FINITE PRESENTATION OF FIBRE PRODUCTS OF METABELIAN GROUPS

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Abstract. We show that if Γ is a finitely presented metabelian group, then the "untwisted" fibre product or pull-back P associated to any short exact sequence $1 \to N \to \Gamma \to Q \to 1$ is again finitely presented. In contrast, if N and Q are abelian, then the analogous "twisted" fibre-product is not finitely presented unless Γ is polycyclic. Also a number of examples are constructed, including a non-finitely presented metabelian group P with $H_2(P, \mathbb{Z})$ finitely generated.

Associated to each pair of short exact sequences of groups $1 \to N_i \to \Gamma_i \xrightarrow{p_i} Q \to 1$, $i = 1, 2$, one has the fibre product $P = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid p_1(\gamma_1) = p_2(\gamma_2)\}$. In this article we shall be concerned entirely with the case $\Gamma_1 = \Gamma_2 = \Gamma$, $N_1 = N_2 = N$, and for the most part we shall focus on the case where $p_1 = p_2$, where we shall call the fibre product *untwisted*.

We are interested in the question of when such fibre products are finitely presented. There have recently been several significant results in this direction. Firstly, if Γ is free and both Q and N are infinite, then P is *never* finitely presented [2, 3]. Likewise if Γ is a surface group [6]. On the other hand, if $p_1 = p_2$ and one knows that N is finitely generated, Γ is finitely presented and Q is of type F_3, then P is *always* finitely presented — this is the 1-2-3 Theorem of [7].

Intrigued by this contrast in behaviour, we shall look at a class of groups Γ that are far from free and which do not fall within the scope of the 1-2-3 Theorem, namely short exact sequences of metabelian groups.

In this context one also finds a contrast in the behaviour of fibre products, even within examples that, superficially, appear very similar

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Example 1. Fix an integer $q > 1$, and let $\Gamma = \langle\, x, t \mid t^{-1}xt = x^q \,\rangle$, let $N = \langle x \rangle^G$, and let $Q = \langle t \rangle$ be infinite cyclic. Define $p_1 = p_2 = p$ to be the homomorphism from Γ to Q with $p(x) = 1$ and $p(t) = t$. Then Γ is the Baumslag-Solitar group $B(1, q)$, and N is isomorphic to the additive group $\mathbb{Z}[1/q]$, where conjugation by t in Γ corresponds to multiplication by q in $\mathbb{Z}[1/q]$.

We claim that the pullback P is isomorphic to the group

$$\hat{P} = \langle\, x_1, x_2, t \mid t^{-1}x_1t = x_1^q,\ t^{-1}x_2t = x_2^q,\ [x_1, x_2] = 1 \,\rangle.$$

This would be clear if we could show that all conjugates $x_1^{t^i}$ of x_1 in $\hat{P}$ commute with all conjugates $x_2^{t^j}$ of x_2. But if $i \leq j$, say, then $x_2^{t^j}$ is a power of $x_2^{t^i}$, and $[x_1, x_2] = 1 \Rightarrow [x_1^{t^i}, x_2^{t^i}] = 1$, so the claim follows.

Example 2. Let Γ, N, Q and p_1 be as in Example 1, but this time define p_2 by $p_2(x) = 1$ and $p_2(y) = t^{-1}$. Then the fibre product P is not finitely presented.

This will follow from a general result proved in Section 5, but we can also prove it directly by showing that $H_2(P)$ is not finitely generated. (See, for example, Theorem 5.3 of [1] for the relevant properties of $H_2(P)$.) To do this, we shall exhibit an extension E of an infinitely generated group Z by P with $Z \subseteq Z(E) \cap [E, E]$. (*Note*: Throughout this paper, an extension of a group X by a group Y will mean a group having a normal subgroup isomorphic to X with quotient group isomorphic to Y.)

Define D to be the group with elements $\{\, (a, b, c) \mid a, b, c \in \mathbb{Z}[1/q] \,\}$ and multiplication $(a, b, c)(a', b', c') = (a + a', b + b', c + c' + a'b)$, let t be the automorphism of D mapping (a, b, c) to $(qa, b/q, c)$, and let E be the semidirect product of D by $\langle t \rangle$ using this action. Let Z be the subgroup $\{\, (0, 0, c) \mid c \in \mathbb{Z}[1/q] \,\}$ of E. Then Z is central in E and is contained in $[E, E]$ (because $[(0, c, 0), (1, 0, 0)] = (0, 0, c)$ in D), and it is easily seen that $E/Z \cong P$. Hence Z is a quotient of $H_2(P)$, which is therefore not finitely generated.

We shall see in Section 4 that Example 2 is typical behaviour for twisted fibre products in the non-polycyclic case.

Example 1 points us in the direction of the following criterion, which is the main result of this paper.

Theorem 1. *If Γ is a finitely presented metabelian group, then the untwisted fibre product associated to any short exact sequence $1 \to N \to \Gamma \to Q \to 1$ is finitely presented.*

1. Decomposition of untwisted fibre products

A key difference between twisted and untwisted fibre products is that the latter have a natural semi-direct product decomposition. The proof of the following lemma is straightforward.

Lemma 2. *Let P be the untwisted fibre product associated to a short exact sequence $1 \to N \to \Gamma \to Q \to 1$. Let $\hat{\Gamma}$ be the diagonal copy of Γ in $\Gamma \times \Gamma$ and let $N_1 = N \times \{1\}$. Then $P = N_1 \rtimes \hat{\Gamma}$.*

Remark 3. *Note that the action of $(\gamma, \gamma) \in \hat{\Gamma}$ on $(n, 1) \in N_1$ is the action of γ by conjugation on $N \subseteq \Gamma$.*

For the sake of notational convenience, we shall drop the decorations on the above subgroups and simply write $P = N \rtimes \Gamma$.

There is a further decomposition of fibre products that we shall need, the existence of which is not sensitive to the (un)twisted nature of the situation.

Lemma 4. *The fibre product associated to any pair of short exact sequences $1 \to N_i \to \Gamma_i \to Q \to 1,\ i = 1, 2$, has the form $1 \to N_1 \times N_2 \to P \to Q \to 1$.*

Proof. It is obvious that $N_1 \times N_2$ is normal in P, and that it is the kernel of the map $(\gamma_1, \gamma_2) \mapsto p_i(\gamma_i)$. □

The following general observation will also be required in the proof of our theorem:

Proposition 5. *If Γ_1, Γ_2 are finitely generated and Q is finitely presented, then the fibre product associated to any pair of short exact sequences $1 \to N_i \to \Gamma_i \xrightarrow{p_i} Q \to 1$ is finitely generated.*

Proof. Let $\rho_1 : F_1 \to \Gamma_1$, $\rho_2 : F_2 \to \Gamma_2$ be epimorphisms of finitely generated free groups onto Γ_1, Γ_2, and let R_1, R_2 be the complete inverse images under ρ_1, ρ_2 in F_1, F_2 of the kernels of the maps from Γ_1, Γ_2 to Q. Then $F_1/R_1 \cong F_2/R_2 \cong Q$, and because Q is finitely presented, R_1 and R_2 are the normal closures in F_1, F_2 of finite subsets S_1, S_2. Let T be a finite generating set for Q and let T_1, T_2 be finite sets of inverse images of T under ρ_1, ρ_2. Then the fibre product P is generated by the finite set

$$\{\, (\rho_1(s_1), 1),\ (1, \rho_2(s_2)),\ (t_1, t_2) \mid s_1 \in S_1,\ s_2 \in S_2,\ t_1 \in T_1,\ t_2 \in T_2,\ p_1\rho_1(t_1) = p_2\rho_2(t_2) \,\}.$$

□

2. Bieri-Strebel Theory for Metabelian Groups

Let Q be a finitely generated abelian group. For any extension Γ of a (not necessarily finitely generated) abelian group A by Q, the conjugation action of Γ on A makes A into a Q–module. It is easy to see that Γ is finitely generated if and only if A is finitely generated as a Q–module. The results of the present paper are applications of a fundamental theorem of Bieri and Strebel [4], which gives a necessary and sufficient condition for Γ to be finitely presented.

A homomorphism of Q to the additive group $\mathbb{R}$ is called a *valuation* of Q. Associated to each such valuation v one has the submonoid of Q

$$Q_v = \{q \in Q \mid v(q) \geq 0\}.$$

For valuations v, v' we write $v \sim v'$ if and only if there exists $\lambda > 0$ such that $v(q) = \lambda v'(q)$ for all $q \in Q$. Let n be the torsion-free rank of Q. Then $\mathrm{Hom}(Q, \mathbb{R}) \cong \mathbb{R}^n$, and there is an obvious identification between the set of equivalence classes of nontrivial valuations of Q and the $(n-1)$-sphere S^{n-1}.

Let A be a finitely generated Q–module. We can view A as a module over the commutative ring $\mathbb{Z}Q_v \subset \mathbb{Z}Q$. Define Σ_A to be the set of $\sim$ classes of valuations v on Q such that A is finitely generated as a $\mathbb{Z}Q_v$–module.

The module A is said to be *tame* if $\Sigma_A \cup -\Sigma_A = S^{n-1}$, in other words, for every valuation v of Q, either A is finitely generated as a Q_v–module, or else it is finitely generated as a Q_{-v}–module.

We can now state the theorem of Bieri and Strebel; this is Theorem A(ii) of [4].

Theorem 6 (Bieri-Strebel)**.** *Consider a short exact sequence* $1 \to A \to \Gamma \to Q \to 1$ *with* A *and* Q *abelian and* Γ *finitely generated. Then* Γ *is finitely presented if and only if* A *is tame as a* Q*–module.*

It is observed in Proposition 2.5 of [4] that all submodules of a tame module A and direct products of a finite number of copies of A are tame. Using these results, we can immediately prove Theorem 1 in the special case when N and Q are both abelian. The fact that Γ is finitely presented tells us that N is a tame Q–module. The fibre product P is an extension of $N \times N$ by Q, where the induced module action on both of the direct factors in $N \times N$ is the same as the original Q-action. Hence $N \times N$ is tame, and so P is finitely presented. This situation occurs in Example 1.

In Example 2, however, we have different induced module actions on the direct factors factors in $N \times N$ in the twisted fibre product P. Since P is not finitely presented in this case, this example shows that the direct product of tame modules need not be tame in general. In Section 4, we shall show that this behaviour is typical of twisted fibre products.

There is one situation in which a product of tame modules is tame, and we shall need this in our proof of Theorem 1. If B is a submodule of A, then $A \times B$ is a submodule of $A \times A$. As observed in [4] submodules and finite direct powers of tame modules are tame, so we have the following:

Lemma 7. *Let A be a tame Q–module where Q is finitely generated abelian, and let B be a submodule of A. Then the Q–module $A \times B$ is tame.* □

3. Proof of Theorem 1

Reducing to the Case Where $N \subseteq \Gamma'$. We are now ready to prove Theorem 1. For untwisted fibre products, there is no loss of generality in assuming that N is a subgroup of Γ and $Q = \Gamma/N$ with p the natural epimorphism. With this assumption, let

$$1 \to N \to \Gamma \to Q \to 1$$

be the short exact sequence with Γ finitely generated metabelian, and let P be the associated fibre product.

We first observe that by Theorem B of [4] Q is finitely presented, and hence P is finitely generated by Proposition 5.

We compare P to the fibre product $\hat{P}$ associated to

$$1 \to [\Gamma, N] \to \Gamma \to \Gamma/[\Gamma, N] \to 1.$$

Lemma 8. *$\hat{P}$ is a normal subgroup of P and the quotient is a finitely generated abelian group.*

Proof. By Lemma 2, there are semi-direct product decompositions $P = N \rtimes \Gamma$ and $\hat{P} = [\Gamma, N] \rtimes \Gamma$. The natural inclusion $\hat{P} \hookrightarrow P$ is that implicit in the notation. So $\hat{P}$ is normal in P and the quotient is naturally isomorphic to $N/[\Gamma, N]$, which is abelian. This quotient is finitely generated, because P is. □

It follows from this lemma that in order to prove our main theorem there is no loss of generality in assuming that $N \subseteq \Gamma'$. For if we replace N by $[\Gamma, N] \subseteq \Gamma'$ and prove the theorem in that context then, from the above lemma, P is an extension of the finitely presented group $\hat{P}$ by the finitely presented group $P/\hat{P}$, and so is itself finitely presented.

Completion of the proof. In the light of the discussion in the previous section we may assume that $N \subseteq \Gamma'$, the commutator subgroup of Γ. We now have a short exact sequence:

$$1 \to N \rtimes \Gamma' \to N \rtimes \Gamma \to \Gamma_{\mathrm{ab}} \to 1,$$

where the middle group, $N \rtimes \Gamma$ is the decomposition of P given in Lemma 2 and the inclusion of the first group is the obvious one. But now, since we are assuming that N is contained in the commutator subgroup of Γ, which is abelian, the first term in this sequence is actually a direct product (with the visible decomposition).

Since Γ is finitely presented, Γ' is a tame module over Γ_{ab} by the "only if" part of Theorem 6. By Lemma 7, $N \times \Gamma'$ is a tame Γ'–module, and the action implicit in the above short exact sequence is indeed the product action of Γ_{ab} on $N \times \Gamma'$. Thus, by the "if" part of Theorem 6, we are done.

4. The Twisted Case

In contrast to our main theorem we have:

Theorem 9. *Let $1 \to N \to \Gamma \xrightarrow{p} Q \to 1$ be a short exact sequence with Q and N abelian. Consider the twisted fibre product $P = \{(\gamma, \gamma') \mid p(\gamma) = -p(\gamma')\}$ associated to this sequence. Then P is finitely presented if and only if Γ is polycyclic.*

Proof. We have a short exact sequence

$$1 \to N \times N \to P \to Q \to 1.$$

The action $Q \to \mathrm{Aut}(N \times N)$ associated to this sequence is, in exponential notation, $(n_1, n_2)^q = (n_1^q, n_2^{-q})$, where $n \mapsto n^q$ is the action on N associated to the short exact sequence $1 \to N \to \Gamma \xrightarrow{p} Q \to 1$.

Thus, given a valuation $v : Q \to \mathbb{R}$, the module $N \times N$ over Q_v is the direct product of the (standard) Q_v–module N (the first visible factor) and the standard Q_{-v} module N, now viewed as a Q_v module via the monoid homomorphism $Q_v \to Q_{-v}$ given by $q \mapsto -q$.

With this structure, $N \times N$ is finitely generated as a Q_v module if and only if N is finitely generated *both* as the standard Q_v module and the standard Q_{-v} module. In other words,

$$\Sigma_{N \times N} = \Sigma_N \cap -\Sigma_N.$$

Similarly,

$$-\Sigma_{N \times N} = \Sigma_N \cap -\Sigma_N.$$

Therefore $N \times N$ is a tame Q–module if and only if $\Sigma_N = -\Sigma_N = S^{n-1}$. But Theorem A(i) of [4] states that this occurs if and only if Γ is polycyclic. $\square$

An anonymous referee has pointed out to us that, by using more of the Σ-theory for metabelain groups, this last result can be considerably extended. Let $1 \to N_i \to \Gamma \xrightarrow{p_i} Q \to 1$ be short exact sequences, $i = 1, 2$, with Q and N_i abelian. The associated fibre product P is an extension of $N_1 \oplus N_2$ by Q. Then P fails to be finitely presented exactly when the complement of the Σ-invariant, that is $\Sigma^c_{N_1 \oplus N_2} = \Sigma^c_{N_1} \cup \Sigma^c_{N_2}$, contains a pair of antipodal points. In the twisted case, we have $p_2 = \alpha \circ p_1$ with $\alpha \in \mathrm{Aut}(Q)$ and so $\Sigma^c_{N_1 \oplus N_2} = \Sigma^c_{N_1} \cup \alpha^*(\Sigma^c_{N_1})$. Thus twisting by a non-trivial automorphism α will often produce antipodal points when $\Sigma^c_{N_1} \neq \emptyset$. In particular, using [5], it can be shown in this way that the example in the next section is not finitely presentable.

5. A Further Example

The final example is a twisted pullback derived from Examples 1 and 2 above, but it does not fall within the scope of Theorem 9 and, unlike Example 2, it has a finitely generated second homology group $H_2(P)$.

Example 3. Fix two coprime integers $q, r > 1$. Let

$$\Gamma = \langle\, x, s, t \mid s^{-1}xs = x^q,\ t^{-1}xt = x^r,\ [s,t] = 1 \,\rangle,$$

let $N = \langle x \rangle^G$, and let $Q = \langle\, s, t \mid [s,t] \,\rangle$ be free abelian of rank 2. Define $p_1 : \Gamma \to Q$ by $p_1(x) = 1$, $p_1(s) = s$, $p_1(t) = t$, and $p_2 : \Gamma \to Q$ by $p_2(x) = 1$, $p_2(s) = s$, $p_2(t) = t^{-1}$.

Then Γ is isomorphic to the semidirect product of the additive group $\mathbb{Z}[\frac{1}{qr}]$ by Q, where s and t act by multiplication by q and r in $\mathbb{Z}[\frac{1}{qr}]$, respectively.

Let P be the twisted pullback corresponding to these groups and maps. We shall show that P cannot be finitely presented by showing that its relation module is not finitely generated. To do this, we shall construct an extension of an abelian group Z by P.

Let $C_r(\infty)$ denote the quotient group $\mathbb{Z}[\frac{1}{qr}]/\mathbb{Z}[\frac{1}{q}]$, which is isomorphic to the multiplicative group of all r^n-th roots of unity, for $n \geq 0$. Note that $C_r(\infty)$ is the union of an infinite ascending chain of characteristic subgroups $C_n\,(n \geq 0)$, where C_n consists of the multiples of $1/r^n$. It follows that $C_r(\infty)$ cannot be finitely generated under any module action.

Define D to be the group with elements

$$\{\, (a, b, c) \mid a, b \in \mathbb{Z}[\tfrac{1}{qr}],\ c \in C_r(\infty) \,\}$$

and multiplication $(a, b, c)(a', b', c') = (a + a', b + b', c + c' + a'b)$, let s be the automorphism of D mapping (a, b, c) to (qa, qb, q^2c), let t be the automorphism of D mapping (a, b, c) to $(ra, b/r, c)$, and let E be the semidirect product of D by Q using this action. Let Z be the subgroup $\{ (0, 0, c) \mid c \in C_r(\infty) \}$ of E. Then, as in Example 2 above, Z is contained in $[D, D]$, Z is normal in E, and it is easily seen that $E/Z \cong P$. Unlike Example 2, Z is not central in E.

Now E is finitely generated, for instance by the four elements $(1, 0, 0)$, $(0, 1, 0)$, s and t since the equation $[(0, c, 0), (1, 0, 0)] = (0, 0, c)$ holds. Let F be a finitely generated free group mapping onto E via $\gamma : F \to E$, and define $R = \gamma^{-1}(Z)$. Then $F/R \cong P$ gives a presentation of P and Z is a quotient of the relation module $R/[R, R]$. Recalling that $Z \cong C_r(\infty)$, we see that $R/[R, R]$ cannot be finitely generated as a P–module, and hence P cannot be finitely presented.

Here is a proof that $H_2(P)$ is finitely generated. P is a split extension of $N \times N \cong \mathbb{Z}[\frac{1}{qr}] \oplus \mathbb{Z}[\frac{1}{qr}]$ by Q, where the actions induced by $s, t \in Q$ are $(a, b) \to (qa, qb)$ and $(a, b) \to (ra, b/r)$, respectively.

Now $H_2(P)$ is filtered by the appropriate E^∞ terms of the Lyndon-Hochschild-Serre Spectral Sequence (see for instance Theorem 6.3 of [1]), and the E^∞ terms are sections of the E^2 terms. So it suffices to show that each of the relevant E^2 terms, namely each of $H_2(Q)$, $H_1(Q, H_1(N \times N)) = H_1(Q, N \times N)$ and $H_0(Q, H_2(N \times N))$, is finitely generated. The first of these is clear, because Q is finitely generated abelian.

By the Universal Coefficient Theorem, $H_2(N \times N)$ is isomorphic to a direct sum of $N \otimes N$ and two copies of $H_2(N)$. It is not hard to calculate that $H_2(N) = 0$ (because N is locally cyclic), and that $N \otimes N \cong N \cong \mathbb{Z}[\frac{1}{qr}]$, where the actions of s and t on $N \otimes N$ correspond respectively to multiplication by q^2 and 1 in $\mathbb{Z}[\frac{1}{qr}]$. So $H_0(Q, H_2(N \times N))$ is isomorphic to the quotient V of $\mathbb{Z}[\frac{1}{qr}]$ by the ideal generated by $(q^2 - 1)$. We claim that V is finite. To see this let $A = \mathbb{Z}/(q^2 - 1)$, which is a finite ring, and observe that V can be viewed as the quotient of the polynomial ring $A[y]$ by the ideal generated by $(qr)y - 1$. Since A is finite, there are natural numbers $0 < n < m$ such that $(qr)^n = (qr)^m$ in A. Then in V we have $1 = (qr)^m y^m = (qr)^n y^n y^{m-n} = y^{m-n}$. Thus V is additively generated by $1, y, \ldots, y^{m-n-1}$ with A coefficients and hence is finite.

The conjugation action of Q on $N \times N$ in P preserves both factors N, so $H_1(Q, N \times N)$ is isomorphic to the direct sum of two groups $H_1(Q, N)$. Although the actions of Q on N are different for the two factors, they are equivalent modulo an automorphism of Q, and so it is sufficient to show that $H_1(Q, N)$ is finitely generated, where Q acts on N as in the group Γ.

We know that Γ is finitely presented, and so $H_2(\Gamma)$ must be finitely generated. The E^2 terms relevant to $H_2(\Gamma)$ in the spectral sequence for the extension $1 \to N \to \Gamma \to Q \to 1$ are $H_2(Q)$, $H_1(Q, N)$, and $H_0(Q, H_2(N))$. Since $Q = \mathbb{Z} \times \mathbb{Z}$ has cohomological dimension 2, $E^3 = E^\infty$ and the d^2 differentials beginning and ending at $H_1(Q, N)$ are both zero maps. So $E^\infty_{1,1} = H_1(Q, N)$ is a section of $H_2(\Gamma)$ and hence $H_1(Q, N)$ is finitely generated as required. This completes the proof.

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