Subgroups of direct products of elementarily free groups

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Abstract

We exploit Zlil Sela's description of the structure of groups having the same elementary theory as free groups: they and their finitely generated subgroups form a prescribed subclass $\mathcal{E}$ of the hyperbolic limit groups. We prove that if $G_1, \ldots, G_n$ are in $\mathcal{E}$ then a subgroup $\Gamma \subset G_1 \times \cdots \times G_n$ is of type FP_n if and only if Γ is itself, up to finite index, the direct product of at most n groups from $\mathcal{E}$. This answers a question of Sela.

Examples of Stallings [24] and Bieri [4] show that finitely presented subgroups of a direct product of finitely many free groups can be rather wild. In contrast, Baumslag and Roseblade [1] proved that the only finitely presented subgroups of a direct product of *two* free groups are the obvious ones: such a subgroup is either free or else has a subgroup of finite index that is the product of its intersections with the factors.

In [9] Miller, Short and the present authors explained this apparent contrast by showing that all of the exotic behaviour among subdirect products of free groups arises from a lack of homological finiteness. More precisely, if a subgroup S of a direct product of n free groups has finitely generated homology up to dimension n, then S has a subgroup of finite index that is isomorphic to a direct product of free groups.

We proved a similar theorem for subdirect products of surface groups. This has implications for the understanding of the fundamental groups of compact Kähler manifolds. Indeed the remarkable work of Delzant and Gromov [11] shows that if the fundamental group Γ of such a manifold is torsion-free and has sufficiently many multi-ended splittings, then there is a short exact sequence $1 \to \mathbb{Z}^n \to \Gamma_0 \to S \to 1$, where S is a subdirect product of surface groups and Γ_0 is a subgroup of finite index in Γ.

Different attributes of surface groups enter the proof in [9] in subtle ways making it difficult to assess which are vital and which are artifacts of the proof. Examples show that the splitting phenomenon for subdirect products does not extend to arbitrary 2-dimensional hyperbolic groups (even small cancellation groups), nor to fundamental groups of higher-dimensional hyperbolic manifolds [5]. However, advances in geometric group theory during the last few years suggest that there is a class of groups which is much more fundamentally tied to surface groups than either of the these classes, namely *limit groups*.

Kharlampovich and Myasnikov have studied limit groups extensively under the name *fully residually free groups* [14], [15]. Remeslennikov [16] had previously referred to them as ∃-free groups, reflecting the fact that these are precisely the groups that have the same existential theory as a free group. The name *limit group* was introduced by Zlil Sela to emphasise the fact that these are precisely the class of groups that arise when one takes limits of stable sequences of homomorphisms $\phi_n : G \to F$, where G is an arbitrary finitely generated group and F is a free group (see section 1).

The existential theory of a group G is the set of first order sentences in the language of group theory that contain only one quantifier ∃ and are true in G. The *elementary theory* of G is the set of all first order sentences that are true in G. Famously, Alfred Tarski asked which groups had the same elementary theory as a free group. This problem was solved by Zlil Sela ([19] to [21]). At about the same time, a somewhat different solution was proposed by Kharlampovich and Myasnikov, but our results here rely on Sela's treatment, in particular his hierarchical description of this class of groups (Definition 1.1). We refer to a group as *elementarily free* if it has the same elementary theory as a free group. This is an important subclass of the class of limit groups.

The following is our main result.

Theorem 0.1 *Let $G_1, \dots, G_n$ be (subgroups of) elementarily free groups and let $\Gamma \subset G_1 \times \cdots \times G_n$ be a subgroup. Then Γ is of type $\mathrm{FP}_n(\mathbb{Q})$ if and only if there are finitely generated subgroups $H_i \subset G_i$ for $i = 1, \dots, n$ such that Γ is isomorphic to a subgroup of finite-index in $H_1 \times \cdots \times H_n$.*

For simplicity of exposition, all homology groups considered in this paper will be with trivial coefficient module $\mathbb{Q}$, so it is natural to use the finiteness condition $\mathrm{FP}_n(\mathbb{Q})$ in the statement of our theorem. However, with minor modifications, our arguments also apply with other trivial coefficient modules, giving corresponding results under the finiteness conditions $\mathrm{FP}_n(R)$ for other suitable rings R.

Theorem 0.1 answers a question of Sela [22, I(12)], who also asked if the stronger version of the above result holds, in which 'elementarily free groups' is replaced by 'limit groups'. We conjecture that the stronger result is indeed true, but it cannot be proved using the methods of this paper alone. In a forthcoming article [8], we use a different technique to prove the stronger conjecture in the case $n = 2$.

By combining a slight refinement of Theorem 0.1 (see 5.1) with Theorem 3 of [7] we obtain:

Theorem 0.2 *Let $G_1, \ldots, G_n$ be (subgroups of) elementarily free groups and $\Gamma \subset G_1 \times \cdots \times G_n$ a subgroup such that $L_i := \Gamma \cap G_i \neq \{1\}$ for $i = 1, \ldots, n$.*

If L_i is finitely generated for $1 \leq i \leq r$ and not finitely generated for $i > r$, then there is a subgroup of finite index $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 = \Gamma_1 \times \Gamma_2$, where Γ_1 is a direct product of r finitely generated subgroups of elementarily free groups, and $H_k(\Gamma_2, \mathbb{Q})$ is infinite dimensional for some $k \leq n - r$ (unless $r = n$).

In [9] we deduced a special case of Theorem 0.1 from a corresponding special case of Theorem 0.2. The latter was proved using an induction that involved the Lyndon-Hochschild-Serre spectral sequence and an analysis of boundary maps using the Fox calculus. The proof of Theorem 0.1 presented here lays bare some of the geometry obscured by the algebraic machinery in [9]. In particular, in the case where S is not of type FP_n, we construct an *explicit family of topological cycles* exhibiting this lack of finiteness.

One of the ingredients in our proof of Theorem 0.1 is worthy of particular mention. Marshall Hall's theorem [13] implies that every non-trivial element of a finitely generated free group is primitive (i.e. generates a free factor) in a subgroup of finite index. Correspondingly, Peter Scott [17] proved that any non-trivial element γ of the fundamental group Σ of a closed surface is represented by a non-separating simple closed curve in some finite-sheeted covering of that surface: algebraically, this means that there is a subgroup of finite index $\Sigma_0 \subset \Sigma$ such that Σ_0 is an HNN extension $S*_C$ where C is infinite cyclic and the stable letter is γ. In Section 3 we prove the following common generalization of these theorems.

Theorem 0.3 *Suppose that Γ splits as an amalgamated free product $G_1 *_A G_2$ or HNN extension G_1*_A, where A is closed in the profinite topology on Γ. If $\gamma \in \Gamma$ is not conjugate to an element in G_1 or G_2, then there is a subgroup of finite index $\Gamma_0 \subset \Gamma$ that splits as an HNN extension $B*_{A'}$, where $A' = A \cap \Gamma_0$ and the stable letter is γ.*

This article is organised as follows. In Section 1 we describe those elements of the structure theory of limit groups that we shall need. In Section 3 we prove Theorem 0.3; our proof relies on the results in Bass-Serre theory presented in Section 2. In Section 4 we prove the Double Coset Lemma (Theorem 4.1), which provides a criterion for homological finiteness in subdirect products of HNN-extensions. In Section 5 we marshal the results of previous sections to prove Theorem 0.1.

1 Limit Groups

Limit groups arise naturally from several points of view. Most geometrically, they are the finitely generated groups whose Cayley graph can be obtained as the pointed Gromov-Hausdorff limit of a sequence of Cayley graphs of a fixed free group (with a varying choice of generating set of fixed finite cardinality) [10]. Limit groups are precisely those finitely generated groups L that are *fully residually free*: for any finite subset $T \subset L$ there exists a homomorphism from L to a free group that is injective on T. It is in this guise that limit groups were studied extensively by Kharlampovich and Myasnikov [14], [15].

The name *limit group* was introduced by Zlil Sela to emphasise the fact that these are precisely the groups that arise when one takes limits of stable sequences of homomorphisms $\phi_n : G \to F$, where G is an arbitrary finitely generated group and F is a free group; *stable* means that for each $g \in G$ either $I_g = \{n \in \mathbb{N} : \phi_n(g) = 1\}$ or $J_g = \{n \in \mathbb{N} : \phi_n(g) \neq 1\}$ is finite, and the *limit* of (ϕ_n) is the quotient of G by $\{g \mid |I_g| = \infty\}$. A good background reference source for limit groups is [3].

1.1 ω-residually free towers

Our results rely heavily on Sela's version ([20], 1.12; cf [15]) of the fundamental theorem that characterizes limit groups and elementarily free groups in terms of ω-residually free towers.

Definition 1.1 *An ω-rft space of* height $h \in \mathbb{N}$ *is defined by induction on h. An ω-rft* group *is the fundamental group of an ω-rft space.*

An ω-rft space of height 0 is the wedge (1-point union) of a finite collection of circles, closed hyperbolic surfaces and tori $\mathbb{T}^n$ (of arbitrary dimension), except that the closed surface of Euler characteristic -1 is excluded[1]*.*

An ω-rft space Y_n of height h is obtained from an ω-rft space Y_{h-1} of height $h-1$ by attaching a block *of one of the following types:*

Abelian: *Y_h is obtained from $Y_{h-1} \sqcup \mathbb{T}^m$ by identifying a coordinate circle in $\mathbb{T}^m$ with any loop c in Y_{h-1} such that $\langle c \rangle \cong \mathbb{Z}$ is a maximal abelian subgroup of $\pi_1 Y_{h-1}$.*

Quadratic: *One takes a connected, compact surface Σ that is either a punctured torus or has Euler characteristic at most -2, and obtains Y_h from $Y_{h-1} \sqcup \Sigma$ by identifying each boundary component of Σ with a homotopically non-trivial loop in Y_{h-1}; these*

[1] dropping this exclusion, and the corresponding one in quadratic blocks, would not affect the results of our paper

identifications must be chosen so that there exists a retraction $r : Y_h \to Y_{h-1}$, and $r_*(\Sigma) \subset \pi_1 Y_{h-1}$ must be non-abelian.

An ω-rft space is called hyperbolic *if no tori are used in its construction.*

Theorem 1.1 ([21]; see also [14], [15]) *A group is elementarily free if and only if it is the fundamental group of a hyperbolic ω-rft space.*

Theorem 1.2 ([20, 1.12]; see also [15]) *Limit groups are precisely the finitely generated subgroups of ω-rft groups.*

A useful sketch proof of the latter theorem can be found in [2].

This powerful theorem allows one to prove many interesting facts about limit groups by induction on height.

Definition 1.2 *The* height *of a limit group S is the minimal height of an ω-rft group that has a subgroup isomorphic to S.*

Our approach to understanding an arbitrary limit group S will be to embed it in the fundamental group of an ω-rft group L, take the splitting of L given by Lemma 1.4 below, and then decompose S as a graph of groups corresponding to its action on the Bass-Serre tree of this splitting (subsection 1.3 below). We refer the reader to [23] and [18] for background on the Bass-Serre theory of groups acting on trees, which is used extensively throughout this paper.

1.2 Elementarily-free versus hyperbolic limit groups

A straightforward application of the local gluing lemma ([6], II.11.3) allows one to deduce from the inductive description given above that every hyperbolic ω-rft supports a locally CAT(-1) metric and every ω-rft supports a locally CAT(0) metric. In particular elementarily free groups are hyperbolic. Moreover, an induction on height, combined with some elementary Bass-Serre theory, shows that any finitely generated subgroup of an elementarily free group L is the fundamental group of a locally CAT(-1) subcomplex of a suitable covering space of the tower space for L. Thus:

Lemma 1.3 *All finitely generated subgroups of elementarily free groups are hyperbolic limit groups.*

Remark 1.1 *We emphasize that the embedding in Theorem 1.2 does not preserve hyperbolicity. For this reason, our proof of Theorem 0.1 does not extend to arbitrary hyperbolic limit groups.*

1.3 Graph of groups decompositions of elementarily free groups

Recall that a graph of groups decomposition is said to be *n-acylindrical* if in the action on the associated Bass-Serre tree, the stabilizer of any edge-path of length greater than n is trivial.

Lemma 1.4 *If L is the fundamental group of an ω-rft space $Y = Y_h$ of height $h \geq 1$, then L is the fundamental group of a 2-vertex graph of groups: one of the vertices is $\pi_1 Y_{h-1}$ and the other is a free or free-abelian group of finite rank at least 2; the edge groups are maximal infinite cyclic subgroups of the second vertex group. This decomposition is 2-acylindrical.*

Proof. The splitting described is that which the Seifert-van Kampen Theorem associates to the decomposition of Y into Y_{h-1} and the final block in the construction.

If the final block is quadratic, then the edge groups are precisely the peripheral subgroups of the surface Σ. As such they are maximal and form a malnormal family: if $C, C' \subset \Sigma$ are two cyclic subgroups in the conjugacy classes of two (not necessarily distinct) edge groups and $x \in \Sigma \smallsetminus C$, then $x^{-1}Cx \cap C' = \{1\}$.

Any edge-path in the Bass-Serre tree of length greater than 2 must contain a vertex with stabilizer a conjugate of Σ, and the intersection of the stabilisers of the incident edges will be of the form $x^{-1}Cx \cap C'$. Thus the splitting is 2-acylindrical.

Suppose now that the final block in the construction of $Y = Y_h$ is abelian. A straightforward use of the ω-residually free condition yields the following facts about limit groups: first, if two non-trivial elements of an ω-residually free group commute and are conjugate, then they are equal; second, if x, y, z are non-trivial and $[x, y] = [y, z] = 1$, then $[x, z] = 1$. It follows from the first of these facts that if A is an abelian subgroup and $y \in A \cap zAz^{-1}$, then $[y, z] = 1$. If $y \neq 1$, it then follows from the second fact that z commutes with A. Thus maximal abelian subgroups of limit groups are malnormal.

By construction, the edge stabilizer in our splitting is maximal-abelian in Y_{h-1}. Hence it is malnormal in Y_{h-1} and the splitting is 2-acylindrical. $\square$

Corollary 1.5 *If Γ is a non-cyclic, finitely generated, freely indecomposable subgroup of an elementarily free group G, then Γ is the fundamental group of a 2-acylindrical graph of groups, in which one of the vertex groups is the fundamental group of a compact surface Σ and the incident edge groups are distinct peripheral subgroups of Σ.*

Proof. By Sela's Theorem 1.1, G is the fundamental group of some ω-rft space Y. If Y can be chosen of height 0, then Γ is the fundamental group of a closed surface Φ of Euler characteristic at most -2. In this case the splitting of Φ along any nontrivial 2-sided simple closed curve induces the desired decomposition of Γ. Otherwise, apply Lemma 1.4 to G, let T be the minimal Γ-invariant subtree of the resulting Bass-Serre tree for G, and take the resulting graph-of-groups decomposition for Γ with underlying graph T/Γ. □

1.4 Subgroup Separability

Let $\mathcal{P}$ be a class of groups, e.g. free or finite. Let Γ be a group. A subgroup $S \subset \Gamma$ is *closed in the pro-$\mathcal{P}$ topology* if for every $x \notin S$ there exists a homomorphism $f : \Gamma \to F$ where $F \in \mathcal{P}$ and $f(x) \notin f(S)$.

If $\{1\}$ is closed in the pro-$\mathcal{P}$ topology then Γ is said to be *residually $\mathcal{P}$*.
Notation: If all infinite cyclic subgroups $S \subset \Gamma$ are closed in the profinite topology, then we say that Γ is $\mathbb{Z}$-separable.

Proposition 1.6 *In a finitely generated free group F, every finitely generated subgroup is closed in the pro-finite topology.*

Proof. This follows easily from Marshall Hall's theorem [13], which states that every finitely generated subgroup is a free factor of a subgroup of finite index in F. □

Corollary 1.7 *If $S \subset \Gamma$ is closed in the pro-free topology then it is closed in the pro-finite topology.*

Lemma 1.8 *If Γ is a residually free group with no non-cyclic abelian subgroups, then Γ is $\mathbb{Z}$-separable.*

Proof. Let $C = \langle c \rangle \subset \Gamma$ and suppose that $\gamma \in \Gamma$ with $\gamma \notin C$. There are two cases to consider: either $[\gamma, c] \neq 1$, or $H = \langle c, \gamma \rangle$ is cyclic.

In the first case, since Γ is residually finite, there exists a homomorphism $f : \Gamma \to Q$ where Q is finite and $f([\gamma, c]) \neq 1$; in particular $\gamma \notin f(C)$.

In the second case, since Γ is residually free, there exists a homomorphism $\phi : \Gamma \to F$ where F is free such that $\phi(\gamma) \neq 1$. Since $\phi|_H$ is injective, $\phi(\gamma) \notin \phi(C)$, and since F is $\mathbb{Z}$-separable, there exists a finite quotient such that the image of γ does not lie in the image of C. □

Corollary 1.9 *Hyperbolic limit groups (in particular elementarily free groups) are $\mathbb{Z}$-separable.*

A slight variation on the preceding proof shows that if a group Γ is residually free, then each of its maximal abelian subgroups is closed in the pro-free topology. Conversations with Zlil Sela and Henry Wilton convinced us that, in the light of Theorem 1.2, it is not difficult to show that all abelian subgroups of limit groups are closed in the pro-finite (indeed pro-free) topology.

2 Some Bass-Serre Theory

As noted in the previous section, an elementarily free group acts 2-acylindrically on the Bass-Serre tree arising from its ω-rf tower decomposition. In this section we deduce from this that every nontrivial normal subgroup contains an element that acts hyperbolically on the tree – a fact that will be important later.

Recall that an automorphism γ of a tree T is *hyperbolic* if it has no fixed points, and that the unique γ-minimal subtree of T is then an isometrically embedded line A_γ called the *axis* of γ. In contrast, an automorphism with at least one fixed point is *elliptic.*

Proposition 2.1 *Let Γ be a group acting on a tree T.*

1. *If $\alpha, \beta \in \Gamma$ are hyperbolic with disjoint axes A_α and A_β, then $\alpha\beta$ is hyperbolic and its axis contains the unique shortest arc from A_α to A_β.*

2. *If $\alpha, \beta \in \Gamma$ are elliptic with $\mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta) = \emptyset$ then $\alpha\beta$ is hyperbolic.*

3. *If a finite family of convex subsets in T intersects pairwise, then the intersection of the entire family is non-empty.*

4. *If Γ is finitely generated then either Γ fixes a point of T, or else Γ contains hyperbolic isometries.*

5. *If the action of Γ is n-acylindrical for some $n \in \mathbb{N}$, then either Γ fixes a point of T, or else Γ contains hyperbolic isometries.*

6. *If Γ contains hyperbolic elements, then the union of the axes of such elements is the unique minimal Γ-invariant subtree of T.*

Proof.

1: Choose an edge e that lies in the arc joining A_α to A_β. Let X, Y denote the components of $T \setminus \{e\}$ containing A_α, A_β respectively. Note that $\alpha^{\pm 1}(Y \cup e) \subset X$ while $\beta^{\pm 1}(X \cup e) \subset Y$. Thus e is contained in the geodesic path from $(\alpha\beta)^{-n}(e)$ to $(\alpha\beta)^m(e)$ for all $n, m > 0$, and the result follows.

2: A similar argument applies, replacing A_α, A_β by $\mathrm{Fix}(\alpha), \mathrm{Fix}(\beta)$ respectively.

3: An inductive argument reduces us to the case of three convex sets. Choose a point in each of the three pairwise intersections, and consider the geodesic triangle in T with these points as vertices. Each of our three sets contains one of the sides of the triangle. And since we are in a tree, the three sides have a common point.

4: By 2, if Γ is generated by a finite set of elliptics then either the product of some pair of these generators is hyperbolic or else the fixed-point sets of each pair intersect non-trivially, in which case it follows from 3 that Γ has a fixed point.

5: Without loss of generality, we may assume that T is a minimal Γ-tree. We will reach a contradiction by assuming that Γ has no hyperbolic elements and no fixed point.

Consider the union U of the sets $\mathrm{Fix}(\gamma)$ as γ ranges over $\Gamma \setminus \{1\}$. We claim that U is convex, hence a subtree. Indeed, given $x_1, x_2 \in U$ there exist $\gamma_1, \gamma_2 \in N$ such that $x_i \in \mathrm{Fix}(\gamma_i)$ for $i = 1, 2$, and since $\gamma_1\gamma_2$ is not hyperbolic, $\mathrm{Fix}(\gamma_1) \cap \mathrm{Fix}(\gamma_2) \neq \emptyset$ by 2. Thus $[x_1, x_2]$ lies in the convex set $\mathrm{Fix}(\gamma_1) \cup \mathrm{Fix}(\gamma_2)$. The tree U is nonempty and Γ-invariant, so $U = T$.

Given an edge e of T, we may choose vertices $x, y \in T$ in distinct components of $T \smallsetminus \{e\}$ a distance at least n from e. Such vertices exist because T does not have vertices of valence 1 (since removing the edges incident at such vertices would yield a proper invariant subtree), and therefore each edge in the complement of e can be extended to an infinite ray in its component of $T \smallsetminus \{e\}$.

Let $\alpha, \beta \in \Gamma$ be non-trivial elements such that $x \in \mathrm{Fix}(\alpha)$ and $y \in \mathrm{Fix}(\beta)$. Then e lies in the convex set $\mathrm{Fix}(\alpha) \cup \mathrm{Fix}(\beta)$. If e lies in the convex set $\mathrm{Fix}(\alpha)$, then so does the segment joining x to e. But then this segment is fixed by α, contradicting the fact that Γ acts n-acylindrically.

6: This follows from 1. □

Corollary 2.2 *If an action of a group Γ on a tree T is minimal and n-acylindrical for some $n > 0$, and Γ has no global fixed point in T, then every non-trivial normal subgroup $N \subset \Gamma$ contains hyperbolic elements.*

Proof. Since N is normal in Γ, the subset $\mathrm{Fix}(N)$ of T is convex and Γ-invariant, so is either the whole of T or is empty. In the first case, T must have finite diameter by the n-acylindrical property. But then Γ has a global fixed point in T, contrary to hypothesis. Hence N has no global fixed point in T, and the result follows from Proposition 2.1(5). □

Remark 2.1 *The following example illustrates the difficulties one faces in trying to sharpen the above statement. Consider the action of the HNN-extension* $B = \langle a, t \mid tat^{-1} = a^2 \rangle$ *of the cyclic group* $\langle a \rangle$ *on its Bass-Serre tree. This action is cocompact with cyclic edge stabilisers. The normal closure* N *of* a *does not contain any hyperbolic elements.*

3 Hyperbolics are almost stable letters

The proof of the following theorem is based on an idea introduced by John Stallings in the context of a free group acting freely on its Cayley tree [25] (see also [17, Theorem 2.2] and [12, Lemma 15.22]). This result is a restatement of Theorem 0.3.

Theorem 3.1 *Let* Γ *be a group that acts minimally on a tree* T *and let* e *be an edge whose stabiliser* $A \subset \Gamma$ *is closed in the pro-finite topology.*

If $t \in \Gamma$ *is a hyperbolic isometry whose axis contains* e*, then there exists a subgroup* $H \subset \Gamma$ *of finite index such that* H *is an HNN-extension with stable letter* t *and amalgamated subgroup* $A \cap H$*.*

Proof. Consider the segment λ of the axis of t that begins with the edge e and ends with the edge $t(e)$. Let

$$e = g_0(e), g_1(e), \ldots, g_n(e) = t(e)$$

be the finitely many edges in λ that belong to the Γ-orbit of e. The elements g_i are not well-defined (unless $A = 1$). But the left cosets $g_i A$ are well-defined and pairwise distinct.

Since $A \subset \Gamma$ is closed in the profinite topology, there exists a finite-index normal subgroup $K \subset \Gamma$ such that the left cosets $g_0(AK), \ldots, g_n(AK)$ are pairwise distinct.

Bass-Serre theory expresses Γ as the fundamental group of a graph of groups, where one of the edge-groups is A. Using the construction described by Scott and Wall in [18, Proposition 3.6], we can construct a classifying space X for Γ as a graph of aspherical spaces.

Thus Γ acts freely on the universal cover $\tilde{X}$ of X, and there is a Γ-equivariant map $f : \tilde{X} \to T$ such that $E = f^{-1}(e)$ is a product ("cylinder") $\tilde{X}_A \times (0, 1)$ for some

$K(A,1)$-space X_A. We identify $\tilde{X}_A$ with $\tilde{X}_A \times \{\frac{1}{2}\} \subset E$, and choose a point $\tilde{x}_0 \in \tilde{X}_A$ as a base-point for $\tilde{X}$. We also choose a path $\tilde{\tau}$ in $\tilde{X}$ from $\tilde{x}_0$ to $t(\tilde{x}_0)$. Since $\tilde{X}_A$, and each of its Γ-translates, has a bicollared neighbourhood in $\tilde{X}$, we may assume that $\tilde{\tau}$ is transverse to $g(\tilde{X}_A)$ for each $g \in \Gamma$. Moreover, if we further assume that the total number of points of intersection of $\tilde{\tau}$ with $\bigcup_g \, g(\tilde{X}_A)$ is as small as possible, then $\tilde{\tau}$ crosses $g(\tilde{X}_A)$ transversely in a single point for $g = g_1, \ldots, g_{n-1}$, meets $g(\tilde{X}_A)$ in one of its endpoints for $g = g_0 = 1$ and $g = g_n = t$, and is disjoint from $g(\tilde{X}_A)$ for all other g.

Let x be the image of $\tilde{x}$ in X, and τ the image of $\tilde{\tau}$ in X. We use x as the base-point for X. Then τ is a closed path in X representing the element t of $\pi_1(X,x) \cong \Gamma$.

Consider the covering space $Y = \tilde{X}/K$ of X corresponding to the normal subgroup K. Let E_0 denote the image of E in Y, and let X'_A be the image of $\tilde{X}_A$. The quotient group Γ/K acts on Y, and for each $g \in \Gamma$, E_0 and $(gK)(E_0)$ either coincide (if $g \in AK$) or are disjoint. In particular, $E_0 = (g_0K)(E_0), E_1 = (g_1K)(E_0), \ldots, E_n = (g_nK)(E_0)$ are pairwise disjoint. Let y_0 be the image in Y of $\tilde{x}_0$, and τ' the image in Y of $\tilde{\tau}$. Then τ' is a path from y_0 to $(tK)(y_0)$ that intersects each $(g_iK)(X'_A)$ in precisely one point.

We cut E_0 and E_n along the subspace X'_A and its translate $(tK)(X'_A) = (g_nK)(X'_A)$, creating four ("boundary") subspaces $\partial_-E_0, \partial_-E_n$ and $\partial_+E_0, \partial_+E_n$, each homeomorphic to X'_A. This cutting transforms Y to a space Y', and τ' becomes a path in Y' from the copy of y_0 in ∂_+E_0 to the copy of $(tK)(y_0)$ in ∂_-E_n.

We form a new space Z from Y' by identifying ∂_-E_0 to ∂_+E_n, and ∂_+E_0 to ∂_-E_n, using the restriction of the homeomorphism $(tK) : E_0 \to E_n$. Note that this identification in particular identifies the two endpoints of τ' to a single point z, which we may regard as the base-point of Z. The image of τ' in Z is thus a closed path $\overline{\tau}$ based at z. The covering map $Y \to X$ induces a covering map $Z \to X$. (This $|G : K|$-fold cover is not necessarily connected or regular.)

Consider the component Z_0 of Z that contains the copy $\mathcal{A}$ of X'_A that is the image of ∂_-E_0 and ∂_+E_n. Then $z \in \mathcal{A} \subset Z_0$.

The restriction of $Z \to X$ to Z_0 is a covering of X corresponding to a finite index subgroup $H = \pi_1(Z_0, z) \subset \pi_1(X,x) = \Gamma$ *that contains* t, since $\overline{\tau}$ is a closed path in Z_0 which projects to τ.

Since this loop $\overline{\tau}$ representing $t \in H = \pi_1(Z_0, z)$ crosses $\mathcal{A}$ transversely precisely once, the Seifert-van Kampen decomposition of $\pi_1(Z_0, z)$ expresses H as an HNN-extension with stable letter t and associated subgroup $\pi_1(\mathcal{A}) = A \cap H$. $\square$

Corollary 3.2 *Let Γ be a group that acts minimally on a tree T. Let e be an edge whose stabiliser $A \subset \Gamma$ is closed in the pro-finite topology.*

If $N \subset \Gamma$ is a normal subgroup that contains a hyperbolic isometry, then there *exists a subgroup $H \subset \Gamma$ of finite index and an element $t \in N \cap H$ such that H is an HNN-extension with stable letter t and amalgamated subgroup $A \cap H$.*

Proof. Since N is normal, the union $U \subset T$ of the axes of the hyperbolic elements in N is Γ-invariant. But U is a subtree (Proposition 2.1(6)) and the action of Γ is minimal, so $U = T$. Hence there exists a hyperbolic element $t \in N$ whose axis contains e. □

3.1 The curve-lifting lemma

We mentioned in the introduction that Theorem 3.1 generalizes Scott's result that every non-trivial element in the fundamental group of a closed surface can be represented by a simple closed curve in some finite-sheeted covering of that surface. In our proof of Theorem 0.1 we shall need the following refinement of this fact.

Lemma 3.3 *Let Σ be a compact surface with non-positive Euler characteristic, X a space with $\mathbb{Z}$-separable fundamental group, and $f : \Sigma \to X$ a π_1-injective map. If w is a non-trivial element of $\pi_1\Sigma$, then there exists a finite-sheeted cover $\bar{X}$ of X, and a simple closed curve α in the induced cover $\bar{\Sigma}$ of Σ, such that the image of α in Σ represents w.*

Proof. Since f is π_1-injective, we can identify $\pi_1(\Sigma)$ with the subgroup $f_*(\pi_1(\Sigma))$ of $\pi_1(X)$. We do so implicitly throughout the proof without further comment.

Suppose first that w is a proper power of some element $u \in \pi_1\Sigma$; say $w = u^n$. By $\mathbb{Z}$-separability, there is a subgroup of finite index in $\pi_1(X)$ that contains $w = u^n$ but contains none of $u, u^2, \ldots, u^{n-1}$. Replacing X and Σ by the corresponding finite covers, we may assume that w is not a proper power in $\pi_1\Sigma$. We fix a constant-curvature Riemannian metric on Σ such that the boundary of Σ is totally geodesic. With respect to this metric, the conjugacy class of w is represented by a closed geodesic β with transverse self-intersection.

If β is not an embedding, we can express it as a concatenation of paths $\beta = \beta_1\beta_2\beta_3$ where β_2 is an embedded closed path, which must be essential in Σ since it is a based geodesic in a non-positively curved metric. Moreover, the conjugacy class represented by β_2 has a trivial intersection with $\langle w \rangle$, because the elements of this cyclic subgroup are represented by closed geodesics β^n which are strictly longer than β_2.

Since $\pi_1\Sigma \to \pi_1X$ is injective, and π_1X is $\mathbb{Z}$-separable, there is a finite sheeted cover of X such that β lifts to a closed geodesic in the induced cover of Σ but the corresponding lift of β_2 is not closed. This lift of β therefore has fewer double points than β.

Repeating this process, we eventually arrive at a finite cover of X such that β lifts to an embedded closed geodesic α in the induced cover of Σ. The proof is complete. $\square$

4 The double-coset lemma

In the introduction, we explained that our proof of Theorem 0.1 has the advantage over [9] that when homological finiteness fails, one can construct explicit topological cycles that demonstrate the lack of finiteness. That construction is contained in the following proof.

Theorem 4.1 (Double Coset Lemma) *For $i = 1, \ldots, n$, let G_i be an HNN-extension with stable letter w_i and associated subgroups A_i and B_i. Let $G = \prod_{i=1}^n G_i$, $A = \prod_{i=1}^n A_i$, and let $L \subset G$ be a subgroup containing $j_i(w_i)$ $(i = 1, \ldots, n)$, where $j_i : G_i \to G$ is the canonical injection. Suppose also that $p_i(L) = G_i$ for all i, where $p_i : G \to G_i$ is the canonical projection. Then, the n-th homology group $H_n(L, \mathbb{Q})$ contains a subgroup isomorphic to*

$$\mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G \otimes_{\mathbb{Q}A} \mathbb{Q}$$

(the $\mathbb{Q}$ vector space with basis the set of double cosets $\{LgA, g \in G\}$).

Proof. We can construct, for each i, a $K(G_i, 1)$-complex X_i by starting with a classifying space for the base of the HNN extension and forming a mapping torus corresponding to the given isomorphism $A_i \to B_i$. In this construction, w_i corresponds to a 1-cell W_i with both endpoints at the base-point of X_i, and W_i appears only in the boundaries of those 2-cells corresponding to defining relations $w_i^{-1} a w_i = b$ for a set of generators a of A_i. There is no loss of generality in assuming that each of the generators a, b corresponds to a single 1-cell, and hence the 2-cells involving w_i have attaching maps of length 4.

Let $X = X_1 \times \cdots \times X_n$, let $\tilde{X}$ be the universal cover of X, upon which G acts on the left, and let $\overline{X} = L \backslash \tilde{X}$ be the covering complex corresponding to the subgroup $L \subset G$.

We work with cellular chains.

Let $L_i := \{g \in G_i \mid j_i(g) \in L\}$. Then $j_i(L_i)$ is the intersection of the kernels of $p_k|_L : L \to G_k$ for $k \neq i$, so is normal in L. Hence $L_i = p_i j_i(L_i)$ is normal in $p_i(L) = G_i$. Since also $j_i(G_i)$ commutes with $j_k(G_k)$ for $i \neq k$, it follows that $j_i(L_i)$ is normal in G.

But $w_i \in L_i$, so all conjugates of $j_i(w_i)$ belong to $j_i(L_i) \subset L$. Hence all the lifts of W_i in $\overline{X}$ are 1-cycles.

Now let W denote the cellular n-cycle $W_1 \times \cdots \times W_n$ in X. By the above, all the lifts of W to $\overline{X}$ are also n-cycles. We can identify these lifts with right cosets of L in G as follows. Choose an n-chain $\tilde{W}$ in $\tilde{X}$ that covers W. Then the orbit of W under the G-action consists of pairwise distinct n-chains $g(\tilde{W})$ for $g \in G$, and $g(\tilde{W})$ and $h(\tilde{W})$ cover the same n-chain in $\overline{X}$ if and only if $Lg = Lh$.

These chains in $\overline{X}$, as has been observed above, are in fact n-cycles, so generate a subgroup M of $H_n(\overline{X}, \mathbb{Q}) = H_n(L, \mathbb{Q})$. This subgroup is then a homomorphic image of the free $\mathbb{Q}$-module on the set of right cosets Lg of L in G, in other words, $\mathbb{Q}(L\backslash G) = \mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G$. The kernel of the corresponding homomorphism is the intersection of this group of n-cycles with the group of (cellular) $(n+1)$-boundaries of $\overline{X}$.

Now the boundary of an $(n+1)$-cell α of X involves W only if that $(n+1)$-cell is a cube formed as the product of a 2-cell β of some X_i whose boundary involves W_i and of the $(n-1)$ 1-cells W_k with $k \neq i$. (We arranged in the first paragraph that β have an attaching map of length 4, with two sides corresponding to W_i.) Therefore a lift $\tilde{\alpha}$ of α in $\tilde{X}$ has the combinatorial structure of a cube, and the coefficient of the n-dimensional face $\tilde{W}$ in the cellular n-chain $\partial(\tilde{\alpha})$ is $1 - a$ for some $a \in A_i$.

Hence M has a homomorphic image isomorphic to the quotient of

$$\mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G$$

by the submodule generated by $\{(\mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G)(1-a) \mid a \in A\}$. But this quotient is just

$$\mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G \otimes_{\mathbb{Q}A} \mathbb{Q}.$$

Since this is a free $\mathbb{Q}$-module, the epimorphism

$$M \to \mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G \otimes_{\mathbb{Q}A} \mathbb{Q}$$

splits, so M, and hence also $H_n(L, \mathbb{Q})$, has a subgroup isomorphic to

$$\mathbb{Q} \otimes_{\mathbb{Q}L} \mathbb{Q}G \otimes_{\mathbb{Q}A} \mathbb{Q},$$

as claimed. □

We will use the double-coset lemma in the proof of our main result, where we shall need the following elementary properties of double cosets in our calculations.

Lemma 4.2 *Let G be a group, g an element of G, and A, B, C subgroups of G such that $B \subset C$. Then the intersection of C and the double coset AgB is either empty or has the form $(A \cap C)cB$ for some $c \in C$.*

Proof. Suppose that $c_1, c_2 \in AgB \cap C$. Then we can write $c_i = a_i g b_i$ for $i = 1, 2$. Then $(a_2 a_1^{-1}) = c_2 b_2^{-1} b_1 c_1^{-1} \in A \cap C$, so $c_2 \in (A \cap C) c_1 B$. Hence, for any $c \in AgB \cap C$, we have $AgB \cap C \subset (A \cap C)cB$. The converse inclusion is immediate, using the equation $AcB = AgB$. □

Corollary 4.3 *If A, B, C are subgroups of a group G such that $B \subset C$ and $|A \backslash G/B| < \infty$, then $|(A \cap C) \backslash C/B| < \infty$.*

Lemma 4.4 *Let G, H be groups, A, B subgroups of G and g an element of G. If $\phi : G \to H$ is a homomorphism, then $\phi(AgB) = \phi(A)\phi(g)\phi(B)$.*

Corollary 4.5 *If A, B are subgroups of a group G such that $|A \backslash G/B| < \infty$, and $\phi : G \to H$ is a homomorphism, then $|\phi(A) \backslash \phi(G)/\phi(B)| < \infty$.*

5 The Main Theorem

In this section we prove the main theorem, Theorem 0.1, in the following somewhat stronger form.

Theorem 5.1 *Let $G_1, \ldots, G_n$ be subgroups of elementarily free groups and let $\Gamma \subset G_1 \times \cdots \times G_n$ be a subgroup. Then the following are equivalent:*

(i) there exist finitely generated subgroups $\hat{G}_i \subset G_i$, for $i = 1, \ldots, n$, such that Γ is isomorphic to a finite-index subgroup of $\hat{G}_1 \times \cdots \times \hat{G}_n$;

(ii) Γ is of type $FP_n(\mathbb{Q})$;

(iii) for each $k = 1, \ldots, n$ and each finite-index subgroup $\Gamma_0 \subset \Gamma$, the k-th homology $H_k(\Gamma_0, \mathbb{Q})$ is finite dimensional over $\mathbb{Q}$.

Proof. Finitely generated subgroups of limit groups are limit groups, and hence of type FP_∞. Thus any subgroup of finite index in a direct product of finitely many such groups is also of type FP_∞, so (i) implies (ii). It is clear that (ii) implies (iii), so it remains only to prove that (iii) implies (i).

Let $G := G_1 \times \cdots \times G_n$, and let $p_i : G \to G_i$ denote the canonical projection onto the i-th factor, for $i = 1, \ldots, n$.

Replacing each G_i with $p_i(\Gamma)$, we may assume that $p_i(\Gamma) = G_i$ for all i. In particular, each $H_1(G_i, \mathbb{Q})$ is a homomorphic image of $H_1(\Gamma, \mathbb{Q})$, and so finite-dimensinal over $\mathbb{Q}$. Hence each G_i is finitely generated, by [7, Theorem 2].

By abuse of notation, we identify G_i with the normal subgroup

$$\{1\} \times \cdots \times \{1\} \times G_i \times \{1\} \times \cdots \times \{1\}$$

of G. Then we define $L_i = G_i \cap \Gamma$, and note that L_i is normal in Γ, and hence that $L_i = p_i(L_i)$ is normal in $G_i = p_i(\Gamma)$.

If $L_n = \{1\}$, then the natural projection $G_1 \times \cdots \times G_n \to G_1 \times \cdots \times G_{n-1}$ is injective on Γ, so Γ is isomorphic to a subgroup of $G_1 \times \cdots \times G_{n-1} \times \{1\}$. An obvious induction reduces us to the case where all of the L_i are nontrivial. We will be done if we can show that $|G_i : L_i| < \infty$ for each i.

If we replace G_n by a finite-index subgroup $\hat{G}_n$, and replace each G_j by $\hat{G}_j = p_j p_n^{-1}(\hat{G}_n)$ and Γ by $\Gamma \cap (\hat{G}_1 \times \cdots \times \hat{G}_n)$, then neither our hypotheses nor our desired conclusion is disturbed. (And likewise with G_i in place of G_n.) We shall take advantage of this freedom several times in the sequel without further comment.

We first use this freedom to reduce to the case where none of the G_i is cyclic. (This could also be done by appealing to [7, Theorem 3].) If G_n is cyclic, then L_n has finite index in G_n since $L_n \neq \{1\}$, so we may assume that $L_n = G_n \cong \mathbb{Z}$. In this case Γ splits as a direct product $\Gamma' \times \mathbb{Z}$ for some $\Gamma' \subset G_1 \times \cdots \times G_{n-1}$. The Künneth formula and the homological hypothesis on Γ tell us that $H_k(\Gamma_0, \mathbb{Q})$ is finite dimensional for each $k = 1, \ldots, n-1$ and for each finite-index subgroup $\Gamma_0 \subset \Gamma'$. By induction on n, we may assume that the theorem is true for Γ_0, from which it follows for Γ.

Henceforth we assume that none of the G_i is cyclic. The structure result for subgroups of elementarily free groups, Corollary 1.5, tells us that each G_i is either freely decomposable or can be expressed as the fundamental group of a 2-acylindrical graph of groups in which one of the vertex groups is the (nonabelian free) fundamental group of a surface Σ_i with boundary, and the incident vertex groups are distinct peripheral subgroups of $\pi_1\Sigma_i$.

If G_i is freely decomposable, we regard it as the fundamental group of a nontrivial graph of groups with trivial edge groups (which is in particular 1-acylindrical).

The next step concerns those G_i which are freely indecomposable, and for which $\pi_1\Sigma_i \cap L_i$ is nontrivial. For such i, we may choose a nontrivial element a_i in $\pi_1\Sigma_i \cap L_i$. By the curve-lifting lemma, Lemma 3.3, we may assume (after replacing G_i by a finite-index subgroup), that a_i is represented by a simple closed curve in Σ_i. Cutting Σ_i along this curve produces a refinement of the graph-of-groups structure of G_i, in which $\langle a_i \rangle$ is an edge group. Moreover, this refinement remains 2-acylindrical.

For those G_i that are freely decomposable, we define $a_i = 1$, while for the remaining G_i we take a_i to be a generator of a peripheral subgroup of $\pi_1\Sigma_i$ that is an edge group in the graph-of-groups decomposition of G_i.

In all cases, G_i has a 2-acylindrical graph-of-groups decomposition with an edge group $\langle a_i \rangle$. Moreover $a_i \in L_i$ whenever either G_i is freely decomposable or $\pi_1\Sigma_i \cap L_i \neq \{1\}$.

By Corollary 1.9, $\langle a_i \rangle$ is closed in the profinite topology on G_i; by Corollary 2.2, the normal subgroup L_i contains an element that acts hyperbolically on the Bass-Serre tree of the decomposition; hence we may apply Corollary 3.2. After replacing the G_i by finite-index subgroups, we may assume that each G_i has an HNN-decomposition with associated subgroup $\langle a_i \rangle$ and stable letter $w_i \in L_i$.

By hypothesis $H_n(\Gamma, \mathbb{Q})$ is finite-dimensional over $\mathbb{Q}$, so it follows from the double-coset lemma, Theorem 4.1, that $|\Gamma \backslash G/A| < \infty$, where

$$A = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle.$$

We now split the proof into two cases.

Case 1. Suppose that $a_i \in L_i$ for each i.

Since L_i is normal in G_i, it follows that $gAg^{-1} \subset \Gamma$ for all $g \in G$, so that $\Gamma g A = \Gamma g$, so $|G : \Gamma| = |\Gamma \backslash G/A| < \infty$, and the result follows.

Case 2. Suppose that (possibly after renumbering) $a_1 \notin L_1$.

Then, by our choice of a_1, there is a free surface group $F = \pi_1\Sigma_1 \subset G_1$ with $a_1 \in F$ and $F \cap L_1 = \{1\}$. Define $B = G_1 \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$. Then $A \subset B$ and

$$|\Gamma \backslash G/A| < \infty,$$

so Corollary 4.3 gives

$$|(\Gamma \cap B) \backslash B/A| < \infty.$$

Hence Corollary 4.5 gives

$$|p_1(\Gamma \cap B) \backslash G_1/\langle a_1 \rangle| < \infty,$$

since $p_1(B) = G_1$ and $p_1(A) = \langle a_1 \rangle$. But $\langle a_1 \rangle \subset F$, so by Corollary 4.3 again we have

$$|[F \cap p_1(\Gamma \cap B)] \backslash F/\langle a_1 \rangle| < \infty.$$

Now G_1 is normal in B with B/G_1 abelian. Hence $L_1 = \Gamma \cap G_1$ is normal in $\Gamma \cap B$ with $(\Gamma \cap B)/L_1$ abelian. Hence $L_1 = p_1(L_1)$ is normal in $p_1(\Gamma \cap B)$ with $p_1(\Gamma \cap B)/L_1$ abelian. Finally, it follows that $F \cap L_1$ is normal in $F \cap p_1(\Gamma \cap B)$, with $[F \cap p_1(\Gamma \cap B)]/(F \cap L_1)$ abelian.

But $F \cap L_1 = \{1\}$. Hence $F \cap p_1(\Gamma \cap B)$ is an abelian subgroup of the free group F, and so cyclic: say $F \cap p_1(\Gamma \cap B) = \langle b \rangle$. Then F is a non-abelian free group, and $a_1, b \in F$ such that $|\langle b \rangle \backslash F/\langle a_1 \rangle| < \infty$, which is absurd.

This contradiction completes the proof.

□

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