Ch. 4; Q. 38:
Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be posets. Define a relation (the product order) T on $X_1 \times X_2$ by $(x_1, x_2)T(x_1', x_2')$ iff $x_1 \leq_1 x_1'$ and $x_2 \leq_2 x_2'$.
Prove that $(X_1 \times X_2, T)$ is a poset.
We prove this more generally for a product of m posets $(X_i, \leq_i)$.
Write $\mathbf{x}$ for $x_1, ..., x_m \in \Pi X_i$, and $\leq$ for the product order.
We have to verify that $\leq$ satisfies reflexivity, transitivity, and anti-symmetry.
Reflexivity: given $\mathbf{x}$, $x_i \leq x_i$ for all i by reflexivity of $\leq_i$, so $\mathbf{x} \leq \mathbf{x}$.
Transitivity: if $\mathbf{x} \leq \mathbf{y} \leq \mathbf{z}$ then, for all i, $x_i \leq y_i \leq z_i$,
so $x_i \leq z_i$ by transitivity of $\leq_i$. So $\mathbf{x} \leq \mathbf{z}$.
Antisymmetry: if $\mathbf{x} \leq \mathbf{y} \leq \mathbf{x}$ then, for all i, $x_i \leq y_i \leq x_i$,
so $x_i = y_i$ by antisymmetry of $\leq_i$. So $\mathbf{x} = \mathbf{y}$.

Ch. 4; Q. 39:
Suppose $|X \models n$. By identifying the subsets of X with n-tuples of 0s and 1s, prove that the partially ordered set $(\mathbb{P}(X), \subseteq)$ can be identified with the n-fold direct product $(\{0,1\}, \leq)^n$.
Enumerate X as $X = \{x_1, ..., x_n\}$. Identify $A \subseteq X$ with $\mathbf{e}^A \in \{0,1\}^n$ where

$e_i^A = 1$ if $x_i \in A$,
$e_i^A = 0$ otherwise.

Write $\leq$ for the product order on $\{0,1\}^n$, as defined in question 38.
Then we must check that $A \subseteq B$ iff $e^A \leq e^B$.
But indeed,

$e^A \leq e^B$ iff $\forall i. e_i^A \leq e_i^B$
iff $\forall i.$ if $x_i \in A$ then $x_i \in B$
iff $A \subseteq B$.

Ch. 5; Q 48:
Use the theorem that in a finite poset, the maximal size of a chain is the minimal size of an antichain partition, to show that if m and n are positive integers, then a poset of $mn + 1$ elements has a chain of size $m + 1$ or an antichain of size $n + 1$.
Let $(X, \leq)$ be a poset with $|X \models mn+1$, and suppose X has no chain of size $m + 1$. Then the maximal size of a chain is $\leq m$, and so, by the theorem, X has a partition into $\leq m$ antichains. Then by the packed pigeonhole principle, since $|X| > nm$, some antichain in this partition must have size greater than n, so at least $n + 1$. Then any subset of this antichain of size exactly $n + 1$ will also be an antichain.

Ch. 5; Q 49:
Use the result of the previous exercise to show that a sequence of $mn + 1$ real numbers either contains a (non-strictly) increasing subsequence of $m+1$ numbers or a decreasing subsequence of $n + 1$ numbers.
Let $a_1, ..., a_{mn+1}$ be a sequence of real numbers. Define an ordering on $\{1, ..., mn + 1\}$ by

$i \leq' j$ iff $i \leq j$ and $a_i \leq a_j$,

where $\leq$ is the usual ordering.
This defines a partial order, so by the previous exercise, there is either a chain of length $m + 1$ or an antichain of size $n + 1$, defining respectively a

(non-strictly) increasing or decreasing subsequence.

Ch. 6; Q 3:

Find the number of integers between 1 an 10,000 which are neither squares nor cubes.

Let S and C be the set of squares and cubes in this range.

$\sqrt{10000} = 100$, so $|S \models 100$.

$10000^{1/3} = 21.5$, so $|C \models 21$.

If $x \in S \cap C$, the exponent of each prime in the prime decomposition of x must be divisible by both 2 and 3, and hence by 6.

So $S \cap C$ is the set of sixth powers in the range $[1, 10000]$,

so since $(10000)^{1/6} = 4.6$, $|S \cap C| = 4$.

By inclusion-exclusion,

$$|S \cup C| = |S| + |C \vdash |S \cap C| = 100 + 21 - 4 = 117.$$

By subtraction, the answer is $10000 - 117 = 9883$.

Ch. 6; Q 6:

A bakery sells chocolate, cinnamon, and plain doughnuts, and now has 6 chocolate, 6 cinnamon, and 3 plain. A box contains 12 doughnuts. How many different options are there for a box?

We want to find the number of 12-combinations of a multiset with multiplicities 6,6, and 3.

By the formula derived in lectures, this is

$$\sum_{I \subseteq \{1,\ldots,t\}} (-1)^{|I|} \binom{r-(\sum_{i \in I}(c_i+1))+t-1}{t-1}.$$

where $r = 12, t = 3, c_1 = c_2 = 6, c_3 = 3$. We calculate:

$$\begin{aligned}
&\sum_{I \subseteq \{1,\ldots,3\}} (-1)^{|I|} \binom{12-(\sum_{i \in I}(c_i+1))+2}{2} \\
&= \binom{12+2}{2} - \binom{12-7+2}{2} - \binom{12-7+2}{2} - \binom{12-4+2}{2} \\
&\quad + \binom{12-(7+7)+2}{2} + \binom{12-(7+4)+2}{2} + \binom{12-(7+4)+2}{2} \\
&\quad - \binom{12-(7+7+4)+2}{2} \\
&= 91 - 21 - 21 - 45 + 0 + 3 + 3 - 0 \\
&= 10
\end{aligned}$$

Ch. 6; Q 9:

Determine the number of integral solutions to

$$x_1 + x_2 + x_3 + x_4 = 20$$

subject to

$$1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6.$$

Set $y_1 := x_1 - 1, y_2 := x_2, y_3 := x_3 - 4, y_4 := x_4 - 2$.

Then solutions as in the question correspond to solutions in non-negative integers to $y_1 + y_2 + y_3 + y_4 = 20 - 1 - 4 - 2 = 13$ with $y_1 \leq 5, y_2 \leq 7, y_3 \leq 4, y_4 \leq 4$.

Applying the formula as in the previous question, we can see that most terms will be zero (a term is zero when $r < \sum_{i \in I}(c_i + 1)$).

We get non-zero terms for $I = \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$.

In this way, we get the answer 96.

Ch. 6; Q 12:

How many permutations of $\{1, 2, ..., 8\}$ have exactly four integers in their natural positions?

If we first pick 4 of the 8 numbers to fix, and then **derange** the other 4, we get such a permutation. Each such permutation arises once in such a way. So by the number of such permutations is

$$\binom{8}{4} * |D_4| = \binom{8}{4} 4! \sum_{i=0}^{4} \frac{(-1)^i}{i!}$$
$$= 70 * (24 * (1 - 1 + 1/2 - 1/6 + 1/24)) = 70 * 9$$
$$= 630$$

Ch. 6; Q 13:

Note: in a previous version of these solutions, I confused Q12 and Q13, so gave the above answer to Q12 (which was not set) as the answer to Q13. Apologies for any resulting confusion.

How many permutations of $\{1, 2, ..., 9\}$ have at least one odd integer in its natural position?

We can do this directly using inclusion-exclusion, as in the proof of the formula for $|D_n|$. The number of permutations which fix exactly k of the numbers is $(9-k)!$; we want to count the number which fix one of 5 numbers, so by inclusion exclusion this is

$$\sum_{k=1}^{5} (-1)^{k+1} \binom{5}{k} (9-k)!$$
$$= 5 * 8! + 10 * 7! + 10 * 6! + 5 * 5! + 1 * 4!$$
$$= 198120$$

Ch. 6; Q 15:

At a party, seven gentlemen check their hats, then hand them over. How many ways can they be returned so that

(a) no gentleman receives his own hat?

(b) at least one receives his own hat?

(c) at least two receive their own hats?

(a) These are just the derangements; $|D_7| = 1854$.

(b) These are the permutations which are not derangements; $7! - |D_7| = 3186$.

(c) These are the permutations which are neither derangements nor permutations in which exactly 1 hat is fixed. As in the previous question, the latter can be counted by first choosing a lucky(?) gentleman to get his own hat back, then considering a derangement of the remaining 6. So the answer is $7! - (|D_7| + 7 * |D_6|) = 1331$.