Linear equations

$3x + 2y = 1$

$3x + \sqrt{2}y - z = 37$

$ax + by + cz = d$

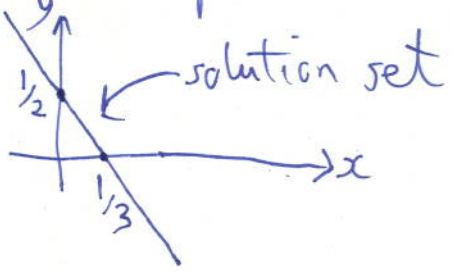
Definition: A linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where the x_i are variables
a_i and b are constants

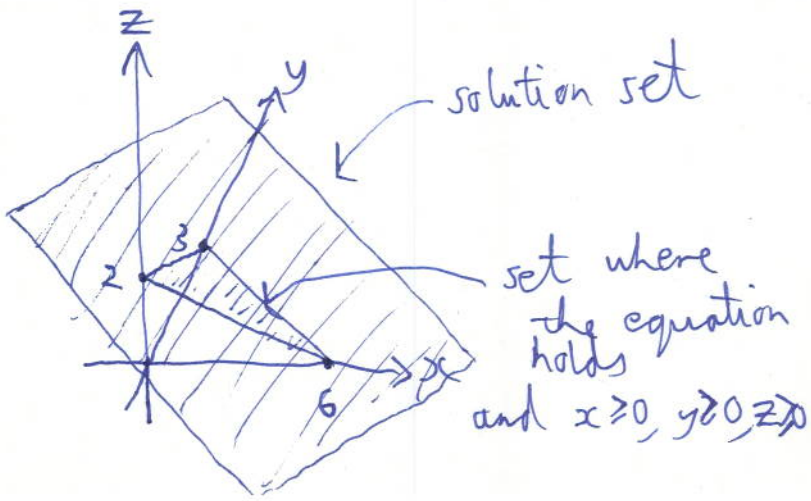
Geometrically: The solutions (x,y) of $ax + by = c$ form a straight line in the plane

e.g. $3x + 2y = 1$



In 3 variables (x,y,z), the solutions of a linear equation $ax + by + cz = d$ form a plane

e.g. $x + 2y + 3z = 6$



Systems of linear equations

Given a system of linear equations

e.g. $ax + by + cz = d$

$ex + fy + gz = h$

want to solve the system

i.e. describe the solutions

i.e. determine which (x, y, z) satisfy all the equations

Geometrically: we want to find the intersection of the two planes

In this case, we have three cases.

(i) The intersection is a plane

e.g. $x + 2y + 3z = 6$

$2x + 4y + 6z = 12$

(equations define same plane)

(ii) The intersection is empty

i.e. the system has no solutions

e.g. $x + 2y + 3z = 6$

$2x + 4y + 6z = 11$

(iii) The intersection is a line

e.g. $x = 1$

$y = 2$

The solutions are $(1, 2, t)$ where t is any number

Solving systems algebraically

Example 1:

$$2x + 3y = 4$$
$$4x - y = 15$$

For any c, if we replace the second equation with the second equation plus c times the first equation, getting

$$2x + 3y = 4$$
$$(4x - y) + c(2x + 3y) = 15 + 4c$$

then the new system has the same solutions as the original system.

Why? If (x, y) is a solution to the original system, then it clearly satisfies the first equation of the new system, and it is easy to see that it satisfies the second.

Moreover, if (x, y) satisfies the second system, it satisfies the original system:

$$\begin{aligned} 4x - y &= (4x - y) + c(2x + 3y) - c(2x + 3y) \\ &= 15 + 4c - c(2x + 3y) \\ &= 15 + 4c - 4c \\ &= 15. \end{aligned}$$

Now let's take $c := -2$, get:

$$2x + 3y = 4$$
$$-7y = 7$$

Now multiply through second equation by $-1/7$

$$2x + 3y = 4$$
$$y = -1$$

(still haven't changed the solution set)

Now add (-3) times second to first

$$2x = 7$$
$$y = -1$$

so

$$\boxed{\begin{aligned} x &= 7/2 \\ y &= -1 \end{aligned}}$$