Ch.7 Q.11:

The Lucas numbers $l_0, l_1, \ldots,$ *are defined by*

$l_{n+2} = l_n + l_{n+1}; l_0 = 2; l_1 = 1.$

Prove that

(a) $l_n = f_{n-1} + f_{n+1}$ *for* $n \geq 1$*;*

(b) $\sum_{i=0}^{n} l_i^2 = l_n l_{n+1} + 2$ *for* $n \geq 0$.

(a) $l_2 = 3$, so this relation holds for $n = 1$ and $n = 2$.

Let $d_n := l_n - f_{n-1} - f_{n+1}$.
Then $d_1 = 0 = d_2$, and the d_n also satisfy the Fibonacci recurrence relation,
$d_{n+2} = d_n + d_{n+1}$ for all $n \geq 0$,
by linearity.

So $d_n = 0$ for all $n \geq 0$.

Hence $l_n = f_{n-1} + f_{n+1}$ for all $n \geq 0$.

(b) This holds for $n = 0$.

Suppose inductively that $\sum_{i=0}^{n} l_i^2 = l_n l_{n+1} + 2$.
Then

$$\begin{aligned}\sum_{i=0}^{n+1} l_i^2 &= (\sum_{i=0}^{n} l_i^2) + l_{n+1}^2 \\ &= l_n l_{n+1} + 2 + l_{n+1}^2 \\ &= l_{n+1}(l_n + l_{n+1}) + 2 \\ &= l_{n+1} l_{n+2} + 2.\end{aligned}$$

So the formula holds for all $n \geq 0$.

Ch.7 Q.14 a,b:

Determine the generating function for the number h_n *of* n*-combinations from the multiset* $\{\infty * e_1, \infty * e_2, \infty * e_3, \infty * e_4\}$ *with the following added restrictions:*

(a) Each e_i *occurs an odd number of times;*

(b) Each e_i *occurs a multiple of 3 number of times.*

(a)

$$\begin{aligned}g(x) &= (\sum_{i=0}^{\infty} x^{2i+1})^4 \\ &= (x \sum_{i=0}^{\infty} x^{2i})^4 \\ &= (\tfrac{x}{1-x^2})^4\end{aligned}$$

(b)

$$\begin{aligned}g(x) &= (\sum_{i=0}^{\infty} x^{3i})^4 \\ &= (1 - x^3)^{-4}\end{aligned}$$

Ch.7 Q.18:

Find the generating function for the number h_n *of non-negative integral solutions of* $2e_1 + 5e_2 + e_3 + 7e_4 = n$.

h_n is the number of n-combinations from the multiset $\{\infty * a_1, \infty * a_2, \infty * a_3, \infty * a_4\}$ where a_1 occurs an even number of times, a_2 a multiple of 5 times, and a_4 a multiple of 7 times.

So, as in the previous question, the generating function is
$$(1-x^2)^{-1}(1-x^5)^{-1}(1-x)^{-1}(1-x^7)^{-1}.$$

Ch.7 Q.29:

Derive, without using exponential generating functions, the formula
$$h_n = \frac{5^n+2*3^n+1}{4}$$
for the number h_n of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times.

This one is tricky! Here's the neatest solution I could find.

Call a number "healthy" if 1 and 3 occur an even number of times. Call a number "sick" if both 1 and 3 occur an odd number of times. Call it "unwell" if it is neither healthy nor sick.

Let h_n, s_n, and u_n be the number of healthy, sick, and unwell, respectively, n-odd-digit numbers.

Consider the operation of changing the last digit in a number which is either a 1 or a 3 to a 3 if it was a 1 and to a 1 if it was a 3. This operation changes a healthy number into a sick number, a sick into a healthy, and an unwell into another unwell. It is defined except on those numbers with no 1s or 3s, which must be healthy.

Since the operation is its own inverse, it provides a bijection between the sick numbers and those healthy numbers which contain a 1 or a 3.

The number of n-odd-digit numbers which don't contain a 1 or a 3, being the n-digit numbers where each digit is one of the 3 numbers 5,7, or 9, is 3^n, so
$$h_n - 3^n = s_n.$$

Now define the complement of a decimal digit d to be $10-d$, and consider the operation of changing the last digit in a number which is not a 5 into its complement.
This operation changes a healthy or sick number into an unwell number, and an unwell number into a healthy or sick number.
It is defined on all odd-digit numbers except those of the form 55...5.

Since again this operation is its own inverse, we obtain
$$u_n = h_n + s_n - 1.$$

Now since the total number of n-odd-digit numbers is 5^n,
$$u_n + h_n + s_n = 5^n.$$

So now we can solve these 3 linear equations to find h_n:
$$h_n + s_n - 1 = u_n = 5^n - h_n - s_n$$
$$\rightarrow 2h_n + 2s_n = 5^n + 1$$
$$\rightarrow 2h_n + 2(h_n - 3^n) = 5^n + 1$$
$$\rightarrow h_n = \frac{5^n+2*3^n+1}{4}.$$

Ch.8 Q.6:

Let $h_n = 2n^2 - n + 3$. *Find the difference table, and find a formula for* $\sum_{k=0}^{n} h_n$.

```
3   4   9   18  31  48 ...
  1   5   9   13  17 ...
    4   4   4   4  ...
      0   0   0  ...
```

As we saw in lectures, we obtain from the initial diagonal the formula

$$h_n = 3 + n + 4\binom{n}{2}$$

and so

$$\sum_{k=0}^{n} h_n = 3(n+1) + \binom{n+1}{2} + 4\binom{n+1}{3}$$

which we could expand out as a polynomial if we so desired,

$$\begin{aligned}\sum_{k=0}^{n} h_n &= 3(n+1) + \tfrac{n(n+1)}{2} + 4\tfrac{(n-1)n(n+1)}{6} \\ &= 3 + (3 + \tfrac{1}{2} - \tfrac{2}{3})n + \tfrac{1}{2}n^2 + \tfrac{2}{3}n^3 \\ &= 3 + \tfrac{17}{6}n + \tfrac{1}{2}n^2 + \tfrac{2}{3}n^3.\end{aligned}$$

Ch.8 Q.7:

Suppose $h_n = f(n)$ *with* f *a cubic polynomial. Suppose the first four entries of the 0th row of its difference table (i.e.* h_0, h_1, h_2, h_3*) are 1,-1,3,10. Determine* h_n *and a formula for* $\sum_{k=0}^{n} h_n$.

We can draw the start of the difference triangle:

```
1   -1   3   10
  -2   4   7
     6   3
       -3
```

.

Since the polynomial is of degree 3, we know that all successive rows are zero.

So just as in the previous question, we have

$$h_n = 1 - 2n + 6\binom{n}{2} - 3\binom{n}{3}$$

and

$$\sum_{k=0}^{n} h_n = (n+1) - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4}.$$

Were we so inclined, we could expand these out as polynomials, hence determining the polynomial f and giving a polynomial expression for the partial sums.