CENTRES OF SKEWFIELDS AND COMPLETELY FAITHFUL IWASAWA MODULES

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ABSTRACT. Let H be a torsion-free compact p-adic analytic group whose Lie algebra is split semisimple. We show that the quotient skewfield of fractions of the Iwasawa algebra Λ_H of H has trivial centre and use this result to classify the prime c-ideals in the Iwasawa algebra Λ_G of $G := H \times \mathbb{Z}_p$. We also show that a finitely generated torsion Λ_G-module having no non-zero pseudo-null submodule is completely faithful if and only if it is has no central torsion. This has an application to the study of Selmer groups of elliptic curves.

1. INTRODUCTION

1.1. **Iwasawa algebras.** Let p be a prime number. In this note we are concerned with modules over the Iwasawa algebra

$$\Lambda_G := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N]$$

of a compact p-adic analytic group G. These groups frequently occur as images of Galois representations on p-power division points of abelian varieties and act on various arithmetic objects of interest such as ideal class groups and Selmer groups. These arithmetic objects then naturally become modules over the associated Iwasawa algebra. By examining the structure of these modules in detail, it is sometimes possible to obtain precise arithmetic information.

1.2. **Selmer groups.** As a concrete example of the philosophy outlined above in action, consider the following situation. Let E be an elliptic curve over a number field F and assume E has no complex multiplication. Let E_{p^∞} denote the group of all p-power division points on E, let $F_\infty = F(E_{p^\infty})$ and let $G = \mathrm{Gal}(F_\infty/F)$. By a well-known theorem of Serre [18], G is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. In this setting, Coates, Schneider and Sujatha [11, §8] study the Pontryagin dual $X(E/F_\infty)$ of the *Selmer group* $\mathcal{S}(E/F_\infty)$; see [9, §1] for a definition. They show that $X(E/F_\infty)$ is always a finitely generated Λ_G-module, and also that under suitable assumptions on E and p, $X(E/F_\infty)$ is finitely generated and torsion-free over Λ_H, where H is a certain closed normal subgroup of G satisfying $G/H \cong \mathbb{Z}_p$. They also posed a number of questions concerning the Λ_G-module structure of $X(E/F_\infty)$, including the following.

- Let Z denote the centre of G. Assuming $X(E/F_\infty)$ is finitely generated and torsion-free over Λ_H, is $X(E/F_\infty)$ torsion-free as a Λ_Z-module?
- Under the same assumptions, is $q(X(E/F_\infty))$ completely faithful?

1.3. **Completely faithful modules.** Let $\mathcal{M}$ denote the category of all finitely generated Λ_G-modules, let $\mathcal{C}$ denote the full subcategory of all pseudo-null modules in $\mathcal{M}$ and let $q : \mathcal{M} \to \mathcal{M}/\mathcal{C}$ denote the quotient functor. Recall [11, §5] that an object $q(M)$ in the quotient category $\mathcal{M}/\mathcal{C}$ is said to be *completely faithful* if $\text{Ann}(N) = 0$ for any $N \in \mathcal{M}$ such that $q(N)$ is isomorphic to a non-zero subquotient of $q(M)$. The purpose of this note is to prove the following

Theorem. *Let $p \geqslant 5$ and let H be a torsion-free compact p-adic analytic group, whose Lie algebra $\mathcal{L}(H)$ is split semisimple over $\mathbb{Q}_p$. Let $G = H \times Z$ where $Z \cong \mathbb{Z}_p$, and let M be a finitely generated torsion Λ_G-module, which has no non-zero pseudo-null submodules. Then $q(M)$ is completely faithful if and only if M is torsion-free over Λ_Z.*

The proof is given in §4.9. We should point out that if M is a finitely generated Λ_G-module which is also finitely generated over Λ_H, then M is automatically Λ_G-torsion. Moreover, in this situation, M has no non-zero pseudo-null submodules if and only if M is torsion-free over Λ_H.

Theorem 1.3 should be compared with [19, Theorem 6.3], which states that if instead G is a non-abelian semidirect product of two copies of $\mathbb{Z}_p$, then $q(M)$ is completely faithful whenever M is a finitely generated Λ_G-module which is finitely generated over Λ_H.

The assumptions on the group G in Theorem 1.3 are fairly mild. For example, any open pro-p subgroup G of $\text{GL}_2(\mathbb{Z}_p)$ satisfies them whenever $p \geqslant 3$: just take $H = G \cap \text{SL}_2(\mathbb{Z}_p)$. Moreover, if G is just assumed to have a closed normal subgroup H such that $\mathcal{L}(H)$ is semisimple and such that $G/H \cong \mathbb{Z}_p$, then by passing to an open subgroup we can ensure that in fact $G = H \times \mathbb{Z}_p$. This is because the only extension of a semisimple Lie algebra by a one-dimensional Lie algebra is the trivial extension.

1.4. **Selmer groups revisited.** Returning to our GL_2 example, Theorem 1.3 implies that $q(X(E/F_\infty))$ must necessarily be completely faithful, provided $X(E/F_\infty)$ is finitely generated over Λ_H and torsion-free over both Λ_H and Λ_Z. Although $X(E/F_\infty)$ is known to be finitely generated and torsion-free over Λ_H in many circumstances, there are at present no known examples where one can prove that $X(E/F_\infty)$ is torsion-free over Λ_Z. Some weak information about the Λ_Z-torsion part of $X(E/F_\infty)$ in a particular case is given in [11, Proposition 8.10].

We should remark at this point that Coates *et al* conjecture that the p-torsion-free part $Y(E/F_\infty)$ of $X(E/F_\infty)$ is finitely generated over Λ_H whenever $p \geqslant 5$ and E has good ordinary reduction at p [10, Conjecture 5.1].

1.5. **Centres of skewfields and prime c-ideals.** In order to prove Theorem 1.3, we compute the centre $Z(Q(\Lambda_H))$ of the quotient skewfield of fractions $Q(\Lambda_H)$ of Λ_H, thereby answering a question of Venjakob [19, Question 6.4]. We also classify *all* the prime c-ideals in both Λ_H and Λ_G:

Theorem. *Under the assumptions of Theorem 1.3, the centre of $Q(\Lambda_H)$ is equal to $\mathbb{Q}_p$, the only prime c-ideal of Λ_H is $p\Lambda_H$ and every prime c-ideal of Λ_G apart from $p\Lambda_G$ is generated by a distinguished polynomial in Λ_Z.*

The proof is given in §4.4 and §4.8 and follows from a very recent result of F. Wei, J. J. Zhang and the author [6, Theorem 7.3], which essentially states that $\Omega_H = \Lambda_H/p\Lambda_H$ has no nontrivial prime c-ideals.

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2. Preliminaries

2.1. **Fractional ideals.** Let R be a Noetherian domain. It is well-known that R has a skewfield of fractions Q. Recall that a right R-submodule I of Q is said to be a *fractional right ideal* if I is non-zero and $I \subseteq uR$ for some non-zero $u \in Q$. Fractional left ideals are defined similarly. If I is a fractional right ideal, then

$$I^{-1} := \{q \in Q : qI \subseteq R\}$$

is a fractional left ideal and there is a similar definition of I^{-1} for fractional left ideals I. There is a natural isomorphism between I^{-1} and $I^* = \operatorname{Hom}_R(I, R)$. The following elementary result will be very useful for computing I^{-1}.

Lemma. *Let R be a Noetherian domain and let I be a non-zero right ideal of R. Then $I^{-1}/R \cong \operatorname{Ext}^1_R(R/I, R)$ as left R-modules.*

Recall that the fractional right ideal I is said to be *reflexive* if

$$I = \overline{I} := (I^{-1})^{-1}.$$

Equivalently, the canonical map $I \to I^{**}$ is an isomorphism. Let I be a fractional right ideal which is also a fractional left ideal. We say that I is a *fractional c-ideal* if it is reflexive on both sides, and we say that I is a *c-ideal* if I is contained in R. Of particular interest are the c-ideals which happen to be prime ideals of R.

2.2. **Two results about prime c-ideals.** Let R be a Noetherian domain and let $x \in R$ is a non-zero central element such that R/xR is a domain. The following facts are well-known, but we include the proofs for the convenience of the reader because we will use these results repeatedly.

Lemma. *Suppose that I is a proper c-ideal of R containing x. Then $I = xR$.*

Proof. Since $I^{-1}I \subseteq R$ and $I \supseteq xR$, we see that $I^{-1}x \subseteq R$. Because x is central, $I^{-1}x = xI^{-1}$ is a two-sided ideal of R. Now $(xI^{-1})I \subseteq xR$ and xR is a prime ideal by assumption, so either $xI^{-1} \subseteq xR$ or $I \subseteq xR$. Since I is proper and reflexive, R is properly contained in I^{-1} and therefore $I = xR$. $\square$

Proposition. *Suppose that x lies in the Jacobson radical of R and that every non-zero two-sided ideal J of $T := R/xR$ satisfies $J^{-1} = T$. Then xR is the only prime c-ideal of R.*

Proof. Let I be a prime c-ideal of R. By the above lemma, it will be enough to show that $x \in I$. Let $M = R/I$ and let J be the image of I in T, so that $M/xM \cong T/J$ as right T-modules. The short exact sequence of right R-modules

$$0 \longrightarrow M \overset{x}{\longrightarrow} M \longrightarrow T/J \longrightarrow 0$$

gives rise to the following long exact sequence of left R-modules:

$$\cdots \longrightarrow \operatorname{Ext}^1_R(M, R) \overset{x}{\longrightarrow} \operatorname{Ext}^1_R(M, R) \longrightarrow \operatorname{Ext}^2_R(T/J, R) \longrightarrow \cdots.$$

Suppose for a contradiction that $x \notin I$. Because I is prime and x is central, the module M is x-torsion-free. By the well-known Rees Lemma (see, for example, [1, Lemma 1.1]), we have

$$\operatorname{Ext}^2_R(T/J, R) \cong \operatorname{Ext}^1_T(T/J, T).$$

Because J is a non-zero two-sided ideal of T, we must have $J^{-1} = T$ by assumption. Since T is a domain, $\mathrm{Ext}_T^1(T/J, T) \cong J^{-1}/T = 0$ by Lemma 2.1, so the finitely generated left R-module $E := \mathrm{Ext}_R^1(M, R)$ satisfies $E = xE$. Because x lies in the Jacobson radical of R, Nakayama's Lemma implies that $E = 0$. Hence $I^{-1} = R$, again by Lemma 2.1. But I is reflexive by assumption, so $I = (I^{-1})^{-1} = R$, contradicting the assumption that I is a prime ideal. $\square$

2.3. **Maximal orders.** Recall that R is said to be a *maximal order* in Q if whenever S is a subring of Q containing R such that $uSv \subseteq R$ for some non-zero $u, v \in Q$, we must have $S = R$. Asano [7] showed that the set $G(R)$ of fractional c-ideals for a maximal order R is an abelian group when equipped with the product

$$I \cdot J := \overline{IJ}$$

and inverse $I \mapsto I^{-1}$ defined in §2.1. He also proved the following

Theorem. $G(R)$ *is a free abelian group on the set of prime c-ideals of R.*

Proof. See, for example, [14, II.1.8 and II.2.6]. $\square$

3. A USEFUL ISOMORPHISM

3.1. **Venjakob's localisation of Iwasawa algebras.** If G is a compact p-adic analytic group, we write Λ_G and Ω_G for the completed group algebras of G with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$, respectively. See [5] for more information about these algebras.

Let H be a closed normal subgroup of G. Recall from [4, §1.5] the two-sided augmentation ideal

$$w_{H,G} := (H - 1)\Omega_G = \ker(\Omega_G \to \Omega_{G/H}).$$

Let N be an open pro-p subgroup of H which is normal in G, and let $P_H = \sqrt{w_{N,G}}$ be the prime radical of $w_{N,G}$. This is a semiprime ideal of Ω_G which only depends on H, see [4, Lemma 3.2]. It was shown in [4, Theorem D] that P_H is a *localisable ideal* in Ω_G, meaning that

$$\{s \in \Omega_G : s \text{ is regular modulo } P_H\}$$

is a two-sided Ore set in Ω_G. We will denote the Ore localisation of Ω_G at this Ore set by $\Omega_{G,H}$. It follows fairly directly from the above definitions that $\Omega_{G,H} = \Omega_{G,L}$ whenever L is a normal subgroup of G which is open in H.

3.2. **Lemma.** *Suppose that $G = Z \times H$ where H is finite and $Z \cong \mathbb{Z}_p^n$ for some $n \geqslant 0$. Then $\Omega_{G,H} \cong K[H]$, where K is the field of fractions of Ω_Z.*

Proof. Because H is finite, we see that $\Omega_{G,H} = \Omega_{G,1}$. Now Z is an open normal uniform subgroup of G, and in this situation [4, Lemma 5.1] shows that $\Omega_{G,H}$ is a crossed product of $\Omega_{Z,1}$ with the finite group G/Z:

$$\Omega_{G,H} = \Omega_{G,1} = \Omega_{Z,1} * (G/Z).$$

Clearly $\Omega_{Z,1}$ is the field of fractions K of Ω_Z, and $\Omega_G = \Omega_Z[H]$ because $G = Z \times H$. Now inspecting the proof of [4, Lemma 5.1] shows that the crossed product on the right hand side is in fact just the ordinary group ring $K[H]$. $\square$

3.3. **A completion of $\Omega_{G,H}$.** Now let H be an arbitrary compact p-adic analytic group and let $Z \cong \mathbb{Z}_p^n$. Letting K denote the field of fractions of Ω_Z, we can form the completed group algebra of H with coefficients in K:

$$K[[H]] := \varprojlim K[H/U],$$

where U runs over all the open normal subgroups of U. We can now state the main result of this section.

Theorem. *Let $G = Z \times H$, let N be an open normal uniform subgroup of H and let $\mathfrak{m}$ be the augmentation ideal $w_{N,G}\Omega_{G,H}$ of $\Omega_{G,H}$. Then*

(a) the $\mathfrak{m}$-adic filtration on $\Omega_{G,H}$ is Zariskian, and
(b) the completion of $\Omega_{G,H}$ with respect to this filtration is isomorphic to $K[[H]]$.

Proof. (a) We refer the reader to the book [13] for information about Zariskian filtrations. Because N is normal in G and has finite index in H, $\Omega_{G,H} = \Omega_{G,N}$ as above. Now ZN is an open normal uniform subgroup of G and $G/ZN \cong H/N$, so [4, Lemma 5.1] shows that

$$\Omega_{G,H} = \Omega_{G,N} \cong \Omega_{ZN,N} * (H/N).$$

Let $J = (N-1)\Omega_{ZN,N}$, so that $\mathfrak{m} = J \cdot \Omega_{G,N}$ and $\mathfrak{m}^k = J^k \cdot \Omega_{G,N}$ for all k. We can therefore compute the Rees ring of $\Omega_{G,N}$ with respect to the $\mathfrak{m}$-adic filtration as follows:

$$\widetilde{\Omega_{G,N}} = \bigoplus_k \mathfrak{m}^{-k} t^k = \left(\bigoplus_k J^{-k} t^k\right) \otimes_{\Omega_{ZN,N}} \Omega_{G,N} \cong \widetilde{\Omega_{ZN,N}} * (H/N).$$

It was shown in [4, Proposition 5.3] that the Rees ring $\widetilde{\Omega_{ZN,N}}$ is Noetherian, so $\widetilde{\Omega_{G,N}}$ is also Noetherian since H/N is finite.

Now $P_H\Omega_{G,H}$ is the Jacobson radical of $\Omega_{G,H}$ by [15, Theorem 3.2.3(a)], and $\mathfrak{m} = w_{N,G}\Omega_{G,H} \subseteq P_H\Omega_{G,H}$.

(b) Let R denote the completion of $\Omega_{G,N}$ at the $\mathfrak{m}$-adic filtration. By [12, Lemma 7.1], the families of ideals $\{w_{N,N}^k : k \geqslant 0\}$ and $\{w_{U,N} : U \triangleleft_o N\}$ are cofinal in Ω_N. It follows that the families of ideals

$$\{\mathfrak{m}^k : k \geqslant 0\} \quad \text{and} \quad \{w_{U,G}\Omega_{G,N} : U \triangleleft_o H, U \leqslant N\}$$

are cofinal in $\Omega_{G,N}$, so R is isomorphic to the completion of $\Omega_{G,N}$ at the second family.

Let U be an open normal subgroup of H which is contained in N. Then $G/U \cong Z \times (H/U)$ and H/U is finite. Now $\Omega_G/w_{U,G} \cong \Omega_{G/U}$, so by Lemma 3.2 we have

$$\Omega_{G,N}/w_{U,G}\Omega_{G,N} \cong \Omega_{G/U,N/U} = \Omega_{G/U,H/U} \cong K[H/U].$$

Hence $R \cong \varprojlim K[H/U] \cong K[[H]]$. $\qquad\qquad\qquad\square$

3.4. **An application.** Let I_H denote the inverse image of P_H in Λ_G. By [4, Theorem G], I_H is a localisable ideal in Λ_G and we denote the corresponding localisation by $\Lambda_{G,H}$. The element p lies in the Jacobson radical of $\Lambda_{G,H}$ and

$$\Lambda_{G,H}/p\Lambda_{G,H} \cong \Omega_{G,H}.$$

Proposition. *Let $G = Z \times H$ where $Z \cong \mathbb{Z}_p^n$ for some n and H is a torsion-free compact p-adic analytic group. Let K be the field of fractions of Ω_Z and suppose that $J^{-1} = K[[H]]$ for every non-zero two-sided ideal J of $K[[H]]$. Then the only prime c-ideal of $\Lambda_{G,H}$ is $p\Lambda_{G,H}$.*

Proof. Because H is torsion-free, $K[[H]]$ is a Noetherian domain by the proof of [4, Theorem C]. In view of Proposition 2.2, it will be enough to show that every non-zero two-sided ideal J of $T := \Omega_{G,H}$ satisfies $J^{-1} = T$.

Fix an open normal uniform subgroup N of H as in Theorem 3.3. By that result, the $\mathfrak{m}$-adic filtration on T is Zariskian and the completion of T with respect to this filtration is isomorphic to $K[[H]]$. Hence $K[[H]]$ is a faithfully flat (right and left) T-module by [13, Chapter II, §2.2.1, Theorem 2], and therefore

$$K[[H]] \otimes_T \mathrm{Ext}^1_T(T/J, T) \cong \mathrm{Ext}^1_{K[[H]]}((T/J) \otimes_T K[[H]], K[[H]])$$

by [6, Proposition 1.2]. Now $(T/J) \otimes_T K[[H]] \cong K[[H]]/J \cdot K[[H]]$ and $J \cdot K[[H]]$ is just the completion $\widehat{J}$ of J with respect the $\mathfrak{m}$-adic topology on T by [13, Chapter II, §1.1.2, Theorem 10(5)]. As such, $J \cdot K[[H]]$ is a *two-sided* ideal of $K[[H]]$ and therefore $(J \cdot K[[H]])^{-1} = K[[H]]$ by our assumption.

It follows from Lemma 2.1 that $K[[H]] \otimes_T \mathrm{Ext}^1_T(T/J, T) = 0$, but $K[[H]]$ is a faithfully flat left T-module, so $\mathrm{Ext}^1_T(T/J, T) = 0$. Therefore $J^{-1} = T$. $\square$

4. The centre of the skewfield of fractions of Λ_H

4.1. **Iwasawa algebras are maximal orders.**

Lemma. *Let G be a torsion-free compact p-adic analytic group. Then Λ_G is a maximal order.*

Proof. It is well known that Λ_G is a domain. Moreover, Ω_G is also a domain [5, Theorem 4.3] and is a maximal order in its skewfield of fractions by [5, Corollary 4.7]. If we filter Λ_G p-adically, we get a Zariskian filtration whose associated graded ring is isomorphic to the polynomial ring $\Omega_G[t]$. This is a maximal order by [14, V.2.5]. Because $\Omega_G[t]$ is a domain we may apply [14, X.2.1] and deduce that Λ_G is also a maximal order. $\square$

If one is happy to pass to open subgroups of G then one can assume that the group G is p-valued in the sense of Lazard. In this case, Λ_G is already known to be a maximal order [11, Proposition 7.2].

4.2. **Two-sided ideals in $K[[H]]$.** From now on, we will assume that our group G satisfies the conditions of Theorem 1.3:

- $G = H \times Z$ for some closed subgroups H and Z,
- $Z \cong \mathbb{Z}_p$,
- H is torsion-free pro-p,
- $\mathcal{L}(H)$ is split semisimple over $\mathbb{Q}_p$,
- $p \geqslant 5$.

We should remark that the hypotheses made on H are only there to ensure that the conclusion of following theorem holds. In particular, the hypothesis that $p \geqslant 5$ can probably be dispensed with altogether.

Theorem. *For any field K of characteristic p, if I is a non-zero two-sided ideal of the completed group algebra $K[[H]]$, then $I^{-1} = K[[H]]$.*

Proof. It is shown in [6, Theorem 7.3] that $K[[H]]$ has no non-trivial reflexive two-sided ideals. As I is non-zero, it follows that $\overline{I} = K[[H]]$. Hence $I^{-1} = \overline{I}^{-1} = K[[H]]$. □

4.3. Here is our first application of Theorem 4.2.

Theorem. *Let I be a prime c-ideal of Λ_G. Then either $I = p\Lambda_G$ or Λ_G/I is finitely generated over Λ_H.*

Proof. Let S be the Ore set in Λ_G that gives rise to the localisation $\Lambda_{G,H}$:

$$S = \{x \in \Lambda_G : x \text{ is regular modulo } I_H\}.$$

Suppose first that $I \cap S = \emptyset$. By [16, Proposition 2.1.16(vii)], $I_S := I \cdot \Lambda_{G,H}$ is a prime ideal of $\Lambda_{G,H}$ and $I = I_S \cap \Lambda_G$. Because I_S is reflexive by [6, Proposition 1.2(a)], it follows from Proposition 3.4 and Theorem 4.2 that $I_S = p\Lambda_{G,H}$. Hence $p \in I_S \cap \Lambda_G = I$, and therefore $I = p\Lambda_G$ by Lemma 2.2.

On the other hand, if $I \cap S \neq \emptyset$, then the right Λ_G-module Λ_G/I is S-torsion and is therefore finitely generated over Λ_H by [10, Propositions 2.3 and 2.6]. □

4.4. Centre of $Q(\Lambda_H)$. To obtain our second application of Theorem 4.2, we combine it with Proposition 2.2:

Proposition. *Let I be a prime c-ideal of Λ_H. Then $I = p\Lambda_H$.*

We can now compute the centre of the skewfield of fractions Q of Λ_H.

Theorem. *The centre of Q is equal to $\mathbb{Q}_p$.*

Proof. Recall that Λ_H is a maximal order in Q by Lemma 4.1. Let q be a non-zero central element of Q. Then $q\Lambda_H$ is a fractional c-ideal and hence can be written as a product of prime c-ideals of Λ_H (and their inverses) by Theorem 2.3. Using the proposition, we see that any such product must be equal to $p^n\Lambda_H$ for some $n \in \mathbb{Z}$. Hence $q = p^n r$ for some $r \in \Lambda_H$ and this r must clearly be a central element of Λ_H. Our assumptions on H force the centre of H to be trivial, so $Z(\Lambda_H) = \mathbb{Z}_p$ by [2, Corollary A]. Hence $q \in \mathbb{Q}_p$. □

Fix a topological generator $g \in Z$ such that $Z = \overline{\langle g \rangle}$ and write $z = g - 1 \in \Lambda_Z$. Motivated in part by Theorem 4.3, we now move on to study the prime ideals in $\Lambda_G \cong \Lambda_H[[z]]$ that are finitely generated over Λ_H.

4.5. Proposition. *Let I be a prime ideal in Λ_G such that Λ_G/I is finitely generated over Λ_H. Then either $I \cap \Lambda_H \neq 0$ or $I \cap \Lambda_Z \neq 0$.*

Proof. Let $A_n := \bigoplus_{i=0}^{n} \Lambda_H z^i$ — this is a finitely generated Λ_H-submodule of Λ_G and we have an increasing chain

$$\Lambda_H = A_0 \subset A_1 \subset A_2 \subset \cdots .$$

The image of this chain inside Λ_G/I must stabilize since Λ_G/I is a Noetherian Λ_H-module by assumption, and it follows that $I \cap A_n \neq 0$ for some n.

Let n be minimal subject to having $I \cap A_n \neq 0$: note that if $n = 0$ then $I \cap \Lambda_H \neq 0$ and we're done. So assume that $n \neq 0$; we can then find some non-zero polynomial

$$a = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in I.$$

Note that $a_n \neq 0$ by the minimality of n. Writing Q for the skewfield of fractions of Λ_H, we can form the polynomial ring $Q[z]$, which can be viewed as an Ore

localisation of the polynomial ring $\Lambda_H[z]$. Note that $Q(I \cap \Lambda_H[z])$ is a two-sided ideal of $Q[z]$ by [16, Proposition 2.1.16], and

$$a_n^{-1}a = z^n + (a_n^{-1}a_{n-1})z^{n-1} + \cdots + (a_n^{-1}a_1)z + a_n^{-1}a_0 \in Q(I \cap \Lambda_H[z]).$$

Let $u \in Q$ and consider the commutator $[u, a_n^{-1}a] = ua_n^{-1}a - a_n^{-1}au$. This lies in $Q(I \cap \Lambda_H[z])$ and has degree strictly smaller than n. Clearing denominators we obtain an element of $I \cap A_{n-1}$ which is zero by the minimality of n. Therefore $[u, a_n^{-1}a] = 0$ for any $u \in Q$ meaning that each $a_n^{-1}a_i$ is central in Q. By Theorem 4.4, $a_n^{-1}a_i \in \mathbb{Q}_p$ for all $i < n$, so we can find some $m \geqslant 0$ such that $p^m a_n^{-1}a \in \Lambda_Z$.

Now $p^m a = a_n \cdot (p^m a_n^{-1}a) \in I$ and I is prime. Because Λ_Z is central in Λ_G and $a_n \notin I$ (otherwise $I \cap \Lambda_H \neq 0$), we see that $I \cap \Lambda_Z \neq 0$. $\qquad\square$

In the case when the ideal I is actually a prime c-ideal, we will be able to refine the above. First, some preparatory results.

4.6. Lemma. Let I be a prime ideal in Λ_G. Then
(a) $I \cap \Lambda_Z$ is a prime ideal in Λ_Z,
(b) $J := I \cap \Lambda_H$ is a prime ideal in Λ_H,

Proof. Let us remark that in the general setting of noncommutative rings, the inverse image of a prime ideal under a ring homomorphism need not be prime (or even semiprime!). This is clearly illustrated by the inclusion

$$\begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subset M_2(k)$$

for any field k. However in our setting there is enough commutativity to enable the obvious proofs of parts (a) and (b) to work. For part (b) let A, B be ideals of Λ_H such that $AB \subseteq J$; then $B\Lambda_G = B[[z]] = \Lambda_G B$ so $(A\Lambda_G)(B\Lambda_G) \subseteq J\Lambda_G \subseteq I$ and either $A\Lambda_G \subseteq I$ or $B\Lambda_G \subseteq I$ because I is prime. Hence $A \subseteq J$ or $B \subseteq J$ whence J is prime. The proof of part (a) is similar but easier. $\qquad\square$

4.7. Distinguished polynomials. Recall [20, §7.1] that a polynomial

$$f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{Z}_p[z]$$

is said to be *distinguished* if $a_n = 1$ and $a_i \in p\mathbb{Z}_p$ for all $i < n$.

Lemma. *Let $J \neq 0$ be an ideal in Λ_Z such that Λ_Z/J is p-torsion-free. Then*
(a) *J is generated by a distinguished polynomial f,*
(b) *if J is a prime ideal of Λ_Z, then $J\Lambda_G = f\Lambda_G$ is a prime ideal of Λ_G.*

Proof. (a) This is a well-known consequence of the p-adic Weierstrass preparation theorem [20, Theorem 7.3].

(b) Let K denote the field of fractions of the domain Λ_Z/J. K is a finite field extension of $\mathbb{Q}_p$ because Λ_Z/J is finitely generated over $\mathbb{Z}_p$. By part (a), the image of z in K satisfies a monic polynomial with coeffcients in $\mathbb{Z}_p$, so Λ_Z/J is contained in the ring of integers $\mathcal{O}$ of K.

Now Λ_G is isomorphic to the completed group ring of H with coefficients in Λ_Z:

$$\Lambda_G \cong \varprojlim_{N \triangleleft_o H} \Lambda_Z[H/N]$$

and for every open normal subgroup N of H we have an exact sequence

$$0 \to \Lambda_Z[H/N] \xrightarrow{f} \Lambda_Z[H/N] \to \mathcal{O}[H/N].$$

Since inverse limits are left exact, we obtain an exact sequence

$$0 \to \Lambda_G \xrightarrow{f} \Lambda_G \to \mathcal{O}[[H]]$$

where $\mathcal{O}[[H]]$ denotes the completed group ring of H with coefficients in $\mathcal{O}$. The latter is a domain by [3, Proposition 2.5(iii)] so $\Lambda_G/f\Lambda_G$ is also a domain and $f\Lambda_G$ is a prime ideal. $\qquad\square$

We are now ready to give proofs of our main results.

4.8. Theorem. Let I be a prime c-ideal of Λ_G. Then $I = f\Lambda_G$ where $f = p$ or $f \in \mathbb{Z}_p[z]$ is a distinguished polynomial which is irreducible in Λ_Z.

Proof. Suppose that $J := I \cap \Lambda_H$ is non-zero. J is prime by Lemma 4.6(b) and reflexive by [6, Proposition 1.2(b)], so $J = p\Lambda_H$ by Proposition 4.4. Hence $p\Lambda_G \subseteq I$. Because $\Lambda_G/p\Lambda_G = \Omega_G$ is a domain by [5, Theorem 4.3], $I = p\Lambda_G$ by Lemma 2.2.

Now suppose that $J = 0$; in particular, $p \notin I$. By Theorem 4.3, Λ_G/I is finitely generated over Λ_H, so Proposition 4.5 implies that $I \cap \Lambda_Z \neq 0$. Because $p \notin I$, Lemma 4.7(a) implies that $I \cap \Lambda_Z = f\Lambda_Z$ for some distinguished polynomial $f \in \mathbb{Z}_p[z]$. The polynomial f must be irreducible in Λ_Z by Lemma 4.6(a). Now $f\Lambda_G$ is a prime ideal in Λ_G by Lemma 4.7(b), so $I = f\Lambda_G$ by Lemma 2.2. $\qquad\square$

Corollary. Let I be a reflexive two-sided ideal of Λ_G. Then $I = f\Lambda_G$ for some $f \in \mathbb{Z}_p[z]$.

Proof. Apply Proposition 4.1, Theorem 2.3 and Theorem 4.8. $\qquad\square$

4.9. Proof of Theorem 1.3. Let N be the Λ_Z-torsion submodule of M. If $N \neq 0$, then $q(N)$ is a non-zero subobject of $q(M)$ because the largest pseudo-null submodule M_o of M is zero by assumption. Moreover, $\mathrm{Ann}(N) \neq 0$ since Λ_Z is central in Λ_G, so $q(M)$ is not completely faithful.

Conversely, suppose that $N = 0$. Recall from Lemma 4.1 that Λ_G is a maximal order. By [8, Proposition 4.2.2], $q(M)$ decomposes uniquely in the quotient category $\mathcal{M}/\mathcal{C}$ as a direct sum $q(M) = A \oplus B$ of a completely faithful object A and a *locally bounded* object B. Since B is a subobject of $q(M)$ and $M_o = 0$, we can find a submodule X of M such that $q(X) \cong B$ by general properties of quotient categories.

Now, since $X_o \leqslant M_o = 0$, the annihilator ideal $I := \mathrm{Ann}(B)$ equals $\mathrm{Ann}(X)$ by [17, Lemma 2.5]. Because X is locally bounded, $I \neq 0$ and therefore X is a finitely generated, torsion, bounded Λ_G-module. Therefore $I = \mathrm{Ann}(q(X))$ is a non-zero reflexive two-sided ideal of Λ_G by [11, Lemma 5.3(i)], and hence $I = f\Lambda_G$ for some non-zero $f \in \Lambda_Z$ by Corollary 4.8. Hence $X \subseteq N = 0$, so $B = q(X) = 0$ and $q(M) = A$ is completely faithful. $\qquad\square$

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