Ch. 3; Q. 9:

There are 10 people in a room, with integral ages between 1 and 60 inclusive. Prove that there are two disjoint groups of people with equal total age. Can 10 be reduced?

There are $2^{10} = 1024$ subsets of people. The total age of the room is at most $600 < 1024$, so by the pigeonhole principle some two distinct subsets have the same total age. These subsets might not be disjoint - but if we remove the intersection from both, the resulting subsets will be disjoint and still have equal total ages.

This argument won't go through if 10 is reduced, since $2^9 = 512 < 600$.

(That doesn't really answer the question, though... although this simple pigeonhole argument doesn't show it, the result might still hold for 9. I don't know whether it does or not!)

Ch. 3; Q. 11:

A student has 37 days to revise for an exam. She needs no more than 60 hours. She wants to study for at least an hour each day, and for an integral number of hours each day. Show that there will necessarily be a succession of days during which she will have studied exactly 13 hours.

Let $r_i :=$ the number of hours she will have spent revising at the end of the ith day. So $1 \leq r_1 < r_2 < ... < r_{37} \leq 60$.

Consider the 74 numbers

$$r_1, ..., r_{37}, r_1 + 13, ..., r_{37} + 13.$$

Each takes one of the 73 values between 1 and 60+13=73 inclusive, so by the pigeonhole principle two must be equal. By assumption, we have no equalities $r_i = r_j$ with $i \neq j$, and so no equalities $r_i + 13 = r_j + 13$, so we must have $r_i = r_j + 13$ for some i, j. But this means she studies for exactly 13 hours on the days after the jth up to and including the ith.

Ch. 3; Q. 15:

Prove that, for any integers $a_1, ..., a_{n+1}$, there exist $i \neq j$ such that n divides $a_i - a_j$.

Let $r_i \in \{0, ..., n-1\}$ be the remainder on dividing a_i by n.

Then by the pigeonhole principle, some $r_i = r_j$ with $i \neq j$.

But then the remainder on dividing $a_i - a_j$ by n is $r_i - r_j = 0$.

Ch. 3; Q. 23:

Prove $r(3, 4) \leq 10$

We showed in lectures that

$$r(m, n) \leq r(m-1, n) + r(m, n-1)$$

and

$$r(3, 3) = 5$$

so it suffices to see that

$$r(2, 4) \leq 5, \text{ i.e. } K_5 \to K_2, K_4.$$

So consider a red-blue-coloured K_5, and suppose it contains no red K_2.

This means that it has no red edges at all, so every edge is blue.

So certainly it contains a blue K_5.

(Note that actually this argument shows $r(2, n) = n = r(n, 2)$.)

Ch. 5; Q. 6:
What is the coefficient of x^5y^{13} in the binomial expansion of $(3x - 2y)^{18}$? What is the coefficient of x^8y^9?
By the binomial theorem, the coefficient of $(3x)^5(-2y)^{13}$ is $\binom{18}{5}$.
So the coefficient of x^5y^{13} is
$\binom{18}{5}3^5(-2)^{13} = -17055940608$
Meanwhile, $8 + 9 \neq 18$, so no terms of the form cx^8y^9 arise in the binomial expansion, so in other words the coefficient of x^8y^9 is 0.

Ch. 5; Q. 10:
Prove by combinatorial reasoning the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$.
Let S be a set of size n.
Consider colouring $k - 1$ of the elements turquoise, and one of the elements magenta. We will count in two ways the number of such colourings.
One way to perform the colouring is to first pick k of the elements ($\binom{n}{k}$ choices), then pick one of the k to colour magenta (k choices), and colour the remaining $k - 1$ turquoise.
So there are $k\binom{n}{k}$ colourings.
Another way to do it is to first pick one of the n to colour magenta (n choices), then pick $k-1$ of the remaining $n-1$ elements to colour turquoise ($\binom{n-1}{k-1}$ choices).
So there are $n\binom{n-1}{k-1}$ colourings.
So $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Ch. 5; Q. 37:
Use the multinomial theorem to show that, for positive integers n and t,
$t^n = \sum \binom{n}{n_1 n_2 ... n_t}$,
where the sum extends over the nonnegative integral solutions to $n_1 + ... + n_t = n$.

$$t^n = (1 + 1 + ... + 1)^n$$

$$= \sum \binom{n}{n_1 n_2 ... n_t} 1^{n_1} * ... * 1^{n_t}$$

$$= \sum \binom{n}{n_1 n_2 ... n_t},$$

where the sums are as in the question.

Ch. 5; Q. 47:
Use Newton's binomial theorem to approximate $10^{\frac{1}{3}}$.
First, we should isolate a factor of the form $(1 + z)^\alpha$ with $|z| < 1$, so we can apply Newton's binomial theorem.
We can do this by noting that $8 = 2^3$, so
$10^{1/3} = (8 + 2)^{1/3} = (8(1 + \frac{1}{4}))^{1/3} = 2(1 + \frac{1}{4})^{1/3}$.
Now by Newton's binomial theorem,
$(1 + \frac{1}{4})^{1/3} = \sum_{i=0}^{\infty} \binom{\frac{1}{3}}{i} (\frac{1}{4})^i$

A few terms should be enough to give us a good approximation.
Using the definition, we calculate

$$\binom{\frac{1}{3}}{0} = 1$$

$$\binom{\frac{1}{3}}{1} = \frac{1}{3}$$

$$\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3} * \frac{-2}{3}}{2!} = \frac{-2}{18}$$

$$\binom{\frac{1}{3}}{3} = \frac{\frac{1}{3} * \frac{-2}{3} * \frac{-5}{3}}{3!} = \frac{10}{162},$$

So $(1 + \frac{1}{4})^{1/3} \approx 1 + \frac{1}{3} * \frac{1}{4} - \frac{2}{18} * \frac{1}{4}^2 + \frac{10}{162} * \frac{1}{4}^3 = \frac{5585}{5184}$,
so $10^{\frac{1}{3}} = 2 * (1 + \frac{1}{4})^{1/3} \approx 2 * \frac{5585}{5184} = \frac{5585}{2592}$.
Indeed, $\frac{5585}{2592}^3 = \frac{174208576625}{17414258688} = 10.004$, so this is a pretty good approximation!