2015 Sample

Math 3U03

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This examination is 3 hours in length. Attempt all questions. The total number of available points is 54. Marks are indicated next to each question. Write your answers in the booklets provided. You must show your work to get full credit. **Good Luck.**

1. (6 points)

   (a) Compute

   $$\sum_{k=0}^{n} \binom{n}{k}$$

   (b) How many subsets of a set of size n are there of even size?

2. (6 points) Show that if you pick any 11 integers between 1 and 100 then there must be 2 whose difference is less than 10.

3. Use combinatorial reasoning to prove that

   $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

4. (6 points)

   (a) State Dilworth's theorem.

   (b) Consider the partial order P given by the integers 1 through 24 and the order given by divisibility (i.e. $m \leq n$ iff m divides n). What is the size of a maximal antichain in P?

   (c) Produce a chain partition of P with the number of parts you found in part (b).

5. (6 points) Solve

   $$h_n = 5h_{n-1} - 6h_{n-2}$$

   subject to $h_0 = -2$ and $h_1 = -1$.

6. (6 points) Determine the number of n digit numbers made up of the digits $1, 2$ and 3 and where 1 and 2 appear an odd number of times (Hint: Use exponential generating functions).

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7. (6 points) Suppose that one has a sequence h_n for all $n \geq 0$ such that

$$h_0 = 1, \Delta h_0 = 1, \Delta^2 h_0 = 2, \Delta^3 h_0 = 3 \text{ and } \Delta^k h_0 = 0 \text{ for all } k \geq 4$$

Give a general formula for h_n.

8. (6 points) Show that $p_n > p_{n-1}$ for $n \geq 2$ where p_m denotes the number of different partitions of the integer m.

9. (6 points) Suppose that $A_1, \ldots, A_n$ are non-empty subsets of a set X which has a system of distinct representatives (SDR). Choose $x \in A_1$. Show that you can always obtain an SDR for this collection of subsets which contains x but show by example that x need not represent A_1.

8. In the game of Impartial Domineering, players take turns to place 1×2 dominoes on a square-tiled board, placing the dominoes horizontally or vertically at their choosing. By considering it as a sum, show that the second player wins when the game is played on the board below, and say what more the second player should make if the first player starts by placing a horizontal domino in the indicated position.