Solutions to Midterm 2

1. (a) [diagram omitted]

   (b) $A := \{8, 12, 9, 5, 7, 11\}$ is an antichain of size 6, and the chains
   $C_1 := \{1, 2, 4, 8\}$
   $C_2 := \{3, 6, 12\}$
   $C_3 := \{5, 10\}$
   $C_4 := \{7\}$
   $C_5 := \{9\}$
   $C_6 := \{11\}$
   form a chain partition of X into 6 chains.

   Since the maximal size of an antichain is equal to the minimal size of a chain partition, there can be no larger antichain and no smaller chain partition.

2. Let F_i be the set of permutations of $(1, ..., n)$ which fix i, i.e. permutations $(a_1, ..., a_n)$ with $a_i = i$.

   Then $\bigcap_{i\in I} F_i$ is the set of permutations which fix all $i \in I$, which can be identified with the set of permutations of $\{1, ..., n\} \setminus I$.

   So $|\bigcap_{i\in I} F_i| = (n - |I|)!$.

   So by the inclusion-exclusion formula,

$$\begin{aligned}
|\textstyle\bigcup_i F_i| &= \textstyle\sum_{\emptyset\neq I\subseteq\{1,...,n\}}(-1)^{|I|} \mid \bigcap_{i\in I} F_i| \\
&= \textstyle\sum_{\emptyset\neq I\subseteq\{1,...,n\}}(-1)^{|I|}(n-|I|)! \\
&= \textstyle\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)! \\
&= \textstyle\sum_{k=1}^{n}(-1)^{k+1}\frac{n!}{(n-k)!k!}(n-k)! \\
&= \textstyle\sum_{k=1}^{n}(-1)^{k+1}\frac{n!}{k!} \\
&= n!\textstyle\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}
\end{aligned}$$

   Now the derangements are the permutations which are not in $\bigcup_i F_i$, so

$$\begin{aligned}
|D_n| &= n! - |\textstyle\bigcup_i F_i| \\
&= n! - n!\textstyle\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} \\
&= n!\textstyle\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}
\end{aligned}$$

3. (a) Taking just those terms in the power series for e^x corresponding to the allowed numbers of the letters,

$$\begin{aligned}
g^{(e)}(x) &= e^x(1+x)(1+\tfrac{x^2}{2!}+\tfrac{x^4}{4!}+...) \\
&= e^x(1+x)\tfrac{e^x+e^{-x}}{2} \\
&= \tfrac{e^{2x}+e^0+xe^{2x}+xe^0}{2} \\
&= \tfrac{1+x+e^{2x}+xe^{2x}}{2}
\end{aligned}$$

   (b) By the definition of $g^{(e)}(x)$, $\frac{h_n}{n!}$ is the coefficient of x^n in the power series expansion of $\frac{1+x+e^{2x}+xe^{2x}}{2}$.
   Expanding e^{2x}, we see that for $n > 1$ this coefficient is

$$\begin{aligned}
&\tfrac{1}{2}(\tfrac{2^n}{n!}+\tfrac{2^{n-1}}{(n-1)!}) \\
&= \tfrac{1}{2}\tfrac{2^n+n2^{n-1}}{n!}
\end{aligned}$$

So $h_n = 2^{n-1} + n2^{n-2}$, and in particular $h_{13} = 30720$.