Prime ideals in noncommutative Iwasawa algebras

By KONSTANTIN ARDAKOV
Christ's College, Cambridge CB2 3BU

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Abstract

We study the prime ideal structure of the Iwasawa algebra Λ_G of an almost simple compact p-adic Lie group G. When the Lie algebra of G contains a copy of the two-dimensional non-abelian Lie algebra, we show that the prime ideal structure of Λ_G is somewhat restricted. We also provide a potential example of a prime c-ideal of Λ_G in the case when the Lie algebra of G is $\mathfrak{sl}_2(\mathbb{Q}_p)$.

1. *Introduction*

Let p be a prime and let G be a compact p-adic Lie group. The Iwasawa algebra of G

$$\Lambda_G = \mathbb{Z}_p[[G]] := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N]$$

is of interest in number theory and arithmetic geometry, particularly when G is an open subgroup of $GL_2(\mathbb{Z}_p)$. When G is torsion free pro-p, Λ_G is also a concrete example of a complete local (noncommutative in general) Noetherian integral domain with good homological properties ([**10**]).

Recently, J. Coates, P. Schneider and R. Sujatha ([**5**]) developed a structure theory for finitely generated modules over Λ_G. One of the main features of this theory is the notion of a *prime c-ideal*, this being a nonzero prime ideal of Λ_G which is reflexive as a right (and left) Λ_G-module. This raises interest in the prime ideal structure of Λ_G in general.

If H is a closed normal subgroup of G such that G/H is torsion free pro-p, the kernel of the natural map $\Lambda_G \twoheadrightarrow \Lambda_{G/H}$ is an obvious example of a prime ideal of Λ_G. Let $\Omega_G = \mathbb{F}_p[[G]] := \varprojlim_{N \triangleleft_o G} \mathbb{F}_p[G/N]$ denote the $\mathbb{F}_p$ version of Iwasawa algebras. Since $\Omega_G \cong \Lambda_G / p\Lambda_G$ has no zero divisors when G is a uniform pro-p group ([**6**],7.26), we see that $p\Lambda_G$ is also an example of a prime ideal in this case. In fact, Ω_G has no zero divisors whenever G is torsion free pro-p, see [**3**].

Suppose G is almost simple so that any infinite closed normal subgroup of G is open. If G is torsion free pro-p, the above discussion produces two prime ideals of Ω_G, namely the zero ideal and the maximal ideal. This prompts the following question:

Question. Are these the only prime ideals of Ω_G?

In [**7**], M. Harris claimed that the two-sided annihilator of the induced module $\mathbb{Z}_p \otimes_{\Lambda_H} \Lambda_G$ for Λ_G is nonzero, whenever H is a suitably large subgroup of G. If true, this would provide an concrete example of a nontrivial two-sided ideal of Λ_G (and of Ω_G). Unfortunately, this interesting paper contains a gap and the author would like to thank J. Ellenberg for pointing this out to him.

We are unable to answer this question at present. However, we show that the two-sided ideal structure of Ω_G is somewhat restricted. More precisely, we prove

THEOREM A. *Let G be an almost simple pro-p group of finite rank. Suppose that the Lie algebra of G contains a copy of the two-dimensional nonabelian Lie algebra and that I is a two-sided ideal of Ω_G. Then*

$$\mathcal{K}(\Omega_G/I) \neq 1.$$

Here $\mathcal{K}$ denotes the Krull (-Gabriel-Rentschler) dimension of modules for Ω_G, studied in greater detail in [**1**]. Recall that a module M is said to be *1-critical* if M is not Artinian but every proper factor of M is Artinian.

THEOREM B. *Let G be a compact p-valued p-adic Lie group with Lie algebra $\mathfrak{sl}_2(\mathbb{Q}_p)$. Let M be a finitely generated Λ_G-module such that M/pM is 1-critical and let $I = \operatorname{Ann}_{\Lambda_G}(M)$. Then if I is nonzero, I is a prime c-ideal of Λ_G.*

This applies in particular when $G = \ker(SL_2(\mathbb{Z}_p) \to SL_2(\mathbb{F}_p))$ and $M = \mathbb{Z}_p \otimes_{\Lambda_B} \Lambda_G$ is the induced module from a Borel subgroup B of G. Thus if I is nonzero in this case, it is an explicit example of a prime c-ideal in Λ_G distinct from $p\Lambda_G$.

2. *Endomorphism rings of 1-critical modules*

In this section we obtain some information about endomorphism rings of 1-critical modules for Ω_G. When G is pro-p, Ω_G is local with unique simple module isomorphic to $\mathbb{F}_p$. One can therefore think of the 1-critical modules as a substitute for the simple modules for Ω_G and as such their endomorphism rings are natural to consider.

THEOREM 2·1. *Let H be an almost simple pro-p group of finite rank. Suppose that the Lie algebra of H contains a copy of the two-dimensional nonabelian Lie algebra. Let M be a finitely generated 1-critical Ω_H-module and let $R = \operatorname{End}_{\Omega_H}(M)$. Then R is a finite field extension of $\mathbb{F}_p$. Moreover, if M is cyclic over Ω_H, $R \cong \mathbb{F}_p$.*

We begin with a very useful primality result. Recall that a two-sided ideal I of a (not necessarily commutative) ring R is said to be *prime* if whenever A, B are two-sided ideals of R strictly containing I, AB also strictly contains I.

Recall also that a ring R is called *semi-local* if R/J is Artinian, where J is the Jacobson radical of R.

PROPOSITION 2·2. *Let R be a semi-local Noetherian ring. Suppose M is a finitely generated 1-critical R-module. Then the global annihilator $I = \operatorname{Ann}_R(M)$ of M is prime.*

Proof. Let $S = \{\operatorname{Ann}_R(T) : 0 \neq T \triangleleft M\}$. Since R is right Noetherian, S has a maximal element $Y = \operatorname{Ann}_R(N)$ say, for some nonzero submodule N of M. It's clear that as $MI = 0$, $I \subseteq Y$.

We claim that Y is prime. If this is false, we can find ideals A and B of R such that $Y \subsetneq A$ and $Y \subsetneq B$ but $AB \subseteq Y$. Now $NA \neq 0$ since $Y = \operatorname{Ann}_R(N)$ and $Y \subsetneq A$; thus $\operatorname{Ann}_R(NA) \in S$. But $NAB = 0$ so $Y \subsetneq B \subseteq \operatorname{Ann}_R(NA)$, contradicting the maximality of Y.

Now, N is a nonzero submodule of the 1-critical module M, so M/N is Artinian and $MJ^n \subseteq N$ for some integer n. Hence $MJ^nY = 0$ and $J^nY \subseteq I$.

Since R is left Noetherian, Y/I is a finitely generated left R/J^n-module, which is an Artinian ring because R is semi-local (and since J is finitely generated as a left ideal). Hence Y/I has finite length as a left R-module.

Since R is right Noetherian, Y/I must be Artinian as a right R-module by Theorem 4.1.6 of [**9**], so $YJ^m \subseteq I$ for some integer m. It follows that $MYJ^m \subseteq MI = 0$ and so MY is Artinian, being a finitely generated right module over the Artinian ring R/J^m. Because M is 1-critical, $MY = 0$ and so $Y = I$ is prime. □

Note that the condition that R is semi-local cannot be removed from the statement of this result, as Theorem 4.2 of [**4**] shows.

The main step comes next.

PROPOSITION 2·3. *Let H be as in Theorem 2·1 an let $G = H \times Z$ where $Z = \overline{\langle\theta\rangle} \cong \mathbb{Z}_p$. Write $z = \theta - 1 \in \Omega_G$. Let M be a finitely generated 1-critical Ω_G-module. Then either*
(i) *$M.z = 0$, or*
(ii) *$M \cong \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$.*

Proof. Since G is pro-p of finite rank, Ω_G is Noetherian and local with unique maximal ideal J_G, say.

Note that as $\theta \in Z(G)$, z acts by Ω_G-module endomorphisms on M.

As M is 1-critical, any non-zero endomorphism of M must be an injection ([**9**], 6.2.3). Assume that $M.z \neq 0$; then z acts injectively on M.

Let $A = \mathbb{F}_p[[z]] \subseteq Z(\Omega_G)$. Now, as M is 1-critical and $M.z \neq 0$, $M/M.z$ is finite dimensional over $\mathbb{F}_p$. Because z acts injectively on M, $M.z^n/M.z^{n+1} \cong M/M.z$ for all $n \geqslant 1$, which means that the graded module of M with respect to the z-adic filtration is finitely generated over $\operatorname{gr} A \cong \mathbb{F}_p[t]$.

As M is a finitely generated module over Ω_G, M is complete with respect to the J_G-adic filtration; in particular, $\cap_{n=0}^{\infty} M.J_G^n = 0$. Hence $\cap_{n=0}^{\infty} M.z^n = 0$, so the z-adic filtration on M is separated.

Because A is complete with respect to the z-adic filtration, M is finitely generated over A, by Theorem 5.7 of Chapter I of [**8**]. Also, z acts injectively on M, so $A \hookrightarrow \operatorname{End}_A(M)$. These facts mean that $\operatorname{End}_A(M)$ is finitely generated as a module over A, a commutative subring. It follows from Corollary 13.1.13(iii) of [**9**] that $\operatorname{End}_A(M)$ is a PI ring.

Now, let $\mathfrak{b}$ be the two-dimensional nonabelian Lie algebra over $\mathbb{Q}_p$ and let $\mathcal{L}(H)$ denote the Lie algebra of H. By assumption on H, $\mathfrak{b} \hookrightarrow \mathcal{L}(H)$, so we can find a closed subgroup B of H with Lie algebra $\mathfrak{b}$. By passing to a subgroup of finite index if necessary, we can write $B = X \rtimes Y$ where $X, Y \cong \mathbb{Z}_p$. This is clearly a uniform pro-p group.

Because the centre of $\mathfrak{b}$ is trivial, so is the centre of B. It follows that $Z(\Omega_B) = \Omega_{\{1\}} = \mathbb{F}_p$, by Corollary A of [**2**]. Hence $\Omega_B.S^{-1} \cong \Omega_B$, where $S = Z(\Omega_B) - \{0\}$.

Let $P = \operatorname{Ann}_{\Omega_G}(M)$ and suppose that $P \cap \Omega_B = 0$. Then $\Omega_B \hookrightarrow \operatorname{End}_A(M)$. It follows that Ω_B is a prime PI-ring. By Posner's Theorem ([**9**], 13.6.5), $\Omega_B.S^{-1} \cong \Omega_B$ is a central simple algebra. This contradicts the fact that J_B is a non-trivial two-sided ideal of Ω_B. Hence $P \cap \Omega_B \neq 0$.

Now, by a result of Venjakob (Theorem 7.1 of [**11**]), the only nonzero prime ideals of Ω_B are $x.\Omega_B$ and J_B where $\Omega_X \cong \mathbb{F}_p[[x]]$. The nonzero two-sided ideal $P \cap \Omega_B$ of Ω_B contains a product of nonzero prime ideals as Ω_B is Noetherian. Since $x \in J_B$, we see that $x^{p^k} \in P$ for some $k \geqslant 1$ and hence $(1 + P) \cap X \neq 1$.

But $(1+P)\cap H$ is then an infinite normal subgroup of H and hence must be open in H, since H is almost simple. This forces $J_H^m \subseteq P$ for some $m \geqslant 1$.

Let $Q = \ker(\Omega_G \twoheadrightarrow A)$. Then it's easy to see that $Q = J_H.\Omega_G = \Omega_G.J_H$, so $Q^m = (J_H.\Omega_G)^m \subseteq P$. By Theorem 2·2, P is prime, so $Q \subseteq P$.

Hence $A \cong \mathbb{F}_p \otimes_{\Omega_H} \Omega_G \twoheadrightarrow \Omega_G/P \twoheadrightarrow M$; since A is itself a 1-critical Ω_G-module, we must have $A \cong M$, so (ii) holds. □

Proof of Theorem 2·1 Let G and $z \in \Omega_G$ be as in Proposition 2·3; it's easy to see that $\Omega_G \cong \Omega_H[[z]]$.

Let $\varphi \in \operatorname{Hom}_{\Omega_H}(M, MJ)$, where $J = J_H$. Then we can make M into an Ω_G-module by setting

$$m.\sum_{n=0}^{\infty} r_n z^n = \sum_{n=0}^{\infty} \varphi^n(m).r_n.$$

The right hand side of this expression makes sense because $\varphi(M) \subseteq MJ$, so $\varphi^n(M) \subseteq MJ^n$ for all n. It's clear that this defines an action of Ω_G on M which extends the action of Ω_H and such that z acts as φ. It's easy to check that M must be 1-critical as an Ω_G-module.

By Proposition 2·3, either $M.z = 0$ (so $\varphi = 0$), or $M \cong \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$, in which case $M.J_H = 0$. As M is finitely generated over Ω_H, the latter case forces M to be finite dimensional over $\mathbb{F}_p$, contradicting the 1-criticality of M. Hence $\varphi = 0$, and therefore $\operatorname{Hom}_{\Omega_H}(M, MJ) = 0$.

Now, as MJ is a characteristic Ω_H-submodule of M, we have the exact sequence

$$0 \to \operatorname{Hom}_{\Omega_H}(M, MJ) \to \operatorname{Hom}_{\Omega_H}(M, M) \to \operatorname{Hom}_{\Omega_H}(M/MJ, M/MJ)$$

which shows that R embeds into $\operatorname{End}_{\mathbb{F}_p}(M/MJ)$, which is finite dimensional over $\mathbb{F}_p$. Note that if M is cyclic, $R \cong \mathbb{F}_p$ because $M/MJ \cong \mathbb{F}_p$.

Now, as M is critical, any nonzero endomorphism of M is an injection. This means that R is a domain, and is hence a finite division ring. By Wedderburn's Theorem, R is a finite field extension of $\mathbb{F}_p$. □

3. *Main results*

We now have enough information to give a proof of Theorem A.

Proof of Theorem A Let $R = \Omega_G$ and let I be a two-sided ideal of R with $\mathcal{K}(R/I) = 1$. Pick a 1-critical quotient $M = R/L$ of R/I for some right ideal L of R and let $P = \operatorname{Ann}_R(M)$. Clearly, $I \subseteq P$.

Let $\bar{}$ denote the natural projection of R onto R/P. Note that $\bar{R}$ is a prime ring, by Proposition 2·2. Since $\bar{R} \twoheadrightarrow M$ and $\mathcal{K}(M) = 1$, $\bar{R}$ is infinite dimensional over $\mathbb{F}_p$.

Let Q be the quotient ring of $\bar{R}$; by Goldie's Theorem ([**9**], 2.3.6) we know that Q is simple Artinian, because $\bar{R}$ is prime Noetherian. Say $Q \cong M_n(D)$ for some division ring D and integer $n \geqslant 1$. Here $D = \operatorname{End}_Q(V)$ where V is the unique simple Q-module. In what follows, we use the fact that Q is a flat $\bar{R}$-module.

Suppose $\bar{L}Q < \bar{R}Q$, i.e. $MQ \neq 0$. Since M is finitely generated over $\bar{R}$, MQ is finitely generated over Q and is hence isomorphic to a direct sum of k copies of V for some integer $k > 0$. Hence $\operatorname{End}_Q(MQ) \cong M_k(D)$.

Let N be the torsion submodule of M with respect to $\mathcal{C}_{\bar{R}}(0)$ (the set of regular elements of $\bar{R}$), so that M/N is torsion free with respect to $\mathcal{C}_{\bar{R}}(0)$ and $(M/N)Q \cong MQ$. Now if $N \neq 0$, M/N is finite dimensional over $\mathbb{F}_p$ (because M is 1-critical) and so $\mathrm{End}_R(M/N)$ must also be finite dimensional over $\mathbb{F}_p$; if $N = 0$, $\mathrm{End}_R(M/N)$ is finite dimensional over $\mathbb{F}_p$ by Theorem 2·1.

As M/N is finitely generated over $\bar{R}$, torsion free with respect to $\mathcal{C}_{\bar{R}}(0)$ and as $\bar{R}$ is prime Goldie, M/N is torsionless, by Theorem 3.4.7 of [**9**]. Hence by Theorem 3.4.6 of [**9**], $\mathrm{End}_R(M/N)$ is a right order in $\mathrm{End}_Q((M/N)Q) = \mathrm{End}_Q(MQ) \cong M_k(D)$, so $M_k(D)$ and hence $Q \cong M_n(D)$ must be finite dimensional over $\mathbb{F}_p$.

This is impossible as $\bar{R} \hookrightarrow Q$ with $\bar{R}$ infinite dimensional over $\mathbb{F}_p$. So in fact $MQ = 0$ and hence $\bar{L}$ must contain a regular element $\bar{x}$ of $\bar{R}$. Now we get a chain

$$\bar{R} > \bar{x}\bar{R} > \bar{x}^2\bar{R} > \ldots > \bar{0}$$

of right ideals of $\bar{R}$ with each quotient isomorphic to $\bar{R}/\bar{x}\bar{R}$, because $\bar{x}$ is regular in $\bar{R}$. Hence

$$\mathcal{K}(R/I) \geqslant \mathcal{K}(\bar{R}) \geqslant \mathcal{K}(\bar{R}/\bar{x}\bar{R}) + 1 \geqslant \mathcal{K}(R/L) + 1 = 2,$$

a contradiction. □

To prove Theorem B, we first prove an analogue of Proposition 2·2 for Λ_G. First, an elementary Lemma.

Lemma 3·1. *Let G be a compact p-adic Lie group. Let M be a finitely generated p-torsion free Λ_G-module and let $I = \mathrm{Ann}_{\Lambda_G}(M)$. Then:*
(i) *Λ_G/I is p-torsion free,*
(ii) *If M has finite rank over $\mathbb{Z}_p$, so does Λ_G/I.*

Proof. (i) If $px \in I$, $Mx\Lambda_G.p = 0$. Because M is p-torsion free, this forces $Mx = 0$ and hence $x \in I$. (ii) Say $M \cong \mathbb{Z}_p^d$ for some integer d. The action of Λ_G on M gives rise to a $\mathbb{Z}_p$-module homomorphism $\Lambda_G \to \mathrm{End}_{\mathbb{Z}_p}(M) \cong M_d(\mathbb{Z}_p)$ with kernel precisely I. Since $M_d(\mathbb{Z}_p)$ also has finite rank over $\mathbb{Z}_p$, the result follows. □

Proposition 3·2. *Let G be a pro-p group of finite rank and let M be a finitely generated Λ_G-module such that M/pM is 1-critical. Then the global annihilator $I = \mathrm{Ann}_{\Lambda_G}(M)$ is prime.*

Proof. Since G is pro-p of finite rank, Λ_G is Noetherian and local with unique maximal ideal J, say.

Let $Y = \mathrm{Ann}_{\Lambda_G}(N)$ be a maximal element of the set $\{\mathrm{Ann}_{\Lambda_G}(L) : 0 \neq L \leqslant M\}$ for some nonzero submodule N of M. The same proof as the one used in Proposition 2·2 shows that Y is a prime ideal containing I.

Let T/N be the p-torsion part of M/N, so $p^nT \subseteq N$ for some integer n and M/T is p-torsion free. If $0 \neq x \in N$, we can write $x = p^k y$ with $y \notin pM$, because the J-adic filtration on M is separated and $p \in J$. Then $p^k(y+T) = 0$ so $y \in T - pM$ as M/T is p-torsion free, which means that pM is strictly contained in $pM + T$.

Since M/pM is 1-critical, this forces $M/(pM+T) \cong (M/T)/p(M/T)$ to be finite dimensional over $\mathbb{F}_p$, and hence M/T must have finite rank over $\mathbb{Z}_p$.

Let $U = \mathrm{Ann}_{\Lambda_G}(M/T)$. By Lemma 3·1 (ii), Λ_G/U has finite rank over $\mathbb{Z}_p$. Now $(M/T).U = 0$ so $MU \subseteq T$. Hence $MUp^n \subseteq Tp^n \subseteq N$ and so $MUp^nY \subseteq NY = 0$. This means that $Up^nY = p^nUY \subseteq I$. As Λ_G/I is p-torsion free by Lemma 3·1(i), $UY \subseteq I$.

Because Λ_G is left Noetherian, Y/I is hence a finitely generated left Λ_G/U-module and hence must have finite rank over $\mathbb{Z}_p$. Moreover, Λ_G/I is p-torsion free and hence so is Y/I.

Let $V = \mathrm{Ann}_{\Lambda_G}(Y/I)$, where Y/I is viewed as a *right* Λ_G-module. We have $(Y/I)V = 0$ so $YV \subseteq I$, i.e. $MYV \subseteq MI = 0$. Hence MY is a finitely generated module over the ring Λ_G/V (which has finite rank over $\mathbb{Z}_p$ by Lemma 3·1(ii)), and so MY also has finite rank over $\mathbb{Z}_p$.

Now, the natural map $MY \to M \to M/pM$ has kernel $MY \cap pM \supseteq pMY$ and MY/pMY is finite dimensional over $\mathbb{F}_p$ because MY is free of finite rank over $\mathbb{Z}_p$. Since M/pM is 1-critical, this map must be 0 and so $MY \subseteq pM$. An obvious induction argument shows that $MY \subseteq p^n M$ for all integers $n \geqslant 0$, whence $MY = 0$.

Hence $Y \subseteq I$, but $I \subseteq Y$ so $Y = I$ is prime. □

We will assume for the remainder of this paper that G is a compact p-valued p-adic Lie group.

The following basic Lemma is fundamental to everything that follows.

Lemma 3·3. *Equip Λ_G with the filtration constructed in Proposition 7.2 of* [**5**]. *Let M be a finitely generated Λ_G-module equipped with the filtration deduced from Λ_G, and let* gr M *denote the associated graded module. Then*

$$\mathcal{K}(M) \leqslant \mathcal{K}(\mathrm{gr}\ M).$$

Proof. The filtration on Λ_G was observed to be Zariskian in Lemma 4.1 of [**5**], so the result follows from Theorem 7.1.3 of Chapter I of [**8**]. □

We recall that a finitely generated Λ_G module M is said to be *pseudo-null* if

$$\mathrm{Hom}_{\Lambda_G}(L, S/\Lambda_G) = 0$$

for any submodule L of M; here S denotes the skew-field of fractions of Λ_G. It is shown in Theorem 4.10 (3) of Chapter III of [**8**] that M is pseudo-null if and only if the graded module gr M for the commutative ring gr Λ_G satisfies

$$\mathcal{K}(\mathrm{gr}\ M) \leqslant \mathcal{K}(\mathrm{gr}\ \Lambda_G) - 2.$$

Since gr Λ_G is a polynomial ring in $\dim G + 1$ variables over $\mathbb{F}_p$, we have

Lemma 3·4. *Let M be a finitely generated Λ_G-module such that $\mathcal{K}(M) \geqslant \dim G$. Then M is not pseudo-null.*

The following Lemma is due to Peter Schneider and we are grateful to him for allowing us to include it here.

Lemma 3·5. *Suppose that M is a finitely generated bounded Λ_G-module which is not pseudo-null and whose annihilator ideal P is prime; then P is a prime c-ideal.*

Proof. Let M_0 denote the maximal pseudo-null submodule of M, so that $M \neq M_0$. By Lemma 4.3(i) of [**5**], the annihilator ideal I of M/M_0 is a nonzero proper reflexive ideal in Λ_G.

As such it is a nontrivial product of prime c-ideals $P_1, \ldots, P_r$ under the product on the set of fractional c-ideals defined in section 4 of [**5**].

Since obviously $P \subseteq I$ we have $P \subseteq P_1 \cdot \ldots \cdot P_r \subseteq P_1 \cap \ldots \cap P_r$. But the P_i are of height one, hence $P = P_i = I$ is a prime c-ideal. □

We can now give a proof of Theorem B.

Proof of Theorem B Note first that $\mathfrak{sl}_2(\mathbb{Q}_p)$ is a simple Lie algebra containing a copy of the two-dimensional nonabelian Lie algebra $\mathfrak{b}$, so Theorem A applies.

Any compact p-valued p-adic Lie group is automatically pro-p of finite rank, so I is prime by Proposition 3·2. It remains to show that I is a c-ideal.

By assumption, $I \neq 0$, so Λ_G/I is bounded. By Lemma 3·5 applied to Λ_G/I, we see that it's sufficient to prove that Λ_G/I is not pseudo-null.

Now, Λ_G/I is p-torsion free by Lemma 3·1 (ii), so $p+I$ is a regular element of the ring Λ_G/I. It follows that

$$\mathcal{K}(\Lambda_G/I) > \mathcal{K}(\Lambda_G/(I + p\Lambda_G)) = \mathcal{K}(\Omega_G/Q),$$

where Q is the image of I in Ω_G.

Since M is finitely generated, $(\Omega_G/Q)^k$ surjects onto M/pM for some integer k. Since $\mathcal{K}(M/pM) = 1$, Q is a two-sided ideal of Ω_G such that $\mathcal{K}(\Omega_G/Q) \geqslant 1$.

By Theorem A, $\mathcal{K}(\Omega_G/Q) \geqslant 2$, so $\mathcal{K}(\Lambda_G/I) \geqslant 3 = \dim G$. By Lemma 3·4, Λ_G/I is not pseudo-null, as required. □

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REFERENCES

[**1**] K. Ardakov, *Krull dimension of Iwasawa Algebras*, to appear in J. Algebra.
[**2**] K. Ardakov, *The centre of completed group algebras of pro-p groups*, submitted.
[**3**] K.Ardakov, K.A. Brown, paper in preparation.
[**4**] K.A.Brown, R.B. Warfield, Jr, *The Influence of Ideal Structure on Representation Theory*, J. Algebra **116** (1988), 294-315.
[**5**] J. Coates, P. Schneider, R. Sujatha, *Modules over Iwasawa algebras*, Journal of the Inst. of Math. Jussieu (2003), **2**(1), 73-108.
[**6**] J.D. Dixon, M.P.F. Du Sautoy, A. Mann, D. Segal, *Analytic pro-p groups*, 2nd edition, CUP (1999).
[**7**] M. Harris, *The annihilators of p-Adic Induced Modules*, J.Algebra **67**, 68-71 (1980).
[**8**] L. Huishi and F. van Oystaeyen, *Zariskian filtrations*, K-monographs in mathematics, vol.2 (1996).
[**9**] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian rings*, AMS Graduate Studies in Mathematics, vol. 30.
[**10**] A. Neumann, *Completed group algebras without zero divisors*, Arch. Math. (Basel) **51** (1988), no. 6, 496-499.
[**11**] O. Venjakob, *A noncommutative Weierstrass Preparation Theorem and applications to Iwasawa Theory*, J. Reine Angew. Math. **559** (2003), 153-191.