

Subgroups of direct products of two limit groups

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Abstract

If Γ_1 and Γ_2 are limit groups and $S \subset \Gamma_1 \times \Gamma_2$ is of type FP_2 then S has a subgroup of finite index that is a product of at most two limit groups.

1 Introduction

In [7] Miller, Short and the present authors described the homological finiteness properties of subgroups of direct products of free and surface groups. This completed a programme of research that began with celebrated examples of Stallings [24] and Bieri [4] and took an unexpected turn when Baumslag and Roseblade [2] showed that the only finitely presented subdirect¹ products of two free groups are the obvious ones, namely free groups and subgroups of finite index.

This last theorem admits instructive proofs from several perspectives [9], [23], [15]; each of these proofs lends new insight and leads to an interesting generalisation, but each relies heavily on the fact that one is dealing with only two factors. The search for a higher-dimensional analogue culminated in the Main Theorem of [7], which shows in particular that if S is a subgroup of a direct product of n free or surface groups and the homology of S (with trivial coefficients) is finitely generated up to dimension n , then S has a subgroup of finite index that is isomorphic to a direct product of free or surface groups.

¹Recall that $S \subset A \times B$ is termed a subdirect product if its projection to both A and B is surjective.

Upon the successful completion of his project to characterise those finitely generated groups that have the same elementary or existential theories as free groups ([18] to [20]; cf. [13, 14]), Zlil Sela proposed a list of central questions concerning these *limit groups* [21]. In particular he asked whether the results of [7] could be extended to subdirect products of limit groups. In the first two papers in this series [5], [6], we established results concerning the subgroup structure of limit groups that enabled us to answer this question in the case where the limit groups have the same elementary theory as a free group. In the present article we turn our attention to the subdirect products of general limit groups. We use our previous results in tandem with calculations in group homology and geometrical arguments based on Bass-Serre theory to prove the following analogue of the Baumslag-Roseblade theorem.

Theorem 1.1. *Let $S \subset \Gamma_1 \times \Gamma_2$ be a subdirect product of non-abelian limit groups. If S is of type $\text{FP}_2(\mathbb{Q})$ and $L_i = S \cap \Gamma_i$ is non-trivial for $i = 1, 2$, then S has finite index in $\Gamma_1 \times \Gamma_2$.*

Some simple reductions (4.1) yield:

Corollary 1.2. *Let Γ_1 and Γ_2 be limit groups. If $S \subset \Gamma_1 \times \Gamma_2$ is finitely presented then S has a subgroup of finite index that is a product of at most two limit groups.*

This is a special case of the following result, which is proved in section 5.

Corollary 1.3. *Let $\Gamma_1, \dots, \Gamma_n$ be limit groups, with $\Gamma_3, \dots, \Gamma_n$ abelian, and let $S \subset \Gamma_1 \times \dots \times \Gamma_n$ be a subdirect product such that S is of type $\text{FP}_m(\mathbb{Q})$, where $m = \min(n, 3)$. Then S is isomorphic to a subgroup of finite index in a product of at most m limit groups.*

For simplicity of exposition, the homology of a group G in this paper will always be with coefficients in a $\mathbb{Q}G$ -module – typically the trivial module \mathbb{Q} . With minor modifications, our arguments also apply with other coefficient modules, giving corresponding results under the finiteness conditions $\text{FP}_n(R)$ for other suitable rings R .

The final section of this paper contains some speculation concerning the subgroups of the direct product of arbitrarily many limit groups.

2 Limit groups, their decomposition and subgroup structure

Since this is the third in a series of papers (following [5, 6]), we shall only recall the minimal necessary amount of information about limit groups. The reader unfamiliar with this fascinating class of groups can consult the seminal papers of Sela [18] to [20] or the excellent introduction [3]. The papers of Kharlampovich and Myasnikov [13, 14] approach the subject from a perspective more in keeping with traditional combinatorial group theory, and a further perspective is developed in [11].

2.1 Limit groups

A finitely generated group Γ is a *limit group* if for every finite set $T \subset \Gamma$ there exists a homomorphism ϕ from Γ to a free group so that $\phi|_T$ is injective.

Our results rely on the fact that limit groups are the finitely generated subgroups of ω -residually free tower (ω -rft) groups [19]. Such tower groups are the fundamental groups of tower spaces that are assembled from graphs, tori and surfaces in a hierarchical manner. The number of stages in the process of assembly is the *height* of the tower. Each stage in the construction involves the attachment of a surface along its boundary, or the attachment of an n -torus T along an embedded circle representing a primitive element of $\pi_1 T$.

The *height* of a limit group S is the minimal height of an ω -rft group that has a subgroup isomorphic to S . Limit groups of height 0 are free products of finitely many free abelian groups (each of finite rank) and surface groups of Euler characteristic at most -2 .

The splitting described in the following proposition is obtained as follows: embed Γ in an ω -rft group G , take the graph of groups decomposition that the Seifert-van Kampen Theorem associates to the addition of the final block in the tower, then apply Bass-Serre theory to get an induced graph of groups decomposition of Γ .

Proposition 2.1. *If Γ is a limit group of height $h \geq 1$, then Γ is the fundamental group of a finite graph of groups with the following properties:*

1. *the vertex groups are limit groups of height $\leq h - 1$;*
2. *the edge groups are trivial or infinite-cyclic;*

3. each edge is adjacent to a vertex where the vertex-group is non-abelian.

Note also that any non-abelian limit group of height 0 splits as $A *_C B$ with C infinite-cyclic or trivial.

2.2 Homological finiteness implies finiteness

The following is proved in [5].

Theorem 2.2. *Let Γ be a limit group and $S \subset \Gamma$ a subgroup. If $H_1(S, \mathbb{Q})$ is finite dimensional then S is finitely generated (and hence is a limit group).*

2.3 Normal subgroups

From [5] we need:

Theorem 2.3. *If Γ is a limit group and $H \subset \Gamma$ is a finitely generated subgroup, then either H has finite index in its normaliser or else the normaliser of H is abelian.*

And from [6] we need:

Theorem 2.4. *Let Γ be a limit group that splits as an amalgamated free product or HNN extension with cyclic edge group $C \subset \Gamma$.*

If $N \subset \Gamma$ is a non-trivial normal subgroup then there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index and an element $t \in N \cap \Gamma_0$ such that Γ_0 is an HNN-extension with stable letter t and amalgamated subgroup $C \cap \Gamma_0$.

2.4 Double cosets

We also require the following *Double Coset Lemma* from [6].

Theorem 2.5. *Suppose that, for $i = 1, \dots, n$, $\Gamma_i *_{A_i}$ is an HNN extension with associated subgroup A_i and stable letter t_i . Let $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$, $A = A_1 \times \dots \times A_n$, and let $S \subset \Gamma$ be a subgroup containing t_1, \dots, t_n . Then $H_n(S, \mathbb{Q})$ contains a subgroup isomorphic to*

$$\mathbb{Q} \otimes_{\mathbb{Q}S} \mathbb{Q} \Gamma \otimes_{\mathbb{Q}A} \mathbb{Q},$$

a rational vector space with basis the set $S \backslash \Gamma / A$ of double cosets SgA with $g \in \Gamma$.

In the proof of Theorem 1.1 we will need the above result in the case $n = 2$. In this case, if $H_2(S, \mathbb{Q})$ is finite dimensional then $|S \backslash \Gamma / A| < \infty$. Thus it is of interest to understand when a pair of subgroups in a group admit only finitely many double cosets. The following assertions are proved in [6], and are easily verified.

Proposition 2.6. *Let G be a group, A, B, C subgroups of G with $A \subset C$ and $|B \backslash G / A| < \infty$, and $f : G \rightarrow H$ a homomorphism. Then*

1. $|(B \cap C) \backslash C / A| < \infty$, and
2. $|f(B) \backslash f(G) / f(A)| < \infty$.

3 Some homological algebra

In this section we gather the homological algebra that we shall need in the proof of the main theorem.

3.1 The LHS spectral sequence

Associated to any short exact sequence of groups $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ one has the Lyndon-Hochschild-Serre (LHS) spectral sequence in homology (see [10, p. 171], or [12, p.303]). Given a $\mathbb{Q}B$ -module M , the E^2 page of the spectral sequence has (p, q) -entry $H_p(C, H_q(A, M))$, and the spectral sequence converges to $H_*(B, M)$.

The computations in this article concern $H_i(B)$ with trivial coefficients and $i \leq 2$. The terms of relevance on the E^2 page are

$$H_0(C, H_2(A)), H_1(C, H_1(A)), H_2(C, H_0(A)).$$

In each of our calculations, the first and last of these terms will be finitely generated and we will be concerned with the issue of finite generation for $H_1(C, H_1(A))$. The only non-zero derivative hitting this group is $d_2 : H_3(C, H_0(A)) \rightarrow H_1(C, H_1(A))$. Since the terms $E_{p,q}^\infty$ with $p + q = 2$ are the composition factors of a filtration of $H_2(B)$, we see that $H_2(B)$ will be finitely generated only if the quotient of $H_1(C, H_1(A))$ by the image of d_2 is finitely generated. But $H_3(C, H_0(A))$ will always be finitely generated in our setting, and hence $H_2(B)$ will be finitely generated only if $H_1(C, H_1(A))$ is finitely generated.

3.2 Fox calculus

Recall that a presentation $\langle X \mid R \rangle$ of a group G gives rise to a partial free $\mathbb{Q}G$ -resolution

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \mathbb{Q}G \rightarrow \mathbb{Q}$$

of the trivial G -module \mathbb{Q} . Here F_1 and F_2 are free modules with bases in one-to-one correspondence with X and R respectively.

The boundary maps ∂_1 and ∂_2 are easy to calculate. ∂_1 is given by an $X \times 1$ -matrix where the x -entry is $1 - x$, and ∂_2 is given by an $R \times X$ -matrix where the (r, x) -entry is the Fox derivative $\partial r / \partial x$, defined inductively on the length of r by the rules:

$$\begin{aligned} \frac{\partial}{\partial x}(1) &= 0 = \frac{\partial}{\partial x}(y) = \frac{\partial}{\partial x}(y^{-1}), \quad y \in X \setminus \{x\}, \\ \frac{\partial}{\partial x}(x) &= 1, \quad \frac{\partial}{\partial x}(x^{-1}) = x^{-1}, \quad \frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x}. \end{aligned}$$

Lemma 3.1. *Let $\Gamma = G *_C$ be an HNN extension with stable letter t , and let M be a $\mathbb{Q}\Gamma$ -module on which t acts trivially. Then $H_1(\Gamma, M)$ contains a subgroup that admits an epimorphism onto $H_0(C, M)$.*

Proof. Given a presentation $\langle X \mid R \rangle$ of G , and a set of generators Y for C (expressed as words in the alphabet X), we have a presentation for Γ of the form

$$\mathcal{P} = \langle X, t \mid R, ytu_yt^{-1} \ (y \in Y) \rangle$$

for Γ (where the u_y are words in X).

We calculate $H_1(\Gamma, M)$ by tensoring the resolution associated to this presentation with M and calculating ∂_2 using Fox calculus. Specifically, we have

$$\bigoplus_{y \in Y} M \oplus \bigoplus_{r \in R} M \xrightarrow{\partial_2} M \oplus \bigoplus_{x \in X} M \xrightarrow{\partial_1} M \rightarrow 0.$$

With regard to the visible decompositions, ∂_2 is given by a matrix of the form

$$\begin{pmatrix} \frac{\partial}{\partial t}(ytu_yt^{-1}) & * \\ 0 & * \end{pmatrix},$$

where $\frac{\partial}{\partial t}(ytu_yt^{-1}) = y - 1$, while the restriction of ∂_1 to the first factor is multiplication by $(1 - t)$. But since t acts trivially on M , multiplication by

$(1 - t)$ is the zero-map on M . Therefore the first factor M in dimension 1 is contained in the group of 1-cycles. Let A denote the subgroup of $H_1(\Gamma, M)$ generated by this group of 1-cycles.

Now the first coordinate of the image of ∂_2 is contained in the submodule N of M generated by $\{(y - 1)M, y \in Y\}$. It follows that the kernel of the natural epimorphism $M \rightarrow A$ is contained in N . Thus there is an epimorphism $A \rightarrow M/N$, and since Y generates C , the quotient M/N is the group of coinvariants $H_0(C, M)$. \square

3.3 A mapping cylinder argument

We are grateful to Peter Kropholler for a helpful conversation concerning the following lemma.

Lemma 3.2. *If G is of type $\text{FP}_n(\mathbb{Q})$ and acts cocompactly on a tree with all edge-stabilisers of type $\text{FP}_n(\mathbb{Q})$, then all vertex-stabilisers are also of type $\text{FP}_n(\mathbb{Q})$.*

Proof. If V, E are the vertex and edge sets of the tree, then the hypothesis on edge-stabilisers is equivalent to $\mathbb{Q}E$ having a projective $\mathbb{Q}G$ -resolution that is finitely generated in dimensions $\leq n$, and the assertion of the lemma is that the analogous property for $\mathbb{Q}V$ holds.

Since the action is cocompact, $\mathbb{Q}V$ is finitely generated as a $\mathbb{Q}G$ -module, so there is a projective $\mathbb{Q}G$ -resolution \mathcal{F} for $\mathbb{Q}V$ that is finitely generated in dimension 0.

Suppose inductively that $k < n$ and there is a projective $\mathbb{Q}G$ -resolution \mathcal{F} for $\mathbb{Q}V$ that is finitely generated in dimensions $\leq k$. Let \mathcal{P} be a projective $\mathbb{Q}G$ -resolution for $\mathbb{Q}E$ that is finitely generated in dimensions $\leq n$.

Let $f : \mathcal{P} \rightarrow \mathcal{F}$ be a chain-map lifting the boundary map $\mathbb{Q}E \rightarrow \mathbb{Q}V$, and form the mapping cylinder resolution $\mathcal{M} = \mathcal{M}(f)$ of \mathbb{Q} .

Thus $\mathcal{M}_i = \mathcal{P}_{i-1} \oplus \mathcal{F}_i$ is finitely generated in dimensions $i \leq k$. The boundary maps $D_i : \mathcal{M}_i \rightarrow \mathcal{M}_{i-1}$ are defined by

$$D_i(x, y) := (d^{\mathcal{P}}(x), (-1)^{i-1}f(x) + d^{\mathcal{F}}(y)),$$

where $d^{\mathcal{P}}$ and $d^{\mathcal{F}}$ are the boundary maps in \mathcal{P} and \mathcal{F} respectively.

Since $k < n$ and G is of type $\text{FP}_n(\mathbb{Q})$, the kernel of the k -th boundary map $D_k : \mathcal{M}_k \rightarrow \mathcal{M}_{k-1}$ is finitely generated. Since also \mathcal{P}_{k+1} is finitely generated, there is a finitely generated $\mathbb{Q}G$ -direct summand L of \mathcal{F}_{k+1} such that

(i) $D_{k+1}(\mathcal{P}_k \oplus L) = \text{Ker}(D_k) = D_{k+1}(\mathcal{M}_{k+1})$; and

(ii) $f_{k+1}(\mathcal{P}_{k+1}) \subset L$.

Now suppose that $a \in \text{Ker}(d_k : \mathcal{F}_k \rightarrow \mathcal{F}_{k-1})$. Then $(0, a) \in \text{Ker}(D_k)$, so by (i) we can find $b \in \mathcal{P}_k$ and $c \in L$ such that

$$(0, a) = D_{k+1}(b, c) = (d_k(b), (-1)^k f_k(b) + d_{k+1}(c)).$$

Then $b \in \text{Ker}(d_k) = \text{Im}(d_{k+1})$, so we can find $e \in \mathcal{P}_{k+1}$ such that $b = d_{k+1}(e)$. Then $f_k(b) = f_k(d_{k+1}(e)) = d_{k+1}(f_{k+1}(e))$, and $a = (-1)^k f_k(b) + d_{k+1}(c) = d_{k+1}((-1)^k f_{k+1}(e) + c) \in d_{k+1}(L)$ (using (ii)).

Hence $\text{Ker}(d_k : F_k \rightarrow F_{k-1}) = d_{k+1}(L)$ is finitely generated. It follows that there is a projective $\mathbb{Q}G$ -resolution of $\mathbb{Q}V$ that is finitely generated in dimensions $\leq k + 1$.

By induction, there is a projective $\mathbb{Q}G$ -resolution of $\mathbb{Q}V$ that is finitely generated in dimensions $\leq n$. In other words, the vertex stabilisers of the action are all of type $\text{FP}_n(\mathbb{Q})$, as claimed □

4 The main argument

In this section we prove Theorem 1.1 and Corollary 1.2.

4.1 Reduction to subdirect products

Let Γ_1 and Γ_2 be non-abelian limit groups. Let $S \subset \Gamma_1 \times \Gamma_2$ be a subgroup and let $p_i : S \rightarrow \Gamma_i$ be the natural projection. If S is of type $\text{FP}_2(\mathbb{Q})$, then $H_1(S, \mathbb{Q})$ is finite dimensional, so $H_1(p_i(S), \mathbb{Q})$ is finite dimensional for $i = 1, 2$. By Theorem 2.2, $p_i(S) \subset \Gamma_i$ is a limit group. Replacing Γ_i by $p_i(S)$, we reduce to the case where S projects onto each factor Γ_i .

Let $L_i = S \cap \Gamma_i$. If L_1 were trivial, then $S \cong p_2(S)$ would be a limit group, satisfying the conclusion of Corollary 1.2. Thus we may assume that L_i is non-trivial for $i = 1, 2$ (and Corollary 1.2 becomes a weak form of Theorem 1.1).

Remark 4.1. Since $p_i(S) = \Gamma_i$ and L_i is normal in S , we also have that $L_i = p_i(L_i)$ is normal in Γ_i .

4.2 Subgroups of finite index

Throughout the proof we shall repeatedly pass to subgroups of finite index in one or other of the Γ_i . When we do so, we shall be assuming that S is replaced with the inverse image of this subgroup under the projection $p_i : S \rightarrow \Gamma_i$ and Γ_j ($j \neq i$) is replaced by $p_j p_i^{-1}(H_i)$. This does not affect L_j .

Remark 4.2. It is because of the need to pass to subgroups of finite index that we have assumed that S is of type FP_2 rather than simply that $H_2(S)$ is finitely generated: the former hypothesis passes to subgroups of finite index, whereas the latter does not necessarily.

4.3 Useful lemmas and special cases

Much of our proof will revolve around the LHS spectral sequences associated to the short exact sequences for the projections of S to the two factors $1 \rightarrow L_2 \rightarrow S \rightarrow \Gamma_1 \rightarrow 1$ and $1 \rightarrow L_1 \rightarrow S \rightarrow \Gamma_2 \rightarrow 1$. In particular, following the discussion in subsection 3.1, we shall use the hypothesis that S is of type FP_2 by assuming that $H_1(\Gamma_i, H_1(L_j, \mathbb{Q}))$ is finitely generated for $j \neq i$. In the ensuing argument we shall make several appeals to the following:

Lemma 4.3. *Let S be a subdirect product of two non-abelian limit groups Γ_1 and Γ_2 , such that the intersections $L_1 = S \cap \Gamma_1$ and $L_2 = S \cap \Gamma_2$ are both non-trivial. If either $H_1(L_1, \mathbb{Q})$ or $H_1(L_2, \mathbb{Q})$ is finite dimensional, then S has finite index in $\Gamma_1 \times \Gamma_2$.*

Proof. Suppose that $H_1(L_1, \mathbb{Q})$ is finite dimensional. It follows from Theorem 2.2 that L_1 is finitely generated, and from Theorem 2.3 and Remark 4.1 that L_1 has finite index in Γ_1 . Therefore, after replacing Γ_1 by L_1 , we can split $1 \rightarrow L_2 \rightarrow S \rightarrow \Gamma_1 \rightarrow 1$ to obtain $S = \Gamma_1 \times L_2$. Thus L_2 , being a retract of S , is finitely generated, so of finite index in Γ_2 . \square

Proposition 4.4. *Let G_1 be a group and let Γ_2 be a limit group. Let $S \subset G_1 \times \Gamma_2$ be a subgroup of type $\text{FP}_2(\mathbb{Q})$ that projects onto Γ_2 , and that intersects Γ_2 non-trivially. Suppose that Γ_2 can be expressed as an amalgamated free product or HNN extension with cyclic edge-group C .*

If $C \cap S$ has finite index in C , then $H_1(G_1 \cap S, \mathbb{Q})$ is finite dimensional.

Proof. Let $L_1 = G_1 \cap S$ and let $L_2 = \Gamma_2 \cap S$. Since the cyclic subgroups of Γ_2 are closed in the profinite topology, we may pass to a subgroup of finite index which intersects C in $C \cap L_2$ and hence assume that $C \subset L_2$.

By appealing to Theorem 2.4 and passing to a further subgroup of finite index, we may assume that Γ_2 is an HNN extension $G *_C$ with stable letter $t \in L_2$.

Consider the Lyndon-Hochschild-Serre spectral sequence in \mathbb{Q} -homology corresponding to the group extension

$$1 \rightarrow L_1 \rightarrow S \rightarrow \Gamma_2 \rightarrow 1.$$

The action of S by conjugation on L_1 induces an action of $\Gamma_2 = S/L_1$ on $H_*(L_1, \mathbb{Q})$ in which L_2 acts trivially. In particular t and C act trivially on the Γ_2 -module $H_k(L_1, \mathbb{Q})$, for each k .

The $(1, 1)$ -term $E_{1,1}^3$ on the E^3 -page of this spectral sequence survives to infinity, and so is isomorphic to a subquotient of $H_2(S, \mathbb{Q})$, which is finite-dimensional, by hypothesis. Hence $E_{1,1}^3$ is finite-dimensional. But $E_{1,1}^3$ is the cokernel of the differential $d : E_{3,0}^2 \rightarrow E_{1,1}^2$ from the E^2 -page. Moreover, $E_{3,0}^2 = H_3(\Gamma_2, H_0(L_1, \mathbb{Q})) = H_3(\Gamma_2, \mathbb{Q})$ is finite-dimensional, since the limit group Γ_2 has type $\text{FP}_\infty(\mathbb{Q})$. Hence $E_{1,1}^2 = H_1(\Gamma_2, H_1(L_1, \mathbb{Q}))$ is finite-dimensional.

Since t acts trivially on $H_*(L_1, \mathbb{Q})$, we may apply Lemma 3.1, with $\Gamma = \Gamma_2$ and $M = H_1(L_1, \mathbb{Q})$. It follows that $H_0(C, H_1(L_1, \mathbb{Q}))$ is finite-dimensional. And $H_0(C, H_1(L_1, \mathbb{Q})) \cong H_1(L_1, \mathbb{Q})$ since C acts trivially on $H_*(L_1, \mathbb{Q})$. \square

The spectral sequence argument ending with the penultimate paragraph of the preceding proof admits a straightforward generalisation as follows. If S is of type FP_n , $k \leq n-1$ and $H_j(L_1, \mathbb{Q})$ is finite dimensional for $j < k$, then $E_{1,k}^2 = H_1(\Gamma_2, H_k(L_1, \mathbb{Q}))$ is finite-dimensional. Taking $M = H_k(L_1, \mathbb{Q})$ in the final paragraph then implies $H_k(L_1, \mathbb{Q}) \cong H_0(C, H_k(L_1, \mathbb{Q}))$ is finite-dimensional.

To generalise further, recall that Theorem 2.4 is a weak form of what we actually proved in [6] concerning the existence of stable letters in normal subgroups. Correspondingly, we can weaken the hypotheses on Γ_2 and C in the above proposition, and appeal to [6, Corollaries 2.2 and 3.2] instead of Theorem 2.4. These remarks lead to the following generalisation (which will not be needed in the sequel).

Proposition 4.5. *Let G_1 and G_2 be groups. Suppose G_2 is of type $\text{FP}_{n+1}(\mathbb{Q})$ and admits an l -acylindrical² splitting (for some $l > 0$) over a subgroup C that is closed in the profinite topology.*

²A splitting is l -acylindrical if arcs of length l in the associated Bass-Serre tree have trivial stabilizers.

Let $S \subset G_1 \times G_2$ be a subgroup of type FP_n that intersects G_2 non-trivially and intersects C in a subgroup of finite index. Then $H_i(G_1 \cap S, \mathbb{Q})$ is finite dimensional for $i \leq n - 1$.

In the light of Lemma 4.3, the following is an immediate consequence of Proposition 4.4.

Corollary 4.6. *Let Γ_1 and Γ_2 be non-abelian limit groups. Let $S \subset \Gamma_1 \times \Gamma_2$ be a subdirect product of type $\text{FP}_2(\mathbb{Q})$ that intersects each of the factors non-trivially. Suppose that Γ_2 has a graph-of-groups decomposition with a cyclic edge-group C .*

If $C \cap S$ has finite index in C , then S has finite index in $\Gamma_1 \times \Gamma_2$.

In the case where C is trivial we have:

Corollary 4.7. *Let Γ_1 and Γ_2 be non-abelian limit groups and let $S \subset \Gamma_1 \times \Gamma_2$ be a subdirect product of type $\text{FP}_2(\mathbb{Q})$ that intersects each of the factors non-trivially. If Γ_2 is a nontrivial free product, then S has finite index in $\Gamma_1 \times \Gamma_2$.*

The remaining non-abelian limit groups of height 0 are covered by:

Corollary 4.8. *Let Γ_1 and Γ_2 be non-abelian limit groups with Γ_2 the fundamental group of a compact surface. Let $S \subset \Gamma_1 \times \Gamma_2$ be a subdirect product that intersects each of the factors non-trivially.*

If S is of type $\text{FP}_2(\mathbb{Q})$, then S has finite index in $\Gamma_1 \times \Gamma_2$.

Proof. Passing to an index 2 subgroup of Γ_2 if necessary, we may assume that the surface in question is orientable, and by Corollary 4.7 we may assume that it is closed.

Fix $\alpha \neq 1$ in L_2 . By a theorem of Scott [16], there is a finite-sheeted cover of Σ in which α lifts to a simple closed curve. Replacing Γ_2 by the corresponding subgroup of finite index, we may assume that Γ_2 splits over the cyclic subgroup $\langle \alpha \rangle \subset L_2$. The result then follows from Corollary 4.6. \square

The following observation will be used to control the way in which S intersects each non-abelian vertex group in the decomposition of Γ_2 provided by Proposition 2.1.

Lemma 4.9. *Let Γ_1 and Γ_2 be non-abelian limit groups and $S \subset \Gamma_1 \times \Gamma_2$ a subdirect product of type $\text{FP}_2(\mathbb{Q})$ with $L_i = \Gamma_i \cap S$ non-trivial for $i = 1, 2$. Consider a graph-of-groups decomposition \mathcal{G} of Γ_2 that has cyclic edge groups and a vertex group Γ_v that is a non-abelian limit group. Then $\Gamma_v \cap L_2 \neq \{1\}$.*

Proof. In \mathcal{G} we fix an edge e incident to the vertex v and denote the corresponding edge group $\Gamma_e \subset \Gamma_2$. We also fix a graph-of-groups decomposition of Γ_1 that has a cyclic edge group Γ_f (see Proposition 2.1 and the remark following it).

Using Theorem 2.4, we may replace Γ_1 and Γ_2 by subgroups of finite index and assume that they split as HNN extensions with amalgamated subgroups Γ_e, Γ_f and stable letters in L_1 and L_2 , respectively. The Double Coset Lemma (Theorem 2.5) then tells us that $H_2(S, \mathbb{Q})$ contains a subgroup isomorphic to

$$\mathbb{Q}(S \backslash (\Gamma_1 \times \Gamma_2) / (\Gamma_f \times \Gamma_e)).$$

Since $H_2(S, \mathbb{Q})$ is finitely generated, $|S \backslash (\Gamma_1 \times \Gamma_2) / (\Gamma_f \times \Gamma_e)| < \infty$. So, by Proposition 2.6, $|(S \cap (\Gamma_f \times \Gamma_2)) \backslash (\Gamma_f \times \Gamma_2) / (\Gamma_f \times \Gamma_e)| < \infty$. Applying the projection $p_2 : S \rightarrow \Gamma_2$, we have $|p_2(S \cap (\Gamma_f \times \Gamma_2)) \backslash \Gamma_2 / \Gamma_e| < \infty$, by Proposition 2.6. Hence

$$|\Gamma_v \cap p_2(S \cap (\Gamma_f \times \Gamma_2)) \backslash \Gamma_v / \Gamma_e| < \infty,$$

again by Proposition 2.6. Now, $S \cap (\Gamma_f \times \Gamma_2)$ is generated by $L_2 = p_2(L_2)$ and a cyclic group, so if $\Gamma_v \cap L_2 = \{1\}$ then $C = \Gamma_v \cap p_2(S \cap (\Gamma_f \times \Gamma_2))$ would be a cyclic group.

But if C were cyclic, then in the non-abelian, residually-free group Γ_v we would have two cyclic subgroups, C and Γ_e , such that $|C \backslash \Gamma_v / \Gamma_e| < \infty$. But this is impossible, for there is an epimorphism, ϕ say, from Γ_v onto a non-abelian free group F , where the conclusion (from Proposition 2.6) that the cyclic subgroups $\phi(C)$ and $\phi(\Gamma_e)$ satisfy $|\phi(C) \backslash F / \phi(\Gamma_e)| < \infty$ is absurd. \square

4.4 Completing the proof of Theorem 1.1

We have non-abelian limit groups Γ_1, Γ_2 and a subdirect product $S \subset \Gamma_1 \times \Gamma_2$ of type $\text{FP}_2(\mathbb{Q})$ such that $L_1 = S \cap \Gamma_1$ and $L_2 = S \cap \Gamma_2$ are non-trivial. We want to prove that S has finite index in $\Gamma_1 \times \Gamma_2$. We shall do so by induction on

$$h := \text{height}(\Gamma_1) + \text{height}(\Gamma_2),$$

where *height* is defined as in subsection 2.1. Let $h_i = \text{height}(\Gamma_i)$ for $i = 1, 2$.

Corollaries 4.7 and 4.8 cover the case where $h_2 = 0$ and the case where Γ_2 is freely decomposable. Thus, by Proposition 2.1, we may assume that Γ_2 has a graph-of-groups decomposition $\Gamma_2 \cong \pi_1(\mathcal{G}, X)$ with infinite-cyclic edge

groups such that each edge is incident at a vertex where the vertex group is a non-abelian limit group of height $\leq h_2 - 1$. Let Γ_v be such a vertex group and let Γ_e be an incident edge group. Lemma 4.9 tells us that $\Gamma_v \cap L_2 \neq \{1\}$. Corollary 4.6 tells us that S has finite index in $\Gamma_1 \times \Gamma_2$ unless $\Gamma_e \cap L_2 = \{1\}$. Thus we proceed with the assumption that $\Gamma_e \cap L_2 = \{1\}$ for all edges e in the graph X . (This will ultimately lead to a contradiction.)

Let $p_2 : S \rightarrow \Gamma_2$ be the natural projection and let $S_e = p_2^{-1}(\Gamma_e)$.

Claim: S_e is a limit group.

Since $\Gamma_e \cap L_2$ is trivial, the projection $S \rightarrow \Gamma_1$, which has kernel L_2 , restricts to an injection on S_e . Thus S_e is isomorphic to a subgroup of a limit group and the claim will follow from Theorem 2.2 if we prove that $H_1(S_e, \mathbb{Q})$ is finite dimensional.

By Theorem 2.4, we may assume that Γ_2 splits as an HNN extension with amalgamated subgroup Γ_e and stable letter $t \in L_2$. (This reduction involves passing to a subgroup of finite index $S'_e \subset S_e$, but if $H_1(S'_e, \mathbb{Q})$ is finite dimensional, then $H_1(S_e, \mathbb{Q})$ is.)

Let $M = H_1(L_1, \mathbb{Q})$. Analysing the LHS spectral sequence for

$$1 \rightarrow L_1 \rightarrow S \xrightarrow{p_2} \Gamma_2 \rightarrow 1$$

exactly as in the proof of Proposition 4.4, we see that since $H_2(S, \mathbb{Q})$ is finite dimensional, $H_1(\Gamma_2, M)$ is finite dimensional, and hence so is $H_0(\Gamma_e, M)$, by Lemma 3.1.

We can now calculate $H_1(S_e, \mathbb{Q})$ using the LHS spectral sequence for

$$1 \rightarrow L_1 \rightarrow S_e \xrightarrow{p_2} \Gamma_e \rightarrow 1.$$

The relevant terms on the E^2 page are $H_1(\Gamma_e, H_0(L_1, \mathbb{Q})) = H_1(\Gamma_e, \mathbb{Q}) \cong \mathbb{Q}$ and $H_0(\Gamma_e, M)$, which we have just argued is finitely generated. Thus $H_1(S_e, \mathbb{Q})$ is finite dimensional and our claim is proved.

The graph of groups decomposition $\Gamma_2 \cong \pi_1(\mathcal{G}, X)$ pulls back to a graph of groups decomposition $S \cong \pi_1(\mathcal{S}, X)$ with the same underlying graph X . Here the vertex and edge groups have the form $S_v = p_2^{-1}(\Gamma_v)$ and $S_e = p_2^{-1}(\Gamma_e)$ respectively, as v and e vary over the vertices and edges of X . We have just shown that the edge groups S_e are all limit groups. In particular each S_e has a finite classifying space [1], and so is of type $\text{FP}_\infty(\mathbb{Q})$.

We are assuming that S is of type $\text{FP}_2(\mathbb{Q})$, so by Lemma 3.2, we can now deduce that *the vertex groups S_v are all of type $\text{FP}_2(\mathbb{Q})$* .

For the remainder of the proof we focus our attention on a single non-abelian vertex group $V := \Gamma_v$ in \mathcal{G} and the corresponding vertex group S_v in \mathcal{S} . Let Λ be the projection of S_v to Γ_1 . Since $H_1(S_v, \mathbb{Q})$ is finite dimensional, so is $H_1(\Lambda, \mathbb{Q})$. By Theorem 2.2, Λ is a limit group (of height at most h_1). Moreover, V is a limit group of height strictly less than h_2 (see 2.1).

Thus we have a product of limit groups $\Lambda \times V$ with the sum of heights $< h$, and a subgroup $S_v \subset \Lambda \times V$ that is of type $\text{FP}_2(\mathbb{Q})$ and projects onto each factor. Moreover $S_v \cap \Lambda = L_1$, which is non-abelian since it is non-trivial and normal in the non-abelian limit group Γ_1 . Hence, in particular, Λ is non-abelian.

We also have, by hypothesis, that V is non-abelian and that $V \cap L_2 \neq \{1\}$. Thus we may apply our inductive hypothesis to conclude that S_v has finite index in $\Lambda \times V$. Since $L_1 = S_v \cap \Lambda$, we conclude that L_1 has finite index in Λ and hence is finitely generated. Lemma 4.3 then implies that S has finite index in $\Gamma_1 \times \Gamma_2$. This contradicts the fact that $S \cap \Gamma_e$ was assumed to be trivial for each of the infinite-cyclic edge groups $\Gamma_e \subset \Gamma_2$. \square

5 Abelian factors

Our main theorem concerns a subdirect product of two non-abelian limit groups. The result is false as stated in the case where one or both of the limit groups is allowed to be abelian.

An abelian limit group is just a free abelian group of finite rank. In particular, *every* subgroup of a direct product of two such groups is of type $\text{FP}_\infty(\mathbb{Q})$. More generally, one can easily construct counterexamples to the exact analogue of the main theorem in which one of the factors is allowed to be non-abelian, along the following lines.

Let Γ_1 be a non-abelian limit group, and $f_1 : \Gamma_1 \rightarrow \mathbb{Z}^k$ an epimorphism for some $k \geq 1$. Let $\Gamma_2 = \mathbb{Z}^m$ for some $m > k$, and let $f_2 : \Gamma_2 \rightarrow \mathbb{Z}^k$ be an epimorphism. Let $S \subset \Gamma_1 \times \Gamma_2$ be the fibre product of f_1 and f_2 . Then $S \cong \Gamma_1 \times \mathbb{Z}^{m-k}$ is a subdirect product of type FP_∞ and $S \cap \Gamma_i$ is non-trivial for $i = 1, 2$, but S does not have finite index in $\Gamma_1 \times \Gamma_2$.

It is not difficult to show that *all* counterexamples to the exact analogue of the main theorem are, up to finite index, isomorphic to one of the above types. In particular, in each case the group S under consideration is itself a direct product of limit groups, up to finite index.

This observation is a special case of Corollary 1.3. We are now ready to

prove Corollary 1.3, which we restate here as

Corollary 5.1. *Let $\Gamma_1, \dots, \Gamma_n$ be limit groups, with $\Gamma_3, \dots, \Gamma_n$ abelian, and let $S \subset \Gamma_1 \times \dots \times \Gamma_n$ be a subdirect product such that S is of type $\text{FP}_m(\mathbb{Q})$ (where $m = \min(n, 3)$). Then S is isomorphic to a subgroup of finite index in a product of at most m limit groups.*

Proof. If all the Γ_i are abelian, then S is a free abelian group of finite rank, and hence a limit group. If, on the other hand, none of the Γ_i are abelian, then the result follows from Corollary 1.2. Hence we may assume that $n \geq 2$, that Γ_1 is non-abelian, and that Γ_n is abelian.

If Γ_{n-1} is abelian, let G_{n-1} be the image of S under the direct projection $\Gamma_1 \times \dots \times \Gamma_n \rightarrow \Gamma_{n-1} \times \Gamma_n$. The result for $S \subset \Gamma_1, \dots, \Gamma_n$ follows easily from the corresponding result for $S \subset \Gamma_1 \times \dots \times \Gamma_{n-2} \times G_{n-1}$.

Hence we may assume that there is precisely one abelian factor, namely Γ_n , and hence that $n \in \{2, 3\}$. Let $L_n = S \cap \Gamma_n$. Since Γ_n is abelian, we may assume (up to replacing Γ_n by a finite index subgroup) that L_n is a direct factor of Γ_n . Say $\Gamma_n \cong L_n \oplus M$.

Let $\pi : \Gamma_n \rightarrow L_n$ be the projection with kernel M , and define $\theta := \pi \circ p_n : \Gamma_1 \times \dots \times \Gamma_n \rightarrow L_n$. Then $\theta|_S$ is a splitting of the inclusion map $L_n \rightarrow S$, so $S \cong T \times L_n$, where $T = \text{Ker}(\theta|_S) = S \cap (\Gamma_1 \times \dots \times \Gamma_{n-1} \times M)$.

Now $T \cap M = \{1\}$, and so T is isomorphic to a subgroup of $\Gamma_1 \times \dots \times \Gamma_{n-1}$. Moreover, T is of type $\text{FP}_m(\mathbb{Q})$ since $S \cong T \times L_n$ is of type $\text{FP}_m(\mathbb{Q})$. It follows from the main theorem that T has finite index in $\Gamma_1 \times \dots \times \Gamma_{n-1}$.

Hence S is, up to finite index, a product of at most m limit groups, as claimed. \square

6 The natural conjecture for n factors

The results of this paper, and of [6], encourage us to make the following conjecture, which would give a positive answer to a question of Sela [21].

Conjecture 6.1. *Let $\Gamma_1, \dots, \Gamma_n$ be limit groups, and S a subgroup of $\Gamma_1 \times \dots \times \Gamma_n$ such that S has type $\text{FP}_n(\mathbb{Q})$. Then S has a subgroup of finite index that is itself a direct product of n or fewer limit groups.*

More precisely, we believe:

Conjecture 6.2. *Let $\Gamma_1, \dots, \Gamma_n$ be non-abelian limit groups and $S \subseteq \Gamma_1 \times \dots \times \Gamma_n$ a subdirect product that intersects each Γ_i non-trivially. If exactly $k > 0$ of the intersections $S \cap \Gamma_i$ are not finitely generated, then there exists a subgroup of finite index $S_0 \subset S$ such that $H_k(S_0, \mathbb{Q})$ is infinite dimensional.*

The arguments in [5] and [6] prove these conjectures in the case where the Γ_i are subgroups of elementarily free groups. Those arguments used in a crucial way the fact that the top of the ω -residually free tower decomposition is a quadratic block. It does not seem plausible to extend the methods of [6] to the case where each of the Γ_i has an abelian block at the top of its ω -rft decomposition.

In the present paper, we addressed the case where at most 2 of the Γ_i are non-abelian, which reduces easily to the case of precisely two factors, each non-abelian. This more algebraic approach (which is closer to the argument we used with Miller and Short in the surface case [7]) may eventually lead to a proof in the general case. However, there are considerable difficulties associated to the fact that one must repeatedly pass to subgroups of finite index. It seems possible that these may be overcome by converting many of our arguments based on the finiteness of various homology groups into arguments based on properties of corresponding homology functors, but we do not know how to do this at present.

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