SUBGROUPS OF DIRECT PRODUCTS OF LIMIT GROUPS

MARTIN R. BRIDSON, JAMES HOWIE, CHARLES F. MILLER III, AND HAMISH SHORT

ABSTRACT. If $\Gamma_1, \dots, \Gamma_n$ are limit groups and $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is of type $\mathrm{FP}_n(\mathbb{Q})$ then S contains a subgroup of finite index that is itself a direct product of at most n limit groups. This answers a question of Sela.

1. Introduction

The systematic study of the higher finiteness properties of groups was initiated forty years ago by Wall [29] and Serre [26]. In 1963, Stallings [28] constructed the first example of a finitely presented group Γ with $H_3(\Gamma; \mathbb{Q})$ infinite dimensional; his example was a subgroup of a direct product of three free groups. This was the first indication of the great diversity to be found amongst the finitely presented subgroups of direct products of free groups, a theme developed in [4].

In contrast, Baumslag and Roseblade [3] proved that in a direct product of two free groups the only finitely presented subgroups are the *obvious ones*: such a subgroup is either free or has a subgroup of finite index that is a direct product of free groups. In [11] the present authors explained this contrast by proving that the exotic behaviour among the finitely presented subgroups of direct products of free groups is accounted for entirely by the failure of higher homological-finiteness conditions. In particular, we proved that the only subgroups S of type FP_n in a direct product of n free groups are the obvious ones: if S intersects each of the direct factors non-trivially, it virtually splits as the direct product of these intersections. We also proved that this splitting phenomenon persists when one replaces free groups by the fundamental groups of compact surfaces [11]; in the light of the work of Delzant and Gromov [17], this has significant implications for the structure of Kähler groups.

Date: 18 Sept 2007.

1991 *Mathematics Subject Classification.* 20F65,20F67,20J05 (20E07, 20E08, 20E26).

Key words and phrases. subdirect products, homology of groups, limit groups, residually-free groups.

This work was supported in part by the CNRS (France) and by Alliance Scientifique grant # PN 05.004. Bridson is supported by a Royal Society Nuffield Research Merit Award and an EPSRC Senior Fellowship.

Examples show that the splitting phenomenon for FP_∞ subgroups does not extend to products of more general 2-dimensional hyperbolic groups or higher-dimensional Kleinian groups [7]. But recent work at the confluence of logic, group theory and topology has brought to the fore a class of groups that is more profoundly tied to surface and free groups than either of the above classes, namely *limit groups*.

Limit groups arise naturally from several points of view. They are perhaps most easily described as the finitely generated groups L that are *fully residually free* (or *ω–residually free*): for any finite subset $T \subset L$ there exists a homomorphism from L to a free group that is injective on T. It is in this guise that limit groups were studied extensively by Kharlampovich and Myasnikov [18, 19, 20]. They are also known as *$\exists$-free groups* [21], reflecting the fact that these are precisely the finitely generated groups that have the same existential theory as a free group.

More geometrically, limit groups are the finitely generated groups that have a Cayley graph in which each ball of finite radius is isometric to a ball of the same radius in some Cayley graph of a free group of fixed rank.

The name *limit group* was introduced by Sela. His original definition involved a certain limiting action on an $\mathbb{R}$-tree, but he also emphasized that these are precisely the groups that arise when one takes limits of stable sequences of homomorphisms $\phi_n : G \to F$, where G is an arbitrary finitely generated group and F is a free group; *stable* means that for each $g \in G$ either $I_g = \{n \in \mathbb{N} : \phi_n(g) = 1\}$ or $J_g = \{n \in \mathbb{N} : \phi_n(g) \neq 1\}$ is finite, and the *limit* of (ϕ_n) is the quotient of G by $\{g \mid |I_g| = \infty\}$.

In his account [25] of the outstanding problems concerning limit groups, Sela asked whether the main theorem of [11] could be extended to cover limit groups. The present article represents the culmination of a project to prove that it can. Building on ideas and results from [11, 8, 9, 10, 13] we prove:

Theorem A. *If $\Gamma_1, \dots, \Gamma_n$ are limit groups and $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is a subgroup of type $\mathrm{FP}_n(\mathbb{Q})$, then S is virtually a direct product of n or fewer limit groups.*

Combining this result with the fact that every finitely generated residually free group can be embedded into a direct product of finitely many limit groups ([19, Corollary 2], [22, Claim 7.5]), we obtain:

Corollary 1.1. *Every residually free group of type $\mathrm{FP}_\infty(\mathbb{Q})$ is virtually a direct product of a finite number of limit groups.*

B. Baumslag [2] proved that a finitely generated, residually free group is fully residually free (i.e. a limit group) unless it contains a subgroup isomorphic to $F \times \mathbb{Z}$, where F is a free group of rank 2. Corollary 1.1 together with the methods used to prove Theorem A yield the following generalization of Baumslag's result:

Corollary 1.2. *Let Γ be a residually free group of type* $\mathrm{FP}_n(\mathbb{Q})$ *where* $n \geq 1$, *let F be a free group of rank 2 and let F^n denote the direct product of n copies of F. Either Γ contains a subgroup isomorphic to $F^n \times \mathbb{Z}$ or else Γ is virtually a direct product of n or fewer limit groups.*

We also prove that if a subgroup of a direct product of n limit groups fails to be of type $\mathrm{FP}_n(\mathbb{Q})$, then one can detect this failure in the homology of a subgroup of finite index.

Theorem B. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups and let $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated subgroup with $L_i = \Gamma_i \cap S$ non-abelian for $i = 1, \ldots, n$.*

If L_i is finitely generated for $1 \leq i \leq r$ and not finitely generated for $i > r$, then there is a subgroup of finite index $S_0 \subset S$ such that $S_0 = S_1 \times S_2$, where S_1 is the direct product of the limit groups $S_0 \cap \Gamma_i$, $i \leq r$ and (if $r < n$) $S_2 = S_0 \cap (\Gamma_{r+1} \times \cdots \times \Gamma_n)$ has $H_k(S_2; \mathbb{Q})$ infinite dimensional for some $k \leq n - r$.

In Section 9 we shall prove a more technical version of Theorem B and account for abelian intersections.

Theorems A and B are the exact analogues of Theorems A and B of [11]. In Section 3 we introduce a sequence of reductions that will allow us to deduce both theorems from the following result (which, conversely, is an easy consequence of Theorem B). We remind the reader that a subgroup of a direct product is called a *subdirect product* if its projection to each factor is surjective.

Theorem C. *Let $\Gamma_1, \ldots, \Gamma_n$ be non–abelian limit groups and let $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated subdirect product which intersects each factor non-trivially. Then either :*

(1) *S is of finite index;* or
(2) *S is of infinite index and has a finite-index subgroup $S_0 < S$ such that $H_j(S_0; \mathbb{Q})$ has infinite $\mathbb{Q}$-dimension for some $j \leq n$.*

For simplicity of exposition, the homology of a group G in this paper will almost always be with coefficients in a $\mathbb{Q}G$-module – typically the trivial module $\mathbb{Q}$. But with minor modifications, our arguments also apply with other coefficient modules, giving corresponding results under the finiteness conditions $\mathrm{FP}_n(R)$ for other suitable rings R.

A notable aspect of the proof of the above theorems is that following a raft of reductions based on geometric methods, the proof takes an unexpected twist in the direction of nilpotent groups. The turn of events that leads us in this direction is explained in Section 4 – it begins with a simple observation about higher commutators from [13] and proceeds via a spectral sequence argument.

Several of our results shed light on the nature of arbitrary finitely presented subgroups of direct products of limit groups, most notably Theorem 4.2. These results suggest that there is a real prospect of understanding all such subgroups, i.e. all finitely presented residually free groups. We take up this challenge in [12], where we describe precisely which subdirect products of limit groups are finitely presented and present a solution to the conjugacy and membership problems for these subgroups (cf. [13], [14]). But the isomorphism problem for finitely presented residually free groups remains open, and beyond that lie many further challenges. For example, with applications of the surface-group case to Kähler geometry in mind [17], one would like to know if all finitely presented subdirect products of limit groups satisfy a polynomial isoperimetric inequality.

We are grateful to the many colleagues with whom we have had useful discussions at various times about aspects of this work, particularly Emina Alibegović, Mladen Bestvina, Karl Gruenberg, Peter Kropholler, Zlil Sela and Henry Wilton. We are also grateful to the anonymous referee for his/her helpful comments.

2. Limit groups and their decomposition

Since this is the fourth in a series of papers on limit groups (following [8, 9, 10]), we shall only recall the minimal necessary amount of information about them. The reader unfamiliar with this fascinating class of groups should consult the introductions in [1, 6], the original papers of Sela [22, 23, 24], or those of Kharlampovich and Myasnikov [18, 19, 20] where the subject is approached from a perspective more in keeping with traditional combinatorial group theory; a further perspective is developed in [16].

2.1. **Limit groups.** Our results rely on the fact that limit groups are the finitely generated subgroups of ω-residually free tower (ω-rft) groups [22, Definition 6.1] (also known as NTQ-groups [19]). A useful summary of Sela's proof of this result was given by Alibegović and Bestvina in the appendix to [1] (cf. [23, (1.11), (1.12)]). The equivalent result of Kharlampovich and Myasnikov [19, Theorem 4] is presented in a more algebraic manner.

An ω-rft group is the fundamental group of a tower space assembled from graphs, tori and surfaces in a hierarchical manner. The number of stages in the process of assembly is the *height* of the tower. Each stage in the construction involves the attachment of an orientable surface along its boundary, or the attachment of an n-torus T along an embedded circle representing a primitive element of $\pi_1 T$. (There are additional constraints in each case.)

The *height* of a limit group Γ is the minimal height of an ω-rft group that has a subgroup isomorphic to Γ. Limit groups of height 0 are free products of finitely many free abelian groups (each of finite rank) and surface groups of Euler characteristic at most -2.

The splitting described in the following proposition is obtained as follows: embed Γ in an ω-rft group G of minimal height, take the graph of groups decomposition of G that the Seifert-van Kampen Theorem associates to the addition of the final block in the tower, then apply Bass-Serre theory to get an induced graph of groups decomposition of Γ.

Recall that a graph-of-groups decomposition is termed *k-acylindrical* if in the action on the associated Bass-Serre tree, the stabilizer of each geodesic edge-path of length greater than k is trivial; if the value of k is unimportant, one says simply that the decomposition is *acylindrical*.

Proposition 2.1. *If Γ is a freely-indecomposable limit group of height $h \geq 1$, then it is the fundamental group of a finite graph of groups that has infinite cyclic edge groups and has a vertex group that is a non-abelian limit group of height $\leq h-1$. This decomposition may be chosen to be 2-acylindrical.*

Note also that any non-abelian limit group of height 0 splits as $A *_C B$ with C infinite-cyclic or trivial, and this splitting is 1-acylindrical for surface groups, and 0-acylindrical for free products.

2.2. **The class of groups $\mathcal{C}$.** We define a class of finitely presented groups $\mathcal{C}$ in a hierarchical manner; it is the union of the classes $\mathcal{C}_n$ defined as follows.

At *level* 0 we have the class $\mathcal{C}_0$ consisting of free products $A * B$ of non-trivial, finitely presented groups, where at least one of A and B has cardinality at least 3 – in other words, all finitely presented non-trivial free products with the exception of $\mathbb{Z}_2 * \mathbb{Z}_2$.

A group lies in $\mathcal{C}_n$ if and only if it is the fundamental group of a finite, acylindrical graph of finitely presented groups, where all of the edge groups are cyclic, and at least one of the vertex groups lies in $\mathcal{C}_{n-1}$.

The following is an immediate consequence of Proposition 2.1.

Corollary 2.2. *All non-abelian limit groups lie in $\mathcal{C}$.*

2.3. **Other salient properties.** In the proof of Theorems A and B, the only properties of limit groups Γ that will be needed are the following.

1. Limit groups are finitely presented, and their finitely generated subgroups are limit groups.
2. If Γ is non-abelian, it lies in $\mathcal{C}$ (Corollary 2.2).
3. Cyclic subgroups are closed in the profinite topology on Γ. (This is true for all finitely generated subgroups [30].)
4. If a subgroup S of Γ has finite dimensional $H_1(S;\mathbb{Q})$, then S is finitely generated (and hence is a limit group) [8, Theorem 2].

(5) Limit groups are of type FP_∞ (in fact $\mathcal{F}_\infty$). This follows directly from the fact that the class of limit groups coincides with the class of constructible limit groups, [6, Definition 1.14].

2.4. **Notation.** Throughout this paper, we consistently use the notational convention that S is a subgroup of the direct product of the limit groups Γ_i ($1 \leq i \leq n$), that L_i denotes the intersection $S \cap \Gamma_i$, and that $p_i : \Gamma_1 \times \cdots \times \Gamma_n \to \Gamma_i$ is the coordinate projection.

2.5. **Subgroups of finite index.** Throughout the proof Theorems A, B and C we shall repeatedly pass to subgroups of finite index $H_i \subset \Gamma_i$. When we do so, we shall assume that our original subgroup S is replaced by $p_i^{-1}(H) \cap S$ and that each Γ_j ($j \neq i$) is replaced by $p_j p_i^{-1}(H_i)$. This does not affect the intersections $L_j = S \cap \Gamma_j$ ($j \neq i$).

Recall [15, VIII.5.1] that the property FP_n is inherited by finite-index subgroups and persists in finite extensions. In the proof of Theorem C we detect the failure of property FP_n by considering the homology of subgroups of finite index: if $H_k(S_1; \mathbb{Q})$ is infinite dimensional for some $S_1 < S$ of finite index, then neither S nor S_1 is of type FP_k.

Some care is required here because one cannot conclude in the previous sentence that S has an infinite-dimensional homology group: the finite-dimensionality of homology groups is a property that persists in finite extensions but is not, in general, inherited by finite-index subgroups. In the context of the proof of Theorem C, care has been taken to ensure that each passage to a finite-index subgroup respects this logic.

3. Reductions of the main theorem

The following proposition reduces Theorem A to Theorem C.

Proposition 3.1. *Theorem A is true if and only if it holds under the following additional assumptions.*

(1) $n \geq 2$.
(2) *Each projection* $p_i : S \to \Gamma_i$ *is surjective.*
(3) *Each intersection* $L_i = S \cap \Gamma_i$ *is non-trivial.*
(4) *Each* Γ_i *is a non-abelian limit group.*
(5) *Each* Γ_i *splits as an HNN-extension over a cyclic subgroup* C_i *with stable letter* $t_i \in L_i$.

Proof. (1) The case $n = 0$ of Theorem A is trivial.

In the case $n = 1$, $S < \Gamma_1$ has type $FP_1(\mathbb{Q})$, so is finitely generated. But a finitely generated subgroup of a limit group is again a limit group, and there is nothing more to prove. (The case $n = 2$ was proved in [10] but an independent proof is given below.)

(2) Since S has type $FP_n(\mathbb{Q})$ it is finitely generated, hence so is $p_i(S)$ and we can replace each Γ_i by $p_i(S)$.

(3) If, say, L_n is trivial, then the projection map $q_n : S \to \Gamma_1 \times \cdots \times \Gamma_{n-1}$ is injective, and S is isomorphic to a subgroup $q_n(S)$ of $\Gamma_1 \times \cdots \times \Gamma_{n-1}$. After iterating this argument, we may assume that each L_i is non-trivial.

(4) Suppose that one or more of the Γ_i is abelian. A group is an abelian limit group if and only if it is free abelian of finite rank. Hence a direct product of finitely many abelian limit groups is again an abelian limit group. This reduces us to the case where precisely one of the Γ_i – say Γ_n – is abelian.

Now, replacing Γ_n by a finite index subgroup if necessary, we may assume that $L_n \subset \Gamma_n$ is a direct factor of Γ_n: say $\Gamma_n = L_n \oplus M$. Since $M \cap S$ is trivial, the projection $\Gamma_1 \times \cdots \times \Gamma_n \to \Gamma_1 \times \cdots \times \Gamma_{n-1} \times L_n$ with kernel M maps S isomorphically onto a subgroup T of $\Gamma_1 \times \cdots \times \Gamma_{n-1} \times L_n$. Since $L_n \subset T$, it follows that $S \cong T = U \times L_n$ for some subgroup U of $\Gamma_1 \times \cdots \times \Gamma_{n-1}$. But then U has type $\mathrm{FP}_n(\mathbb{Q})$, since S does, and if Theorem A holds in the case where all the Γ_i are non-abelian, then U is virtually a direct product of $n-1$ or fewer limit groups. But then $S \cong U \times L_n$ is virtually a direct product of n or fewer limit groups, so Theorem A holds in full generality.

(5) The subgroup L_i of Γ_i is normal by (2) and non-trivial by (3). Hence it contains an element t_i that acts hyperbolically on the tree of the splitting described in Proposition 2.1 (see [9, Section 2]). Then by [9, Theorem 3.1], t_i is the stable letter in some HNN decomposition (with cyclic edge-stabilizer) of a finite-index subgroup $\Delta_i \subset \Gamma_i$.

Replacing each Γ_i by the corresponding subgroup Δ_i, and S by $S \cap (\Delta_1 \times \cdots \times \Delta_n)$, gives us the desired conclusion.

(The above argument extends to all groups in $\mathcal{C}$ under the additional hypothesis that the edge groups in the splittings defining $\mathcal{C}$ are all closed in the profinite topology.) □

4. The elements of the proof of Theorem C

We have seen that Theorem A follows from Theorem C. The proof of Theorem C extends from Section 5 to Section 8. In the present section we give an overview of the contents of these sections and indicate how they will be assembled to complete the proof.

In Section 5 we prove the following extension of the basic result that non-trivial, finitely-generated normal subgroups of non-abelian limit groups have finite index [8].

Theorem 4.1. *Let Γ be a group in $\mathcal{C}$, and $1 \neq N < G < \Gamma$ with N normal in Γ and G finitely generated. Then $|\Gamma : G| < \infty$.*

Using this result, together with the HNN decompositions of the Γ_i described in Proposition 3.1, we deduce (Section 6):

Theorem 4.2. *Let $\Gamma_1, \ldots, \Gamma_n$ be non-abelian limit groups. If $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup with $H_2(S_1;\mathbb{Q})$ finite dimensional for all finite-index subgroups $S_1 < S$, and if S satisfies conditions (1) to (5) of Proposition 3.1, then:*

- *the image of each projection $S \to \Gamma_i \times \Gamma_j$ is of finite index in $\Gamma_i \times \Gamma_j$;*
- *the quotient groups Γ_i/L_i are virtually nilpotent of class at most $n-2$.*

We highlight the case $n = 2$.

Corollary 4.3. *If Γ_1 and Γ_2 are non-abelian limit groups, and $S < \Gamma_1 \times \Gamma_2$ is a subdirect product intersecting each factor non-trivially, with $H_2(S_1;\mathbb{Q})$ finite dimensional for all finite-index subgroups $S_1 < S$, then S has finite index in $\Gamma_1 \times \Gamma_2$.*

An important special case of Theorem C, considered in Section 7, arises where S is the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \to \mathbb{Z}$.

Theorem 4.4. *Let $\Gamma_1, \ldots, \Gamma_n$ be non-abelian limit groups, and N the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \to \mathbb{Z}$. Then there is a subgroup of finite index $N_0 \subset N$ such that at least one of the homology groups $H_k(N_0;\mathbb{Q})$ $(0 \le k \le n)$ has infinite $\mathbb{Q}$-dimension.*

We complete the proof of Theorem C in Section 8. We have seen that each of the Γ_i/L_i is virtually nilpotent. Setting $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and noting that S contains the product $L = L_1 \times \cdots \times L_n$, we argue by induction on the difference in Hirsch lengths $d = h(\Gamma/L) - h(S/L)$ to prove that $H_k(S;\mathbb{Q})$ has infinite $\mathbb{Q}$-dimension for some $k \le n$ if $d > 0$. The initial step of the induction is provided by Theorem 4.4, and the inductive step is established using the LHS spectral sequence. Section 9 contains a proof of Theorem B.

5. Subgroups containing normal subgroups

In this section we prove Theorem 4.1. We assume that the reader is familiar with Bass-Serre theory [27], which we shall use freely. All our actions on trees are without inversions.

Lemma 5.1. *Let Δ be a group acting k-acylindrically, cocompactly and minimally on a tree X. Let H be a finitely generated subgroup of Δ. Suppose that $M < H$ is a non-trivial subgroup which is normal in Δ. Then the action of H on X is cocompact.*

Proof. If X is a point there is nothing to prove, so we may assume that X has at least one edge. By hypothesis, Δ has no global fixed point in its action on X. By [9, Corollary 2.2], the non-trivial normal subgroup $M < \Delta$ contains elements which act hyperbolically on X, and the union of the axes of all such elements is the unique minimal

M-invariant subtree X_0 of X. Since M is normal in Δ, the M-invariant subtree X_0 is also invariant under the action of Δ. But X is minimal as a Δ-tree, so $X_0 = X$.

We have shown that M acts minimally on X. Since $M < H$, it follows that H acts minimally on X, so the quotient graph of groups $\mathcal{G}$ has no proper sub-(graph of groups) such that the inclusion induces an isomorphism on π_1. A standard argument in Bass-Serre theory shows that since H is finitely generated, the topological graph underlying $\mathcal{G}$ is compact, as claimed. □

Proposition 5.2. *Let $\Gamma \in \mathcal{C}$, and let C, G be subgroups of Γ with C cyclic and G finitely generated. If $|G\backslash\Gamma/C| < \infty$, then $|\Gamma : G| < \infty$.*

Proof. Let Γ be a group in $\mathcal{C}$. We argue by induction on the level $\ell = \ell(\Gamma)$ in the hierarchy $\mathcal{C} = \cup_n \mathcal{C}_n$ where Γ first appears. By definition, Γ has a non-trivial, k-acylindrical, cocompact action on a tree T, with cyclic edge stabilizers. Without loss of generality we can suppose that this action is minimal.

If $\ell = 0$ there is a single orbit of edges, the edge stabilizers are trivial and the vertex stabilizers are non-trivial. If $\ell > 0$ the edge-stabilizers are non-trivial and the stabilizer of some vertex w is in $\mathcal{C}_{\ell-1}$.

Let c be a generator for C. We treat the initial and inductive stages of the argument simultaneously, but distinguish two cases according to the action of c.

Case 1. Suppose that c fixes a vertex v of T.

Then, by our double-coset hypothesis, the Γ-orbit of v consists of only finitely many G-orbits Gv_i. Since the action of Γ on T is cocompact, there is a constant $m > 0$ such that T is the m-neighbourhood of Γv, and hence the quotient graph $X = G\backslash T$ is the m-neighbourhood of the finitely many vertices Gv_i. In other words, X has finite diameter.

Note also that $\pi_1 X$ has finite rank, because it is a retract of G which is finitely generated.

Finally, note that $X = G\backslash T$ has only finitely many valency 1 vertices. For otherwise, we can deduce a contradiction as follows. Since G is finitely generated, if there are infinitely many vertices of valency 1, then the induced graph-of-groups decomposition of G is degenerate, in the sense that there is a valency 1 vertex $\bar{u}$ with $G_{\bar{u}} = G_{\bar{e}}$, where $\bar{e}$ is the unique edge of $G\backslash T$ incident at $\bar{u}$.

Now $\bar{u} = Gu$ for some vertex u in T, and $\bar{e} = Ge$ for an edge e incident at u. The group $G_{\bar{u}}$ is the stabilizer of u in G, and $G_{\bar{e}}$ is the stabilizer of e in G. The fact that $\bar{u} = Gu$ has valency 1 in $G\backslash T$ means that $G_{\bar{u}}$ acts transitively on the link Lk of u in T. Hence $|\mathrm{Lk}| = |G_{\overline{u}} : G_{\overline{e}}| = 1$, so u is a valency 1 vertex of T. But this contradicts the fact that T is minimal as a Γ-tree.

We have shown that $X = G\backslash T$ has finite diameter, finite rank, and only finitely many vertices of valency 1. It follows that X is a finite graph.

In the case where Γ has level $\ell = 0$, the stabilizer Γ_e of any edge e of T is trivial. The number of edges in $X = G\backslash T$ that are images of edges $\gamma e \in \Gamma e$ can therefore be counted as $|G\backslash\Gamma/\Gamma_e| = |G\backslash\Gamma| = |\Gamma : G|$. Hence, in this case, $|\Gamma : G| < \infty$, as required.

In the case where $\ell > 0$, there is a vertex w of T whose stabilizer Γ_w in Γ is a group in $\mathcal{C}_{\ell-1}$. Let Γ_e denote the stabilizer of some edge e incident at w. Then $|(G \cap \Gamma_w)\backslash\Gamma_w/\Gamma_e|$ is bounded above by the finite number of edges of $X = G\backslash T$ incident at $Gw \in G\backslash T$ that are images of edges $\gamma e \in \Gamma e$. By inductive hypothesis, $G \cap \Gamma_w$ has finite index in Γ_w. Similarly, for each $\gamma \in \Gamma$, $G \cap \gamma\Gamma_w\gamma^{-1}$ has finite index in $\gamma\Gamma_w\gamma^{-1}$. Consider the action of Γ_w by right multiplication on $G\backslash\Gamma$: the orbits are the double cosets $G\backslash\Gamma/\Gamma_w$ and hence are finite in number because they index a subset of the vertices of $X = G\backslash T$; moreover the stabilizer of $G\gamma$ is $\gamma^{-1}G\gamma \cap \Gamma_w$, which we have just seen has finite index in Γ_w. Thus $G\backslash\Gamma$ is finite.

Case 2. Suppose that c acts hyperbolically on T, with axis A say.

Then the double coset hypothesis implies that the axes $\gamma(A)$, for $\gamma \in \Gamma$, belong to only finitely many G-orbits. On the other hand, the convex hull of $\bigcup_{\gamma\in\Gamma} \gamma(A)$ is a Γ-invariant subtree of T, and hence by minimality is the whole of T.

Let T_0 be the minimal G-invariant subtree of T. If $T_0 = T$ then $X = G\backslash T$ is finite since G is finitely generated, and so $|G\backslash\Gamma/\Gamma_e| < \infty$ for any edge-stabilizer Γ_e in Γ. If $\ell = 0$, then Γ_e is trivial, so $|\Gamma : G| < \infty$. Otherwise, choose e incident at a vertex w whose stabilizer Γ is in $\mathcal{C}_{\ell-1}$ and apply the inductive hypothesis as above to deduce that $|\Gamma : G| < \infty$.

It remains to consider the case $T_0 \neq T$.

Now, for any subgraph Y of T, and any $g \in G$, we have

$$d(g(Y), T_0) = d(g(Y), g(T_0)) = d(Y, T_0).$$

Since the Γ-orbit of A contains only finitely many G-orbits, there is a global upper bound K, say, on $d(\gamma(A), T_0)$ as γ varies over Γ.

Since $T \neq T_0$ and T is spanned by the Γ-orbit of A, there is a translate $\gamma(A)$ of A that is not contained in T_0. Recall that the action is k–acylindrical. Choose a vertex u on $\gamma(A)$ with $d(u, T_0) > K+k+2$ and let Γ_u denote its stabiliser in Γ. Let p be the vertex a distance K from T_0 on the unique shortest path from T_0 to u. Since $d(\gamma(A), T_0) \leq K$, the geodesic $[p, u]$ is contained in $\gamma(A)$. Similarly, $[p, u]$ is contained in any translate of A that passes through u. In particular, if $\delta \in \Gamma_u$ then $[p, u] \subset \delta\gamma(A)$, and since δ fixes u we have $\delta(p) = p$ or $\delta(p') = p$, where p' is the unique point of $\gamma(A)$ other than p with $d(u, p) = d(u, p')$.

If δ fixes the edge of $[p, u]$ incident at u, then $\delta(p) = p$ hence δ fixes $[p, u]$ pointwise, which contradicts the k-acylindricality of the action unless $\delta = 1$. Thus the stabiliser of this edge is trivial, which is a contradiction unless $\ell = 0$.

If $\ell = 0$ then, replacing u by an adjacent vertex if necessary, we may assume that $|\Gamma_u| > 2$. Choose distinct non-trivial elements $\delta_1, \delta_2 \in \Gamma_u$. It cannot be that all three of $\delta_1, \delta_2, \delta_1\delta_2^{-1}$ send p' to p. Thus one of them fixes p, hence $[p, u]$, which again contradicts the k-acylindricality of the action.

□

We are now able to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that $\Gamma \in \mathcal{C}$, $G < \Gamma$ is finitely generated, and N is a non-trivial normal subgroup of Γ that is contained in G. Then by definition of $\mathcal{C}$, Γ acts non-trivially, cocompactly and k-acylindrically on a tree T with cyclic edge stabilizers. There is no loss of generality in assuming the action is minimal, so we may apply Lemma 5.1 to see that the action of G is cocompact. The stabilizer Γ_e in Γ of an edge e is cyclic, and the finite number of edges in $G\backslash T$ is an upper bound on $|G\backslash\Gamma/\Gamma_e|$. It follows from Proposition 5.2 that $|\Gamma : G| < \infty$, as claimed. □

6. Nilpotent quotients

In this section we prove Theorem 4.2, which steers us away from the study of groups acting on trees and into the realm of nilpotent groups.

We first prove a general lemma (from [13]) about a subdirect product S of n arbitrary (not necessarily limit) groups $\Gamma_1, \dots, \Gamma_n$. As before, we write L_i for the normal subgroup $S \cap \Gamma_i$ of Γ_i. We also introduce the following notation. We write K_i for the kernel of the i-th projection map $p_i : S \to \Gamma_i$, and $N_{i,j}$ for the image of K_i under the j-th projection $p_j : S \to \Gamma_j$. Thus $N_{i,j}$ is a normal subgroup of Γ_j.

We shall denote by $[x_1, x_2, \dots, x_n]$ the left-normed n-fold commutator $[[..[x_1, x_2], x_3], \dots], x_n]$.

Lemma 6.1. $[N_{1,j}, N_{2,j}, \dots, N_{j-1,j}, N_{j+1,j}, \dots, N_{n,j}] \subset L_j$.

Proof. Suppose that $\nu_{i,j} \in N_{i,j}$ for a fixed j and for all $i \neq j$. Then there exist $\sigma_i \in S$ with $p_i(\sigma_i) = 1$ and $p_j(\sigma_i) = \nu_{i,j}$. Let σ denote the $(n-1)$-fold commutator $[\sigma_1, \dots, \sigma_{j-1}, \sigma_{j+1}, \dots, \sigma_n] \in S$. Then $p_j(\sigma)$ is the $(n-1)$-fold commutator

$$[\nu_{1,j}, \dots, \nu_{j-1,j}, \nu_{j+1,j}, \dots, \nu_{n,j}] \in \Gamma_j.$$

On the other hand, for $i \neq j$, we have $p_i(\sigma) = 1$ since $p_i(\sigma_i) = 1$. Hence $\sigma \in L_j$, and $p_j(\sigma) = \sigma \in L_j$.

Since the choice of $\nu_{i,j} \in N_{i,j}$ was arbitrary, we have

$$[N_{1,j}, N_{2,j}, \dots, N_{j-1,j}, N_{j+1,j}, \dots, N_{n,j}] \subset L_j$$

as claimed. □

We now consider a finitely generated subdirect product S of non-abelian limit groups $\Gamma_1, \dots, \Gamma_n$ such that $H_2(S_1;\mathbb{Q})$ is finite dimensional for every finite-index subgroup $S_1 < S$.

Let L_i, C_i and t_i be as in Proposition 3.1. We consider the image $A_{i,j} := p_j(p_i^{-1}(C_i))$ under the projection p_j of the preimage under p_i of the cyclic group C_i. Clearly $N_{i,j} < A_{i,j} < \Gamma_j$.

In the remainder of this section we shall prove that $N_{i,j} \subset \Gamma_j$ is of finite index for all i and j. Lemma 6.1 then implies that Γ_i/L_i is virtually nilpotent of class at most $n-2$, as is claimed in Theorem 4.2.

As a first step towards showing that $N_{i,j} \subset \Gamma_j$ is of finite index, we prove the following lemma.

Lemma 6.2. *Let $\Gamma_1, \dots, \Gamma_n$ be non-abelian limit groups. If $S < \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup with $H_2(S;\mathbb{Q})$ finite dimensional, and if S satisfies conditions (1) to (5) of Proposition 3.1, then for all i, j:*

1. *$|\Gamma_j : A_{i,j}| < \infty$;*
2. *$A_{i,j}/N_{i,j}$ is cyclic.*

Proof. (1) It suffices to consider the case $i = 1$. The HNN decomposition $\Gamma_1 = B_1 *_{C_1}$ described in Proposition 3.1 (5) pulls back to an HNN decomposition of S with stable letter $\widehat{t_1} = (t_1, 1, \dots, 1)$, base group $\widehat{B_1} = {p_1}^{-1}(B_1)$, and amalgamating subgroup $\widehat{C_1} = {p_1}^{-1}(C_1)$. As C_1 is cyclic, $\widehat{C_1} = K_1 \rtimes \langle \widehat{c_1} \rangle$ where $\widehat{c_1}$ is a choice of a lift of a generator of C_1. Consider the Mayer-Vietoris sequence for the HNN decomposition of S.

$$\cdots \to H_2(S;\mathbb{Q}) \to H_1(\widehat{C_1};\mathbb{Q}) \xrightarrow{\phi} H_1(\widehat{B_1};\mathbb{Q}) \to H_1(S;\mathbb{Q}) \to \cdots$$

The map ϕ is the difference between the map induced by inclusion and the map induced by the inclusion twisted by the action of $\widehat{t_1}$ by conjugation. Notice that $\widehat{t_1}$ commutes with K_1 and so acts trivially on $H_*(K_1;\mathbb{Q})$. Thus ϕ factors through the map $H_1(\widehat{C_1};\mathbb{Q}) \to H_1(\langle \widehat{c_1} \rangle;\mathbb{Q})$, in particular the image of ϕ has dimension at most 1. Since $H_2(S;\mathbb{Q})$ is finite dimensional by hypothesis, it follows that $H_1(\widehat{C_1};\mathbb{Q})$ is finite dimensional. For each j, $A_{1,j} = p_j(\widehat{C_1})$ is a homomorphic image of $\widehat{C_1}$, so $H_1(A_{1,j};\mathbb{Q})$ is finite-dimensional. Since $A_{1,j}$ is a subgroup of the non-abelian limit group Γ_j, it follows that it is finitely generated. Since it contains the non-trivial normal subgroup L_j, Theorem 4.1 now implies that $A_{1,j}$ has finite index in Γ_j, as claimed.

(2) As p_j is surjective, $A_{i,j}/N_{i,j} = p_j(\widehat{C}_i)/p_j(K_i)$ is a homomorphic image of $\widehat{C}_i/K_i$, so it is also cyclic, as claimed. □

The other crucial ingredient in the proof of Theorem 4.2 is the following proposition.

Proposition 6.3. *Let G be an HNN extension of the form $B*_C$ with stable letter t, finitely generated base-group B and infinite-cyclic edge group C. Suppose that G has normal subgroups L and N such that $t \in L$, $C \cap N = \{1\}$ and G/N is infinite-cyclic. Suppose further that $H_1(N;\mathbb{Q})$ is infinite dimensional. Let $\Delta \subset G$ be the unique subgroup of index 2 that contains B. Then, there exists an element $x \in L \cap B \cap N$ such that $R\overline{x} \subset H_1(N \cap \Delta;\mathbb{Q})$ is a free R-module of rank 1, where $R = \mathbb{Q}[\Delta/(N \cap \Delta)]$ and $\overline{x}$ is the homology class determined by x.*

Proof. Let T be the Bass-Serre tree of the splitting $G = B*_C$ and consider the graph of groups decomposition of $N_2 := N \cap \Delta$ with underlying graph $X = N_2\backslash T$; since N_2C has finite index in G, this is a finite graph. Each vertex group in this decomposition is a conjugate of $B \cap N_2$, and the edge groups are trivial since $C \cap N_2 = \{1\}$.

Thus, as an abelian group, $H_1(N_2;\mathbb{Q})$ is the direct sum of $H_1(X;\mathbb{Q})$ and p copies of $H_1(B \cap N_2;\mathbb{Q})$, where p is the index of BN_2 in G. The first of these summands is finite-dimensional, and hence $H_1(B\cap N_2;\mathbb{Q})$ is infinite-dimensional (since $H_1(N;\mathbb{Q})$ is infinite-dimensional, implying that $H_1(N_2;\mathbb{Q})$ is too).

Let τ be a generator of G/N. Then $M := H_1(B\cap N_2;\mathbb{Q})$ is a $\mathbb{Q}[\tau^{\pm p!}]$-module, which is finitely generated because B is finitely generated and $B/(B \cap N_2)$ is finitely presented. Since $\mathbb{Q}[\tau^{\pm p!}]$ is a principal ideal domain, the module M has a free direct summand. We fix $z \in B\cap N_2$ so that $\overline{z} \in M$ generates this free summand. It follows that $R\overline{z}$ has infinite $\mathbb{Q}$-dimension, and so is a free submodule of the R-module $H_1(N_2;\mathbb{Q})$.

Since $t \notin \Delta$, $z_1 := z$ and $z_2 := tzt^{-1}$ belong to distinct vertex groups in X. Hence $x := [z,t] = z_1z_2^{-1} \in L \cap N \cap \Delta$ is such that $\overline{x} = \overline{z}_1 - \overline{z}_2$ generates a free $\mathbb{Q}[\tau^{\pm p!}]$-submodule of $H_1(N_2;\mathbb{Q})$, and hence also a free R-submodule. □

The following proposition completes the proof of Theorem 4.2.

Proposition 6.4. *Let $\Gamma_1,\dots,\Gamma_n$ be non-abelian limit groups. If $S < \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup with $H_2(S_1;\mathbb{Q})$ finite dimensional for each subgroup S_1 of finite index in S, and if S satisfies conditions (1) to (5) of Proposition 3.1, then (in the notation of Lemma 6.2) $N_{i,j} \subset \Gamma_j$ is of finite index for all i and j.*

Proof. It suffices to consider the case $(i,j) = (2,1)$. Let T be the projection of S to $\Gamma_1 \times \Gamma_2$, and define $M_i = T \cap \Gamma_i$ for $i = 1,2$. Notice that $M_1 = N_{2,1}$, the projection to Γ_1 of the kernel of the projection $p_2 : S \to \Gamma_2$, and similarly $M_2 = N_{1,2}$.

Since S projects onto each of Γ_1 and Γ_2, the same is true of T. Hence we have isomorphisms

$$\frac{\Gamma_1}{M_1} \cong \frac{T}{M_1 \times M_2} \cong \frac{\Gamma_2}{M_2}.$$

We will assume that these groups are infinite, and obtain a contradiction.

By Lemma 6.2, $T/(M_1 \times M_2)$ is virtually cyclic, so we may choose a finite index subgroup $T_0 < T$ containing $M_1 \times M_2$ such that $T_0/(M_1 \times M_2)$ is infinite cyclic. Hence $G_i := p_i(T_0)$ is a finite-index subgroup containing M_i for $i = 1, 2$, such that G_i/M_i is infinite cyclic. Choose $\tau \in T_0$ such that $\tau.(M_1 \times M_2)$ generates $T_0/(M_1 \times M_2)$, and let $\tau_i = p_i(\tau) \in G_i$ for $i = 1, 2$.

The HNN-decomposition of Γ_i from Proposition 3.1 (5) induces an HNN decomposition $G_i = B'_i *_{C'_i}$ with stable letter $t'_i \in L_i$, where $C'_i = C_i \cap G_i$ and t'_i an appropriate power of the stable letter t_i of Γ_i. Notice that, by Lemma 6.2, $C'_i \cap M_i = \{1\}$. For each $i = 1, 2$, Proposition 6.3 (with $G = G_i$, $N = M_i$, $L = L_i$, $t = t'_i$, $B = B'_i$, $C = C'_i$) provides an index 2 subgroup Δ_i in G_i and an element $x_i \in M_i \cap \Delta_i \cap L_i$ such that $\overline{x}_i$ generates a free $\mathbb{Q}[\tau_i^{\pm 1}]$-submodule of $H_1(M_i \cap \Delta_i; \mathbb{Q})$.

Now define $M'_i := M_i \cap \Delta_i$. It follows that $\overline{x}_1 \otimes \overline{x}_2$ generates a free $\mathbb{Q}[\tau_1^{\pm 1}, \tau_2^{\pm 1}]$-submodule of

$$H_1(M'_1; \mathbb{Q}) \otimes_{\mathbb{Q}} H_1(M'_2; \mathbb{Q}) \subset H_2(M'_1 \times M'_2; \mathbb{Q}).$$

Let T_1 be the finite-index subgroup of T_0 defined by $T_1 := (M'_1 \times M'_2) \rtimes \langle \tau \rangle$, and let $S_1 < S$ be the preimage of T_1 under the projection $S \to T$. Using the LHS spectral sequence for the short exact sequence $M'_1 \times M'_2 \to T_1 \to \langle \tau \rangle$, we see that

$$H_0(\langle \tau \rangle; H_2(M'_1 \times M'_2; \mathbb{Q})) \subset H_2(T_1; \mathbb{Q})$$

has an infinite dimensional $\mathbb{Q}$-subspace generated by the images of

$$\{(\tau_1^m x_1 \tau_1^{-m}) \otimes (\tau_2^n x_2 \tau_2^{-n});\ m, n \in \mathbb{Z}\}.$$

In particular, the image of the map $H_2(L_1 \times L_2; \mathbb{Q}) \to H_2(T_1; \mathbb{Q})$ induced by inclusion is infinite-dimensional. But this contradicts the hypothesis that $H_2(S_1; \mathbb{Q})$ is finite dimensional, since the inclusion $(L_1 \times L_2) \to T_1$ factors through S_1. This is the desired contradiction which completes the proof. □

7. Normal subgroups with cyclic quotient

Proposition 7.1. *If $\Gamma_1, \dots, \Gamma_n$ are groups of type $\mathrm{FP}_n(\mathbb{Z})$ and $\phi : \Gamma_1 \times \cdots \times \Gamma_n \to \mathbb{Z}$ has non-trivial restriction to each factor, then $H_j(\ker \phi; \mathbb{Z})$ is finitely generated for $j \le n - 1$.*

Proof. We first prove the result in the special case where the restriction of ϕ to each factor is epic. Thus we may write $\Gamma_i = L_i \rtimes \langle t_i \rangle$ where $S = \ker \phi$, $L_i = S \cap \Gamma_i$ is the kernel of $\phi|_{\Gamma_i}$ and $\phi(t_i)$ is a fixed generator of $\mathbb{Z}$.

If $n \ge 2$ and we fix a finite set $A_i \subset L_i$ such that $\Gamma_i = \langle A_i, t_i \rangle$, then S is generated by $A_1 \cup \cdots \cup A_n \cup \{t_1 t_2^{-1}, \dots, t_1 t_n^{-1}\}$.

We proceed by induction on n (the initial case $n = 1$ being trivial), considering the LHS spectral sequence in homology for the projection of S to Γ_n,

$$1 \to S_{n-1} \to S \xrightarrow{p_n} \Gamma_n \to 1,$$

where S_{n-1} is the kernel of the restriction of ϕ to $\Gamma_1 \times \cdots \times \Gamma_{n-1}$. In particular, the inductive hypothesis applies to S_{n-1}.

Since Γ_n is of type $\mathrm{FP}_n(\mathbb{Z})$ and $H_q(S_{n-1}; \mathbb{Z})$ is finitely generated for $q \leq n-2$, by induction, on the E^2 page of the spectral sequence there are only finitely generated groups in the rectangle $0 \leq p \leq n$ and $0 \leq q \leq n-2$. It follows that all of the groups on the E^∞ page that contribute to $H_j(S; \mathbb{Z})$ with $j \leq n-1$ are finitely generated, with the possible exception of that in position $(0, n-1)$.

On the E^2 page, the group in position $(0, n-1)$ is $H_0(\Gamma_n; H_{n-1}(S_{n-1}; \mathbb{Z}))$, which is the quotient of $H_{n-1}(S_{n-1}; \mathbb{Z})$ by the action of Γ_n. This action is determined by taking a section of $p_n : S \to \Gamma_n$ and using the conjugation action of S. The section we choose is that with image $L_n \rtimes \langle t_1 t_n^{-1} \rangle$. Since L_n and t_n commute with S_{n-1}, we have

$$H_0(\Gamma_n; H_{n-1}(S_{n-1}; \mathbb{Z})) = H_0(\langle t_1 \rangle; H_{n-1}(S_{n-1}; \mathbb{Z})) \ .$$

The latter group is the $(0, n-1)$ term on the E^2 page of the spectral sequence for the extension

$$1 \to S_{n-1} \to \Gamma_1 \times \cdots \times \Gamma_{n-1} \xrightarrow{\phi} \mathbb{Z} \to 1.$$

This is a 2-column spectral sequence, so the E^2 page coincides with the E^∞ page. Since $\Gamma_1 \times \cdots \times \Gamma_{n-1}$ is of type FP_{n-1} (indeed of type FP_n), it follows that $H_0(\langle t_1 \rangle; H_{n-1}(S_{n-1}; \mathbb{Z}))$ is finitely generated, and the induction is complete.

For the general case, replace $\mathbb{Z}$ by the finite index subgroup $\phi(\Gamma_1) \cap \cdots \cap \phi(\Gamma_n)$ ($= m\mathbb{Z}$, say); replace each Γ_i by the finite-index subgroup $\Delta_i = \Gamma_i \cap \phi^{-1}(m\mathbb{Z})$, and replace S by the finite-index subgroup $T = S \cap (\Delta_1 \times \cdots \times \Delta_n)$. Since $\phi(\Delta_i) = m\mathbb{Z}$ for each i, the above special-case argument applies to T, to show that $H_j(T; \mathbb{Z})$ is finitely generated for each $0 \leq j \leq n-1$. Moreover, T is normal in S, and we may consider the LHS spectral sequence of the short exact sequence

$$1 \to T \to S \to S/T \to 1.$$

On the E^2 page of this spectral sequence, the terms E^2_{pq} in the region $0 \leq q \leq n-1$ are homology groups of the finite group T/S with coefficients in the finitely generated modules $H_q(T; \mathbb{Z})$, and so they are finitely generated abelian groups. But all the terms that contribute to $H_j(S; \mathbb{Z})$ for $0 \leq j \leq n-1$ lie in this region, so $H_j(S; \mathbb{Z})$ is finitely generated for $j \leq n-1$, as required. □

Theorem 7.2. *Let $\Gamma_1, \ldots, \Gamma_n$ be non-abelian limit groups and let S be the kernel of an epimorphism $\phi : \Gamma_1 \times \cdots \times \Gamma_n \to \mathbb{Z}$. If the restriction of ϕ to each of the Γ_i is epic, then $H_n(S; \mathbb{Q})$ has infinite $\mathbb{Q}$-dimension.*

Proof. The proof is by induction on n. The case $n = 1$ was established in [8]: the group $S = \ker \phi$ is a normal subgroup of the non-abelian limit group Γ_1, and if $H_1(S, \mathbb{Q})$ were finite dimensional then S would be finitely generated, and hence would have finite index in Γ_1.

The preceding proposition shows that $H_j(S; \mathbb{Z})$ is finitely generated, and hence $H_j(S; \mathbb{Q})$ is finite dimensional for $j < n$. As in the proof of that proposition, we consider the LHS spectral sequence for

$$1 \to S_{n-1} \to S \stackrel{p_n}{\to} \Gamma_n \to 1,$$

now with $\mathbb{Q}$-coefficients. There are now only finitely generated $\mathbb{Q}$-modules in the region $0 \le q \le n-2$. (Recall that Γ_i is of type FP_∞.) In particular, the terms on the E^2 page involved in the calculation of $H_n(S; \mathbb{Q})$ are all finitely generated except for

$$H_0(\Gamma_n; H_n(S_{n-1}; \mathbb{Q})) = H_0(\langle t_1 \rangle; H_n(S_{n-1}; \mathbb{Q}))$$

and

$$H_1(\Gamma_n; H_{n-1}(S_{n-1}; \mathbb{Q})).$$

It suffices to prove that the latter is infinite dimensional over $\mathbb{Q}$. (The former is actually finite dimensional, but this is irrelevant.)

The module $M = H_{n-1}(S_{n-1}; \mathbb{Q})$ is a homology group of the kernel of a map from an FP_∞ group to $\mathbb{Z}$. It is thus a homology group of a chain complex of free $R = \mathbb{Q}[t, t^{-1}]$ modules of finite rank. The ring R is Noetherian, so such a homology group is finitely generated as an R-module. By the inductive hypothesis, M has infinite $\mathbb{Q}$-dimension. So by the classification of finitely generated modules over a principal ideal domain, M has a free direct summand, that is $M = M_0 \oplus R$.

The Γ_n-action on M factors through the quotient $\Gamma_n \to \Gamma_n/L_n = \langle t_n \rangle$, since L_n acts trivially, so the direct sum decomposition passes to M considered as a $\mathbb{Q}\Gamma_n$ module. Hence $H_1(\Gamma_n; M) = H_1(\Gamma_n; M_0) \oplus H_1(\Gamma; R)$.

Finally, as a $\mathbb{Q}\Gamma_n$ module, $R = \mathbb{Q}\Gamma_n \otimes_{\mathbb{Q}L_n} \mathbb{Q}$, so by Shapiro's Lemma $H_1(\Gamma_n; R) \cong H_1(L_n; \mathbb{Q})$ (see for instance [15, III.6.2. and III.5]) .

As L_n is an infinite index normal subgroup of a non-abelian limit group, it is not finitely generated, and therefore neither is the $\mathbb{Q}$-module $H_1(L_n; \mathbb{Q})$ [8]. □

Theorem 4.4 follows immediately from Theorem 7.2 in the light of the Künneth formula, after one has passed to a subgroup of finite index to ensure that whenever $\Gamma_i \to \mathbb{Z}$ is non-trivial it is onto.

8. Completion of the proof of the Main Theorem

The following lemma and its corollary provide an extension to the virtual context of known results about finitely generated nilpotent groups. We shall apply them to direct products of the virtually nilpotent quotients of Γ_i/L_i resulting from Theorem 4.2.

Lemma 8.1. *Let G be a finitely generated virtually nilpotent group and let $\overline{S}$ be a subgroup of infinite index. Then there exists a subgroup K of finite index in G and an epimorphism $f : K \to \mathbb{Z}$ such that $(\overline{S} \cap K) \subset \ker(f)$.*

Proof. We argue by induction on the Hirsch length $h(G)$, which is strictly positive, since G is infinite.

In the initial case, $h(G) = 1$ means that G has an infinite cyclic subgroup K of finite index. Since $\overline{S}$ has infinite index in G, $\overline{S}$ is finite, so $(\overline{S} \cap K)$ is trivial, and we can take $f : K \to \mathbb{Z}$ to be an isomorphism.

For the inductive step, let H be a finite index torsion-free subgroup of G, and C an infinite cyclic central subgroup of H. If $C\overline{S}$ has infinite index in G, then the inductive hypothesis applies to H/C and we are done. Otherwise, $\overline{S}$ has infinite index in $C\overline{S}$, so $C \cap \overline{S}$ has infinite index in $C \cong \mathbb{Z}$. But then $C \cap \overline{S} = \{1\}$, and since $C < H$, it follows that $C\overline{S} \cap H = C \times (\overline{S} \cap H)$. Put $K = C\overline{S} \cap H$ and let f be the projection $K \to C$ with kernel $\overline{S} \cap H$. □

We note that Lemma 8.1 would not remain true if one assumed only that G were polycyclic. For example, it fails for lattices $G = \mathbb{Z}^2 \rtimes \langle t \rangle$ in the 3-dimensional Lie group Sol if one takes $S = \langle t \rangle$.

Repeated applications of Lemma 8.1 yield the following.

Corollary 8.2. *Let G be a finitely generated, virtually nilpotent group and let $\overline{S}$ be a subgroup of G. Then there is a subnormal chain $\overline{S}_0 < \overline{S}_1 < \cdots < \overline{S}_r = G$, where $\overline{S}_0$ is a subgroup of finite index in $\overline{S}$ and for each i the quotient group $\overline{S}_{i+1}/\overline{S}_i$ is either finite or cyclic.*

For the benefit of topologists, we should note that the following algebraic argument is modelled on the geometric proof of the Double Coset Lemma in [9].

Proof of Theorem C.

Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. Recall that the Γ_i are non-abelian, the projections $p_i : S \to \Gamma_i$ are surjective, and the intersections $L_i = S \cap \Gamma_i$ are non-trivial. By passing to a subgroup of finite index we may assume that each Γ_i splits as in Proposition 3.1(5). Let $L = L_1 \times \cdots \times L_n$.

We only need consider the case when S has infinite index in Γ. We shall derive a contradiction from the assumption that for all subgroups $S_0 < S$ of finite index and for all $0 \leq j \leq n$, $H_j(S_0, \mathbb{Q})$ is finite-dimensional.

From Theorem 4.2 we know that each of the quotient groups Γ_i/L_i is virtually nilpotent, and hence so is Γ/L.

Since $L \subset S$ and S has infinite index in Γ, the image $\overline{S}$ of S in Γ/L is of infinite index and we may apply Lemma 8.1 with Γ/L in the role of G. Let $\Lambda < \Gamma$ be the preimage of the subgroup K provided by the lemma. Note that Λ has finite index in Γ, contains L, and admits an epimorphism $f : \Lambda \to \mathbb{Z}$ such that $S \cap \Lambda \subset \ker(f)$. As in (2.5), we may replace the groups Γ_i and S by finite-index subgroups so as to ensure that $L \subset S \subset N$, where N is the kernel of an epimorphism $\Gamma \to \mathbb{Z}$. By Theorem 4.4, there is a finite index subgroup $N_0 < N$ and an integer $j \le n$ such that $H_j(N_0; \mathbb{Q})$ is infinite dimensional.

By Corollary 8.2 (applied to the image of $S \cap N_0$ in Γ/L) there is a subgroup S_0 contained in $S \cap N_0$, which has finite index in S, and a subnormal chain of subgroups $S_0 \lhd S_1 \lhd \cdots \lhd S_k = N_0$ with S_{i+1}/S_i either finite or cyclic for each i. We now use the following lemma to contradict the assumption that $H_j(S_0; \mathbb{Q})$ is finite-dimensional.

Lemma 8.3. *Let $S_0 \lhd S_1$ be groups with S_1/S_0 finite or cyclic. If $H_j(S_0; \mathbb{Q})$ is finite dimensional for $0 \le j \le n$, then $H_j(S_1; \mathbb{Q})$ is finite dimensional for $0 \le j \le n$.*

Proof. In the LHS spectral sequence for the group extension $S_0 \to S_1 \to (S_1/S_0)$ we have $E^2_{p,q} = H_p(S_1/S_0; H_q(S_0; \mathbb{Q}))$. By hypothesis, $E^2_{p,q}$ has finite $\mathbb{Q}$-dimension for $q \le n$. Moreover, $E^2_{p,q} = 0$ for $p > 1$, since S_1/S_0 has homological dimension at most 1 over $\mathbb{Q}$. Thus the derivatives on the E^2 page all vanish and the spectral sequence stabilizes at the E^2 page. Hence, for $0 \le j \le n$, we have

$$\dim_{\mathbb{Q}}(H_j(S_1; \mathbb{Q})) = \dim_{\mathbb{Q}}(E^2_{0,j}) + \dim_{\mathbb{Q}}(E^2_{1,j-1}) < \infty,$$

as required. □

Repeatedly applying this lemma to the subnormal sequence $S_0 \lhd S_1 \lhd \cdots \lhd S_k = N_0$ implies that $H_j(N_0; \mathbb{Q})$ is finite dimensional for all $j \le n$, contradicting Theorem 4.4.

□

This completes the proof of Theorem C, from which Theorem A follows immediately.

9. From Theorem C to Theorem B

Let Γ_i, L_i and S be as in the statement of Theorem B, but without necessarily assuming that the L_i are non-abelian for all i. We first discuss how this situation differs from the special case stated in Theorem B.

If some L_i is trivial, then S is isomorphic to a subgroup of the direct product of the Γ_j with $j \ne i$, as in Proposition 3.1 (3). We now assume that $L_i \ne \{1\}$ for each i.

As in Proposition 3.1 (2), we may replace each Γ_i by $p_i(S)$, where $p_i : S \to \Gamma_i$ is the projection, and hence assume that p_i is surjective, and so each L_i is normal in Γ_i.

If some L_i is non-trivial and abelian, then it is free abelian of finite rank, by [6, Corollary 1.23]. Since L_i is normal, it has finite index in

Γ_i, and it follows immediately from the ω-residually free property that Γ_i is itself abelian.

Arguing as in Proposition 3.1 (4), we may assume that only one of the Γ_i is abelian, say Γ_1, and that L_1 is the only non-trivial abelian L_i. We may also assume that L_1 is a direct factor of Γ_1; say $\Gamma_1 = L_1 \times M_1$. But then S virtually splits as a direct product $L_1 \times S'$, where $S' = S \cap (\Gamma_2 \times \cdots \Gamma_n)$.

Note that the above reduction involved only one passage to a finite index subgroup, and that was within the abelian factor Γ_1. The other Γ_i and L_i are left unchanged. In particular, the L_i remain non-abelian.

We have now reduced to the situation of the statement of Theorem B, with the additional hypothesis that each $p_i : S \to \Gamma_i$ is surjective.

In particular, each L_i is normal in Γ_i, and hence is of finite index for $i = 1, \ldots, r$.

Let $\Pi_r : \Gamma_1 \times \cdots \times \Gamma_n \to \Gamma_1 \times \cdots \times \Gamma_r$ be the natural projection, let $\Lambda = L_1 \times \cdots \times L_r$ and let $\hat{S}_0 = S \cap \Pi_r^{-1}(\Lambda)$. Then $\hat{S}_0$ has finite index in S and $\hat{S}_0 = \Lambda \times \hat{S}_2$, where $\hat{S}_2 = \hat{S}_0 \cap (\Gamma_{r+1} \times \cdots \times \Gamma_n)$. Theorem C now says that that $\hat{S}_2$ has a subgroup of finite index S_2 with $H_k(S_2; \mathbb{Q})$ infinite dimensional for some $k \leq n - r$.

References

[1] E. Alibegović and M. Bestvina, *Limit groups are CAT(0)*, J. London Math. Soc. (2) **74** (2006), 259–272.

[2] B. Baumslag, *Residually Free Groups*, Proc. London Math. Soc. (3) **17** (1967); 402–418.

[3] G. Baumslag and J. E. Roseblade, *Subgroups of direct products of free groups*, J. London Math. Soc. (2) **30** (1984), 44–52.

[4] R. Bieri, "Homological dimension of discrete groups", Queen Mary College Mathematics Notes (1976).

[5] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups.*, Invent. Math. **129** (1997), 445–470.

[6] M. Bestvina and M. Feighn, *Notes on Sela's work: Limit groups and Makanin-Razborov diagrams*, To appear in *Geometric and cohomological methods in group theory* (M.R. Bridson, P.H. Kropholler, I.J. Leary, eds.).

[7] M.R. Bridson, *On the subgroups of semihyperbolic groups*, in "Essays on geometry and related topics", Monogr. Enseign. Math., **38**, pp. 85–111, Geneva, 2001.

[8] M.R. Bridson and J. Howie, *Normalizers in limit groups*, Math. Ann. **337** (2007), 385–394.

[9] M.R. Bridson and J. Howie, *Subgroups of direct products of elementarily free groups*, Geom. Func. Anal., **17** (2007), 385–403.

[10] M.R. Bridson and J. Howie, *Subgroups of direct products of two limit groups*, Math. Res. Lett. **14** (2007), 547–558.

[11] M.R. Bridson, J. Howie, C.F. Miller III and H. Short, The subgroups of direct products of surface groups, Geometriae Dedicata **92** (2002), 95–103.

[12] M.R. Bridson, J. Howie, C.F. Miller III and H. Short, *Finitely presented residually free groups*, in preparation.

[13] M.R. Bridson and C.F. Miller III, *Structure and finiteness properties of subdirect products of groups*, arXiv:0708.4331.

[14] M.R. Bridson and H. Wilton, *Subgroup separability in residually free groups*, Math. Z., to appear.

[15] K.S. Brown, Cohomology of groups, Springer-Verlag 1982.

[16] C. Champetier and V. Guirardel, *Limit groups as limits of free groups*, Israel J. Math. **146** (2005), 1–75.

[17] T. Delzant and M. Gromov, *Cuts in Kähler groups*, in: Infinite groups: geometric, combinatorial and dynamical aspects, (L. Bartholdi, T. Ceccherini-Silberstein, T. Smirnova-Nagnibeda and A. Zuk, eds) 31–55, Progr. Math., 248, Birkhäuser, Basel, 2005.

[18] O.G. Kharlampovich and A.G. Myasnikov, *Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz*, J. Algebra **200** (1998), 472–516.

[19] O.G. Kharlampovich and A.G. Myasnikov, *Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups*, J. Algebra **200** (1998), 517–570.

[20] O.G. Kharlampovich and A.G. Myasnikov, *Elementary theory of free non-abelian groups*, J.Algebra **302** (2006), 451–552.

[21] V. N. Remeslennikov, $\exists$-*free groups*, Sibirsk. Mat. Zh. **30** (1989), no. 6, 193–197.

[22] Z. Sela, *Diophantine geometry over groups. I. Makanin-Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 93, 31–105.

[23] Z. Sela, *Diophantine geometry over groups. II. Completions, closures and formal solutions*, Israel J. Math. **134** (2003), 173–254.

[24] Z. Sela, *Diophantine geometry over groups VI: The elementary theory of a free group*, Geom. Funct. Anal. **16** (2006), 707–730.

[25] Z. Sela, *Diophantine geometry over groups: a list of research problems*, preprint, http://www.ma.huji.ac.il/∼zlil/problems.dvi.

[26] J. P. Serre, *Cohomologie des groupes discrets* Ann. of Math. Studies **70** (1971), 77–169.

[27] J.-P. Serre, “Trees”, Springer-Verlag, Berlin-New York, 1980.

[28] J. R. Stallings, *A finitely presented group whose 3–dimensional homology group is not finitely generated*, Amer. J. Math. , **85**, (1963) 541–543.

[29] C. T. C. Wall, *Finiteness conditions for* CW*-complexes*, Ann. of Math. (2) **81** (1965), 56–69.

[30] H. Wilton, *Hall's Theorem for limit groups*, Geom. Func. Anal., to appear.

Martin R. Bridson, Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB, U.K.
E-mail address: bridson@maths.ox.ac.uk

James Howie, Department of Mathematics and Maxwell Institute for Mathematical Sciences, Heriot–Watt University, Edinburgh EH14 4AS
E-mail address: jim@ma.hw.ac.uk

Charles F. Miller III, Department of Mathematics and Statistics, University of Melbourne, Melbourne 3010, Australia
E-mail address: c.miller@ms.unimelb.edu.au

Hamish Short, L.A.T.P., U.M.R. 6632, Centre de Mathématiques et d'Informatique, 39 Rue Joliot–Curie, Université de Provence, F–13453, Marseille cedex 13, France
E-mail address: hamish@cmi.univ-mrs.fr