1. (a) A chain is a subset C such that for any c and c' in C,
$c \leq c'$ or $c' \leq c$.
(Equivalently: a subset $\{c_1, ..., c_k\}$ with $c_1 < c_2 < ... < c_k$.)

(b) An antichain partition is a partition of X into antichains,
i.e. $X = A_1 \cup ... \cup A_n$ with $A_i \cap A_j = \emptyset$ for $i \neq j$,
where for each A_i, distinct elements are incomparable.

(c) An antichain can meet a chain in at most one point,
so an antichain partition of X partitions any chain into subsets of size at most 1,
so no chain can be longer than the size of an antichain partition.

So it suffices to show that if C is a chain of maximal length,
there exists an antichain partition $A_1, ..., A_n$ with $n = |C|$.

Define A_1 to be the set of minimal elements of X, and recursively define A_{k+1} to be the set of minimal elements of $X \setminus \bigcup_{i=1}^{k}$.
Clearly the A_i are disjoint.
The set of minimal elements of any finite poset $(Y; \leq)$ is an antichain, since if y, y' are minimal and comparable, then say $y < y'$, contradicting minimality of y'.
So each A_i is an antichain.
Enumerate $C = \{c_1, ..., c_n\}$ with $c_1 < c_2 < ... < c_n$.
Then clearly $c_i \in A_i$.
Suppose $X \neq \bigcup_{i=1}^{n} A_i$, say $x \in X \setminus \bigcup_{i=1}^{n} A_i$.
Then $x > c_i$, so $C' := C \cup \{x\}$ is a chain,
contradicting maximality of C.
So the A_i form an antichain partition as required.

2. (a) Let D_n be the set of numbers between 1 and 1000 divisible by n.
Then $|D_n| = \lfloor \frac{1000}{n} \rfloor$.
Note $D_n \cap D_m = D_{\mathrm{lcm}(n,m)}$.
So by inclusion-exclusion,

$$|D_5 \cup D_7 \cup D_{11}| = |D_5| + |D_7| + |D_{11}|$$
$$- (|D_{35}| + |D_{55}| + |D_{77}|) + |D_{385}|$$
$$= 200 + 142 + 90 - (28 + 18 + 12) - 2$$
$$= 376$$

(b) Let D'_n be the set of numbers between 1000 and 2000 divisible by n.
Then $|D'_n| = \lfloor \frac{2000}{n} \rfloor - \lfloor \frac{999}{n} \rfloor$.
So by inclusion-exclusion, and using least common multiples,

$$|D'_4 \cup D'_6 \cup D'_{10}| = |D'_4| + |D'_6| + |D'_{10}|$$
$$- (|D'_{12}| + |D'_{20}| + |D'_{30}|) + |D'_{60}|$$
$$= 369$$

3. (a)

$$g(x) = (x + x^3 + x^5 + ...)^3$$
$$= x^3(1 + x^2 + x^4 + ...)^3$$
$$= \frac{x^3}{(1-x^2)^3}$$

(b) Using the formula $\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$,

$$g(x) = \frac{x^3}{(1-x^2)^3}$$

$= x^3(\sum_{n=0}^{\infty} \binom{n+3-1}{3-1} (x^2)^n)$
$= \sum_{n=0}^{\infty} \binom{n+3-1}{3-1} x^{2n+3}$

So $h_n = \binom{k+2}{2}$ if $n = 2k + 3$ for some $k \geq 0$,
and $h_n = 0$ otherwise.