorem 29.1] - a careful inspection of the proof shows that the hypothesis that the underlying field be algebraically closed is unnecessary for this part of the proof presented there. We have so far been unable to find a more modern reference for the whole result.

$(c) \Rightarrow (d)$. Without loss of generality, we may assume that $\Delta^+_{p'}(G) = 1$. Choose an open normal uniform subgroup N of G. Then $N \cap \Delta^+ = 1$ so Δ^+ is isomorphic to a subgroup of the $p-$nilpotent group G/N. Hence Δ^+ is itself $p-$nilpotent. Since $\Delta^+_{p'}(G) = 1$, we see that Δ^+ is a $p-$group. Now G is a pro-p group by [1, Proposition 3.7] and is therefore $p-$nilpotent, as required.

$(d) \Rightarrow (c)$. This is easy. □

11.5. A summary of results involving $p-$nilpotence.

Theorem. *Let G be a compact $p-$adic analytic group with no elements of order p. Then the following conditions are equivalent:*

(a) $\xi_M = 1$ *for all* $M \in \mathcal{C}$ *(8.1),*
(b) $\chi(G, M) = 1$ *for all* $M \in \mathcal{C}$*,*
(c) $\chi(G, M) \in \mathbb{Z}$ *for all* $M \in \mathcal{M}(kG)$*,*
(d) $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$*,*
(e) every block of kG is local,
(f) G is $p-$nilpotent (11.2).

Proof. Apply Theorems 8.2, 8.4, 11.4 and Proposition 9.4. □

12. Ranks of $K_0(kG)$ and $K_0(Q(kG))$

12.1. We are able to explicitly compute the rank of $K_0(kG)$, as well as the rank of $K_0(Q(kG))$ in the case when $p \nmid |\Delta^+|$. First, we must recall some well-known results from the modular representation theory of finite groups. We follow [6, Volume I, §17A, §21B] in our treatment of Brauer characters.

12.2. Galois action. Let H be a finite group and let $m = p^a m'$ be the exponent of H, where $p \nmid m'$. Let $k' = k(\tilde{\omega})$, where $\tilde{\omega}$ is a primitive $m'-$th root of unity over k and let $\mathcal{G}_k$ be the Galois group $\mathrm{Gal}(k(\tilde{\omega})/k)$. If $\sigma \in \mathcal{G}_k$, then $\sigma(\tilde{\omega}) = \tilde{\omega}^{t_\sigma}$ for some $t_\sigma \in (\mathbb{Z}/m'\mathbb{Z})^\times$. This gives an injection $\sigma \mapsto t_\sigma$ of $\mathcal{G}_k$ into $(\mathbb{Z}/m'\mathbb{Z})^\times$.

We can now define a *left* permutation action of $\mathcal{G}_k$ on H by setting $\sigma.h = h^{t_\sigma}$. Note that $h \mapsto \sigma.h$ is invertible because t_σ is coprime to $|H|$. Note also that this action commutes with any automorphism of H, and in particular with conjugation by elements of H. Thus $\mathcal{G}_k$ permutes the conjugacy classes of H.

12.3. $p-$**regular elements.** An element of H is said to be $p-$*regular* if its order is coprime to p. The set of all $p-$regular elements of H will be denoted by H_{reg} - this is a union of conjugacy classes of H. It is clear that the action of $\mathcal{G}_k$ leaves H_{reg} stable.

12.4. Brauer characters. Fix a finite unramified extension K of $\mathbb{Q}_p$ with residue field k. Let $K' = K(\omega)$, where ω is a primitive $m'-$th root of 1. Then the ring of integers of K' is $\mathcal{O}' = \mathcal{O}[\omega]$ where $\mathcal{O}$ is the ring of integers of K. Moreover, reduction modulo p gives an isomorphism of the residue field of K' with k', with ω mapping to $\tilde{\omega}$. Let $\varphi : \langle\omega\rangle \to \langle\tilde{\omega}\rangle$ be the restriction of this isomorphism to the cyclic group of $m'-$th roots of unity in K'.

Now, if V is a finite dimensional $kH-$module and $h \in H_{\mathrm{reg}}$, the eigenvalues of the action of h on V are powers of $\tilde{\omega}$, $\{\xi_1, \ldots, \xi_d\}$ say. Define

$$\chi_V(h) = \sum_{i=1}^{d} \varphi^{-1}(\xi_i) \in K'.$$

The function $\chi_V : H_{\mathrm{reg}} \to K'$ is called the *Brauer character* of V. It has the following properties:

- χ_V is a *class function*; that is $\chi_V(g^{-1}hg) = \chi_V(h)$ for all $g \in H$ and for all $h \in H_{\mathrm{reg}}$,
- $\chi_V = \chi_U + \chi_W$ whenever $0 \to U \to V \to W \to 0$ is a short exact sequence of finite dimensional $kH-$modules.

See [6, Volume I, p. 509] for details and proofs.

12.5. Berman-Witt Theorem. Let $\mathcal{C}(H_{\mathrm{reg}}, K')$ denote the $K'-$vector space of all $K'-$valued class functions on H_{reg}. The group $\mathcal{G}_k$ acts on this space via

$$(f.\sigma)(h) = f(\sigma.h) \quad \text{for all} \quad f \in \mathcal{C}(H_{\mathrm{reg}}, K'), \sigma \in \mathcal{G}_k, h \in H.$$

We will write $\mathcal{C}(H_{\mathrm{reg}}, K')^{\mathcal{G}_k}$ for the fixed points of $\mathcal{G}_k$ under this action. Next, because χ_V is additive on short exact sequences, we obtain a $K'-$linear map

$$\chi : K' \otimes_{\mathbb{Z}} \mathcal{G}_0(kH) \to \mathcal{C}(H_{\mathrm{reg}}, K'),$$

given by $\chi(\lambda \otimes [V]) = \lambda\chi_V$ for all $\lambda \in K'$ and all relevant $kH-$modules V. The following result is essentially due to Berman and Witt.

Theorem. *χ is an isomorphism of $K' \otimes_{\mathbb{Z}} \mathcal{G}_0(kH)$ onto $\mathcal{C}(H_{\text{reg}}, K')^{\mathcal{G}_k}$.*

Proof. This is a rephrasing of [6, Volume I, Theorem 21.25]. □

Corollary. *The number of isomorphism classes of simple $kH-$modules is equal to the number of $\mathcal{G}_k-$orbits on the $p-$regular conjugacy classes of H.*

12.6. A $G-$equivariant version. Now suppose that we have a group G acting on our finite group H by automorphisms. We will suppose that this is a *left* action, and will write ${}^gh$ for the image of $h \in H$ under $g \in G$. Whenever V is a finite dimensional $kH-$module, let Vg be the $kH-$module whose underlying abelian group is V, but H acts via $v.h = v({}^gh)$.

This induces a *right* action of G on $\mathcal{G}_0(kH)$ given by $[V].g = [Vg]$ which permutes the classes of simple modules. There is also a natural right $K'-$linear action of G on $\mathcal{C}(H_{\text{reg}}, K')$ given by

$$(f.g)(h) = f({}^gh) \quad \text{for all} \quad f \in \mathcal{C}(H_{\text{reg}}, K'), g \in G, h \in H.$$

It is now straightforward to verify the following result:

Lemma. *The isomorphism $\chi : K' \otimes_{\mathbb{Z}} \mathcal{G}_0(kH) \to \mathcal{C}(H_{\text{reg}}, K')^{\mathcal{G}_k}$ appearing in Theorem 12.5 is a map of right $K'G-$modules.*

Taking dimensions of the $G-$fixed points of both sides, we obtain

Corollary. *The number of $G-$orbits on the set of simple $kH-$modules equals the number of $G \times \mathcal{G}_k-$orbits on the $p-$regular conjugacy classes of H.*

12.7. We now come to the main result of this section.

Theorem. *Let G be a compact $p-$adic analytic group. Fix an open normal pro-p subgroup N of G. Then*

(a) The rank of $K_0(kG)$ equals the number of $G \times \mathcal{G}_k-$orbits on $(G/N)_{\text{reg}}$.
(b) If $p \nmid |\Delta^+|$, the rank of $K_0(Q(kG))$ equals the number of $G \times \mathcal{G}_k-$orbits on Δ^+.

Proof. Here G acts on G/N and Δ^+ by conjugation. By Corollary 3.3, the rank of $K_0(kG)$ is equal to the number of isomorphism classes of simple $k[G/N]-$modules, which by Corollary 12.5 is the number of $\mathcal{G}_k-$orbits on the conjugacy classes of $(G/N)_{\text{reg}}$, or equivalently, the number of $G \times \mathcal{G}_k-$orbits on $(G/N)_{\text{reg}}$. Part (a) follows.

Now, the conjugation action of G on Δ^+ gives rise to an action on the blocks of $k\Delta^+$, and also to an action on the set of simple $k\Delta^+-$modules described in (12.6). Let b and s denote the numbers of orbits of G under these actions, respectively.

Because $k\Delta^+$ is semisimple, it is easy to see that $b = s$.

By Corollary 12.6, s equals the number of $G \times \mathcal{G}_k-$orbits on the $p-$regular conjugacy classes of Δ^+. Since $p \nmid |\Delta^+|$ and since G contains Δ^+, this also equals the number of $G \times \mathcal{G}_k-$orbits on the whole of Δ^+.

On the other hand, $b = b(kG)$ by Corollary 9.2 and $b(kG) = \operatorname{rk} K_0(Q(kG))$ by Proposition 9.4. Part (b) follows. □

12.8. We end with a second proof of a part of Theorem 11.5.

Proposition. *Let G be a compact $p-$adic analytic group with $p \nmid |\Delta^+|$. Then* $\operatorname{rk} K_0(kG) = \operatorname{rk} K_0(Q(kG))$ *if and only if G is $p-$nilpotent.*

Proof. Let N be an open normal pro-p subgroup of G. Because $p \nmid |\Delta^+|$, we see that $N \cap \Delta^+ = 1$, so Δ^+ embeds into $\overline{G} = G/N$. It is clear that this embedding, ι say, is a map of $G \times \mathcal{G}_k-$spaces.

By Theorem 12.7, the two ranks are equal if and only if $\overline{G}_{\text{reg}} = \iota(\Delta^+)$. Now, every element x of $\overline{G}$ can be written as $x = x_u x_s$ where x_s is $p-$regular and x_u has order a power of p. This shows that $\overline{G}_{\text{reg}} = \iota(\Delta^+)$ if and only if every element of $\overline{G}/\iota(\Delta^+)$ has order a power of p, that is, if and only if $G/N\Delta^+$ is a $p-$group. This happens if and only if G/Δ^+ is a pro-p group, as required. □

References

[1] K. Ardakov, K. A. Brown, *Primeness, semiprimeness and localisation in Iwasawa Algebras*, submitted.

[2] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Algebra **4**(1966) 442-470.

[3] J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob, *The GL_2 main conjecture for elliptic curves without complex multiplication*, preprint, arXiv math NT/0404297.

[4] J. Clark, *Auslander-Gorenstein rings for beginners*, International Symposium on Ring Theory (Kyongju, 1999), 95-115.

[5] J. Coates, P. Schneider, R. Sujatha, *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu **2** (2003), no. 1, 73-108.

[6] C.W. Curtis, I. Reiner, *Methods of Representation Theory*, J. Wiley (1981).

[7] J.D. Dixon, M.P.F. Du Sautoy, A. Mann, D. Segal, *Analytic pro−p groups*, 2nd edition, CUP (1999).

[8] S. Howson, Euler characteristics as invariants of Iwasawa modules, Proc. London Math. Soc. (3) **85** (2002) 634-658.

[9] G. Karpilovsky, *Structure of blocks of group algebras*, Longman, Essex (1987).

[10] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian rings*, AMS Graduate Studies in Mathematics, vol. 30.

[11] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften **323** Springer (2000).

[12] M. Osima, *On primary decomposable group rings*, Proc. Phy-Math. Soc. Japan (3) **24** (1942) 1-9.

[13] D. Passman, *Infinite Crossed Products*, Pure and Applied Mathematics vol. 135, Academic Press, San Diego, (1989).

[14] J.-P. Serre, *Sur la dimension homologique des groupes profinis*, Topology **3**, (1965) 413-420.
[15] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics **42** Springer (1977).
[16] R. G. Swan, *Algebraic K-theory*, Lecture Notes in Mathematics **76** Springer (1968).
[17] O. Venjakob, *Characteristic elements in non-commutative Iwasawa theory*, Habilitationsschrift, Heidelberg University (2003).
[18] O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. (3) **4** (2002) 271-311.
[19] C. Weibel, D. Yao, *Localization for the $K-$theory of Noncommutative Rings*, Contemporary Mathematics **126** (1992).

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