THE VIRTUAL FIRST BETTI NUMBER OF SOLUBLE GROUPS

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Abstract. We show that if a group G is finitely presented and nilpotent-by-abelian-by-finite, then there is an upper bound on $\dim_{\mathbb{Q}} H_1(M,\mathbb{Q})$, where M runs through all subgroups of finite index in G.

1. Introduction

The *virtual first betti number* of a finitely generated group G is defined as

$$\mathrm{vb}_1(G) = \sup\{\dim H_1(S,\mathbb{Q}) \mid S \leq G \text{ of finite index }\}.$$

A group is said to be *large* if it has a subgroup of finite index that maps onto a non-abelian free group. If G is large then $\mathrm{vb}_1(G) = \infty$. It is easy to find finitely generated groups G that are not large but have $\mathrm{vb}_1(G) = \infty$. For example, in the metabelian group $\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid \forall n, [a, t^{-n}at^n] = 1\rangle$, the subgroup $S_m < \mathbb{Z} \wr \mathbb{Z}$ generated by t^m and the conjugates of a has index m and $H_1(S_m,\mathbb{Z}) = \mathbb{Z}^{m+1}$. In contrast, no example is known of a *finitely presented* group that is not large but has $\mathrm{vb}_1(G) = \infty$ (cf. [11], [17]). Since amenable groups do not contain non-abelian free subgroups, one might hope to resolve this issue by finding a finitely presented amenable group with $\mathrm{vb}_1(G) = \infty$, but this seems to be a non-trivial matter.

We shall prove in this paper that for large classes of finitely presented soluble groups $\mathrm{vb}_1(G)$ is always finite. One would like to prove that the same is true for all finitely presented soluble groups, but here one faces the profound difficulty of deciding which soluble groups admit finite presentations; this is unknown even for abelian-by-polycyclic and nilpotent-by-abelian groups.

In the case of metabelian groups, finite presentability is completely understood in terms of the Bieri-Strebel invariant [8]. Some sufficient conditions for finite presentability of nilpotent-by-abelian groups were considered by Isaac [15] and later Groves [13]. In the case of S-arithmetic nilpotent-by-abelian groups G one knows more thanks to the work of Abels [1]: if G is an extension of a nilpotent group N by an abelian group Q then G is finitely presented if and only if it is of type FP_2, which it is if and only if $H_2(N,\mathbb{Z})$ is finitely generated as a $\mathbb{Z}Q$-module (where the Q action is induced by conjugation) and[1] G/N' is finitely presented as a group. The first of these conditions is an easy consequence of the fact that $\mathbb{Z}Q$ is a Noetherian ring and the second is a corollary of a result of Bieri and Strebel that every metabelian quotient of a group of type FP_2 that does not contain non-cyclic free subgroups is finitely presented [8]. The case where G is an extension of an abelian normal subgroup A by a polycyclic group Q was approached by Brookes and Groves who studied modules over crossed products of a division ring by a free abelian group [4], [5] and [6].

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[1]throughout this article, H' denotes the commutator subgroup of a group H

Given this background, the natural place to begin our investigation into the virtual first betti number of finitely presented soluble groups is in the setting of metabelian groups. Using methods from commutative algebra, we prove (Theorem 4.3) that if G is finitely presented and metabelian, then $\mathrm{vb}_1(G)$ is finite. (The hypothesis that one actually needs to impose on G is somewhat weaker than finite presentability; see Remark 6.5.) The metabelian case is used in the proof of our main theorem, which is the following.

Theorem A. *Let G be a finitely presented group. If G is nilpotent-by-abelian-by-finite, then* $\mathrm{vb}_1(G)$ *is finite.*

Our proof of this theorem relies on the fact that all metabelian quotients of soluble groups of type FP_2 are finitely presented [8, Thm. 5.5], as well as a technical result concerning the homology of subgroups of finite index (Proposition 6.2). Groves, Kochloukova and Rodrigues [14, Thm. A] proved that if an abelian-by-polycyclic group G is of type FP_3 then it is nilpotent-by-abelian-by-finite, in which case $\mathrm{vb}_1(G)$ is finite by Theorem A. The same is true of all soluble groups of type FP_∞, because they are constructible [16], hence nilpotent-by-abelian-by-finite, but in this case stronger finiteness results were already known: constructible soluble groups are obtained from the trivial group by finite sequences of ascending HNN extensions and finite extensions, from which it follows that they have finite Prüfer rank (i.e. there is an upper bound on the number of generators for the finitely generated subgroups).

It is natural to wonder if Theorem A might remain true when the field of rationals $\mathbb{Q}$ in the definition of virtual betti number is replaced with other coefficient fields, such as the field with p elements $\mathbb{F}_p$. We shall see in Section 5 that it does not.

Conjecture: *If G is finitely presented and soluble, then* $\mathrm{vb}_1(G)$ *is finite.*

It is difficult to construct finitely presented soluble groups that are not nilpotent-by-abelian-by-finite. The examples provided by the constructions of Robinson and Strebel [21] all satisfy the conjecture.

While editing the final version of this work, we learnt that Andrei Jaikin-Zapirain has, in unpublished work, also proved Theorem A in the metabelian case. Higher dimensional analogues of Theorem A are considered in the forthcoming PhD thesis of Mokari [19].

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2. Preliminary results

2.1. **Preliminaries on finitely presented metabelian groups.** We fix a short exact sequence of groups $A \rightarrowtail G \twoheadrightarrow Q$, where A and Q are abelian and G is finitely generated. The action of G on A by conjugation induces an action of Q, which enables us to regard A as a right $\mathbb{Z}Q$-module. Because G is finitely generated and Q is finitely presented, A is finitely generated as a $\mathbb{Z}Q$-module.

Associated to a non-zero real character $\chi : Q \to \mathbb{R}$ one has the monoid

$$Q_\chi = \{g \in Q \mid \chi(g) \geq 0\}.$$

The character sphere $S(Q)$ is the set of equivalence classes in $\mathrm{Hom}(Q, \mathbb{R}) \smallsetminus \{0\}$ under the relation that identifies $\chi_1 \sim \chi_2$ if $\chi_1 = \lambda\chi_2$ for some $\lambda > 0$. We write $[\chi]$ for the class of χ. Following Bieri and Strebel [8], let

$$\Sigma_A(Q) = \{[\chi] \mid A \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module}\}.$$

By definition, the $\mathbb{Z}Q$-module A is *2-tame* if $\Sigma_A(Q)^c = S(Q) \smallsetminus \Sigma_A(Q)$ contains no pair of antipodal points. According to [8, Thm. 5.4], G is finitely presented if and only if A is a 2-tame $\mathbb{Z}Q$-module, and this happens precisely when G is of homological type FP_2. We refer the reader to [10] for general results concerning groups of type FP_m. If $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ is an exact sequence of $\mathbb{Z}Q$-modules, then $\Sigma_{A_2}(Q)^c = \Sigma_{A_1}(Q)^c \cup \Sigma_{A_3}(Q)^c$ (see [8, Prop. 2.2]), hence every quotient of a 2-tame $\mathbb{Z}Q$-module is 2-tame.

2.2. **Tensor products and finite presentability.** Let R be a noetherian commutative ring with unit 1 and let W be a finitely generated RQ-module. As above, we have a Sigma invariant $\Sigma_W(Q) = \{[\chi] \mid W \text{ is finitely generated as } RQ_\chi - \text{module}\}$, and W is defined to be 2-tame as an RQ-module if $\Sigma_W^c(Q) = S(Q) \setminus \Sigma_W(Q)$ has no pair of antipodal points.

The question of when the tensor square $W \otimes_R W$ is finitely generated as an RQ-module (with Q acting diagonally) is addressed in [7], where it is shown that $[\chi]$ lies in $\Sigma_W^c(Q)$ if and only if the ring $S = RQ/\mathrm{ann}_{RQ}(W)$ admits a real valuation $v : S \to \mathbb{R} \cup \{\infty\}$ (in the sense of Bourbaki) that extends χ and is such that the restriction v_0 of v to the image $\overline{R}$ of R in S is non-negative and discrete. By [7], $W \otimes_R W$ is finitely generated as an RQ-module if and only if there is no pair of antipodal elements $[\chi], -[\chi] \in \Sigma_W^c(Q)$ that can be lifted to valuations of S that have the same restriction v_0 to $\overline{R}$, with v_0 discrete and non-negative. (These last conditions on v_0 are automatic if $\overline{R}$ is $\mathbb{Z}$.)

Returning to the context of paragraph (2.1), we apply these general considerations with $W = A \otimes \mathbb{Q}$ and $R = \mathbb{Q}$, in which case $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We deduce that if there exists a group extension $A \rightarrowtail G \twoheadrightarrow Q$, with G finitely presented, then $W = A \otimes \mathbb{Q}$ is 2-tame as a $\mathbb{Q}Q$-module, and $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action.

We shall also need a refinement of this observation that involves the annihilator $\mathrm{ann}_{\mathbb{Z}Q}(A)$ of A in $\mathbb{Z}Q$, which we denote I. In [9, (1.3)] Bieri and Strebel prove that

$$\Sigma_A(Q) = \Sigma_{\mathbb{Z}Q/I}(Q).$$

Thus if A is 2-tame as a $\mathbb{Z}Q$-module, then so is $\mathbb{Z}Q/I$.

Lemma 2.1. *If there exists a group extension $A \rightarrowtail G \twoheadrightarrow Q$ with A and Q abelian and G finitely presented, and $I = \mathrm{ann}_{\mathbb{Z}Q}(A)$, then $(\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} (\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action.*

2.3. **Preliminaries on commutative algebra.** We will need the following basic facts from commutative algebra; for details see, for example, [3], [2] or [12]. Let Q be a finitely generated abelian group and recall that the *Krull dimension* of a commutative ring is the supremum of the lengths of all chains of prime ideals in the ring.

(1) The radical $\sqrt{J}$ of each ideal $J \lhd \mathbb{Q}Q$ is the intersection of the finitely many prime ideals that contain J and are minimal subject to this condition.
(2) Finite dimensional $\mathbb{Q}$-algebras are Artinian and therefore have Krull dimension 0.

Throughout, if R is a commutative ring and m a positive integer, then R^m will denote the subring generated by m-th powers, **except** that $\mathbb{Z}^n$ and $\mathbb{Q}^n$ will denote Cartesian powers. Where no ring is specified, tensor products are assumed to be taken over $\mathbb{Z}$.

3. A finiteness result in commutative algebra

Lemma 2.1 assures us that the following theorem applies to the modules that arise from short exact sequences $N \rightarrowtail G \twoheadrightarrow \mathbb{Z}^n$ associated to finitely presented metabelian groups.

Theorem 3.1. *Let $Q \cong \mathbb{Z}^n$ be a group and let $S = \mathbb{Z}Q/I$ be a commutative ring such that $(S \otimes_{\mathbb{Z}} S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action. Then,*

$$\sup_m \dim_{\mathbb{Q}}(S \otimes_{\mathbb{Z}Q^m} \mathbb{Q}) < \infty.$$

Proof. Let $B = S \otimes \mathbb{Q} = \mathbb{Q}Q/J$ and for each positive integer m define $J_m \lhd \mathbb{Q}Q$ to be $(J, Q^m - 1)$ and

$$B_m := B \otimes_{\mathbb{Q}Q^m} \mathbb{Q} = \mathbb{Q}Q/J_m \cong S \otimes_{\mathbb{Z}Q^m} \mathbb{Q}.$$

As $\mathbb{Q}Q/(Q^m - 1)$ is finite dimensional over $\mathbb{Q}$, so is $B_m = \mathbb{Q}Q/J_m$. Hence B_m has Krull dimension 0, i.e. every prime ideal in B_m is a maximal one. Therefore, the finite collection of primes ideals $P_{m,t}$ whose intersection is $\sqrt{B_m}$ are the only prime ideals in $\mathbb{Q}Q$ above J_m, and each of the quotients $\mathbb{Q}Q/P_{m,t}$ is a field.

We shall establish the theorem by proving the following:

Claim 1. *There exist only finitely many fields F such that for some $m \geq 1$ (depending on F) the field F is a quotient of B_m.*

Claim 1 provides an integer m_0 such that if a field F is a quotient of B_m then the natural map $\mathbb{Q}Q \to F$ factors through $\mathbb{Q}Q/(Q^{m_0} - 1)$.

Claim 2. *If m_0 divides m then $J_m = J_{mr}$ for every $r \in \mathbb{N}$.*

To see that the theorem follows from these claims, note that for an arbitrary positive integer m we have $J_m \supseteq J_{mm_0} = J_{m_0}$, whence

$$\dim_{\mathbb{Q}}(\mathbb{Q}Q/J_m) \leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/J_{m_0}) \leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/(Q^{m_0}-1)) = \dim_{\mathbb{Q}} \mathbb{Q}[Q/Q^{m_0}] = m_0^n.$$

Proof of Claim 1. Our hypothesis on S implies that $B \otimes_{\mathbb{Q}} B$ is finitely generated as $\mathbb{Q}Q$-module via the diagonal Q-action, by d elements say. Let F be a field quotient of B_m and let $\theta : \mathbb{Q}Q \to F$ be the canonical projection; so $Q^m - 1 \subseteq \ker(\theta)$. Then $\theta(Q)$ is a finitely generated multiplicative subgroup of F^* that has finite exponent and F, being finite dimensional over $\mathbb{Q}$, embeds in $\mathbb{C}$. Hence $\theta(Q)$ is a finite cyclic group, generated by a root of unity, ϵ of order s say. Thus we obtain a subgroup $H < Q$ such that Q/H is cyclic of order s and $H - 1 \subseteq \ker(\theta)$. Now, $F \cong \mathbb{Q}[x]/(f)$, where f is the minimal polynomial of ϵ over $\mathbb{Q}$. And f is an irreducible factor of $x^s - 1$ in $\mathbb{Q}[x]$, whose zeroes are distinct roots of unity with order precisely s. Thus $\dim_{\mathbb{Q}} F = \deg(f) = \varphi(s)$, where φ is Euler's totient function. On other hand, $F \otimes_{\mathbb{Q}} F$ is an epimorphic image of the $\mathbb{Q}Q$-module $B \otimes_{\mathbb{Q}} B$ and the action of Q on $F \otimes_{\mathbb{Q}} F$ factors through the action of Q/H, so $F \otimes_{\mathbb{Q}} F$ is generated as $\mathbb{Q}[Q/H]$-module by d elements. Hence

$$\varphi(s)^2 = (\dim_{\mathbb{Q}} F)^2 = \dim_{\mathbb{Q}}(F \otimes_{\mathbb{Q}} F) \leq d \dim_{\mathbb{Q}} \mathbb{Q}[Q/H] = ds.$$

An elementary calculation shows that $\varphi(n)/\sqrt{n} \to \infty$ as $n \to \infty$, so for fixed d there are only finitely many possible values of s and ϵ. Let b be a natural number such that the order of ϵ is at most b. Then the order of ϵ is a divisor of $m_0 = b!$ and

$$F \text{ is a quotient of } \mathbb{Q}Q/(Q^{m_0} - 1).$$

Since $\mathbb{Q}Q/(Q^{m_0} - 1)$ is finite dimensional over $\mathbb{Q}$ it has Krull dimension 0, so has only finitely many prime ideals and finitely many field quotients. This completes the proof of Claim 1.

Proof of Claim 2. Since m_0 divides m we have $J_m \subseteq J_{m_0}$, so the prime ideals containing J_{m_0} also contain J_m. On the other hand, we saw earlier that for each of the prime ideals $P_{m,i}$ containing J_m, the quotient $F_i := \mathbb{Q}Q/P_{m,i}$ is a field. By definition, m_0 is such that $\mathbb{Q}Q \to F_i$ factors through $\mathbb{Q}Q/(Q^{m_0} - 1)$, and therefore $P_{m,i}$ (which already contains $J \subset J_m$) contains $J_{m_0} = (J, Q^{m_0} - 1)$. The radical of J_m is the intersection of the prime ideals containing it, so

$$\sqrt{J_m} = \sqrt{J_{m_0}}.$$

Arguing by induction on r, Claim 2 will follow if we can prove that for every prime number p we have $J_m = J_{mp}$, which is equivalent to the assertion that $q^m - 1 \in J_{mp}$ for all $q \in Q$. **We fix $q \in Q$.**

From the preceding argument we have $\sqrt{J_m} = \sqrt{J_{mp}}$. In particular, $Q^m - 1 \subseteq J_m \subseteq \sqrt{J_m} = \sqrt{J_{mp}}$, so there is a natural number (over which we have no control) s such that

$$(q^m - 1)^s \in J_{mp}. \tag{3.1}$$

As $Q^{mp} - 1 \subseteq J_{mp}$, we also have

$$q^{mp} - 1 \in J_{mp}. \tag{3.2}$$

Let $g(x)$ be the greatest common divisor of $x^{pm} - 1$ and $(x^m - 1)^s$ in $\mathbb{Q}[x]$. In characteristic zero, the polynomial $x^{pm} - 1$ has no repeated roots, so neither does $g(x)$. Since $g(x)$ divides $(x^m - 1)^s$, it must actually divide $x^m - 1$, so in fact $g(x) = x^m - 1$. From (3.1), (3.2) and Bézout's Lemma, we have $g(q) \in J_{pm}$. Since $q \in Q$ is arbitrary, this implies that $J_{mp} = J_m$. □

4. The Main Theorem for Metabelian Groups

In this section we prove that all finitely presented metabelian groups have finite virtual first betti number. The proof relies on the finiteness theorem proved in the previous section and two technical lemmas, the first of which is a simple observation about commensurable groups.

Lemma 4.1. *Let G be a group. If $G_0 < G$ is a subgroup of finite index, then* $\mathrm{vb}_1(G) = \mathrm{vb}_1(G_0)$.

Proof. By definition, $\mathrm{vb}_1(G) = \sup_M \dim H_1(M, \mathbb{Q})$ where the supremum is taken over finite index subgroups of G. If M has finite index in G_0, then it also has finite index in G, so $\mathrm{vb}_1(G) \geq \mathrm{vb}_1(G_0)$. Conversely, if S has finite index in G, then $S_0 = G_0 \cap S$ has finite index in G_0, and since it also has finite index in S we have $\dim H_1(S_0, \mathbb{Q}) \geq \dim H_1(S, \mathbb{Q})$, so $\mathrm{vb}_1(G_0) \geq \mathrm{vb}_1(G)$. □

Lemma 4.2. *Let $A \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with A and Q abelian and let n be the torsion-free rank of Q. Then,*

(a) *writing $[G, A] = \langle \{[g, a] = g^{-1}a^{-1}ga \mid g \in G, a \in A\}\rangle$, we have*

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

In the split case, $G = A \rtimes Q$, we have $H_1(G, \mathbb{Q}) \cong (G/[G, A]) \otimes_{\mathbb{Z}} \mathbb{Q}$, and

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n$$

(b) *If G_m is a subgroup of finite index in G and Q_m is the image of G_m in Q, then*

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$

In the split case, $G_m = (A \cap G_m) \rtimes Q_m$, equality is attained:

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$

(c) *If $G = A \rtimes Q$ and $\mathcal{B}$ denotes the set of subgroups of finite index in Q, then*

$$\mathrm{vb}_1(G) = \sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n.$$

Proof. (a) As $[G, A] \subseteq [G, G]$, we see that $H_1(G, \mathbb{Z}) = G/[G, G]$ is a quotient of $G/[G, A]$. So from the central extension $A/[G, A] \rightarrowtail G/[G, A] \twoheadrightarrow Q$ we get

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + \dim_{\mathbb{Q}}(Q \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

If $G = A \rtimes Q$ then, using that A, Q are abelian and A is normal in G, we get $[G, G] = [AQ, AQ] = [Q, A] \subseteq [G, A] \subseteq [G, G]$, hence $[G, G] = [G, A]$ and $A/[G, A] \rightarrowtail G/[G, G] \twoheadrightarrow Q$ is an exact sequence of abelian groups.

For (b) we consider the short exact sequence $A_m \rightarrowtail G_m \twoheadrightarrow Q_m$, where $A_m = A \cap G_m$. From part (a) we have

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n, \tag{4.1}$$

with equality if the sequence splits. Furthermore, since A/A_m is finite we have

$$0 = \mathrm{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \text{ and } (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0.$$

Thus there is an exact sequence (part of the long exact sequence in Tor associated to $A \cap G_m \rightarrowtail A \twoheadrightarrow A/(A \cap G_m)$)

$$0 = \mathrm{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \to A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to A \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0,$$

whence $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \cong A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$. Thus we may replace $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$ in (4.1) by $A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$, and (b) is proved.

(c) From the first part of (b) we have

$$\mathrm{vb}_1(G) \leq \sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n,$$

and to obtain the reverse inequality we use the second part of (b)

$$\sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n = \sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}} H_1(A \rtimes S, \mathbb{Q}),$$

noting that $A \rtimes S$ has finite index in G. □

Theorem 4.3. *Let $A \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with A and Q abelian. If G is finitely presented then its virtual first betti number $\mathrm{vb}_1(G)$ is finite.*

Proof. By passing to a subgroup of finite index in Q and replacing G by the inverse image of this subgroup, we may assume that Q is free abelian. Lemma 4.1 assures us that it is enough to consider this case, and Lemma 4.2(b) tells us that we will be done if we can establish an upper bound on $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q})$ as Q_m ranges over the subgroups of finite index in Q.

Recall that A is finitely generated as a $\mathbb{Z}Q$-module, say by d elements. Thus, denoting the annihilator $\mathrm{ann}_{\mathbb{Z}Q}(A) = \{\lambda \in \mathbb{Z}Q \mid A\lambda = 0\}$ by I, we have an epimorphism of $\mathbb{Z}Q$-modules

$$(\mathbb{Z}Q/I)^{[d]} = \mathbb{Z}Q/I \oplus \ldots \oplus \mathbb{Z}Q/I \to A$$

that induces an epimorphism of $\mathbb{Q}$-vector spaces

$$((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q})^{[d]} = (\mathbb{Z}Q/I)^{[d]} \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}.$$

Thus

$$\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) \leq d. \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q})$$

and it suffices to show that

$$\sup_m \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) < \infty.$$

For every m there is a natural number α_m such that $Q^{\alpha_m} \subseteq Q_m$, and $\mathbb{Z}Q/I \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$ is a quotient of $\mathbb{Z}Q/I \otimes_{\mathbb{Z}Q^{\alpha_m}} \mathbb{Q}$. Thus

$$\dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) \leq \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q^{\alpha_m}} \mathbb{Q}),$$

and we have reduced to showing that

$$\sup_s \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q^s} \mathbb{Q}) < \infty.$$

The theorem now follows from Lemma 2.1 and Theorem 3.1. □

5. Characteristic p case

In this section we shall construct examples which show that the restriction to fields of characteristic 0 in Theorem A is essential, even in the metabelian case. To this end, we consider the mod p *virtual first betti number* of a finitely generated group G,

$$\mathrm{vb}_1^{(\mathrm{p})}(G) = \sup\{\dim H_1(S, \mathbb{F}_p) \mid S < G \text{ of finite index }\}.$$

Proposition 5.1. *For every prime p there exist finitely presented metabelian groups Γ such that $\mathrm{vb}_1^{(\mathrm{p})}(\Gamma)$ is infinite.*

Proof. Let Q be a free abelian group with generators x and y and let $A = \mathbb{F}_pQ/I$, where I is the ideal of $\mathbb{F}_pQ$ generated by $y - x^2 + x - 1$. Then

$$A \cong \mathbb{F}_p[x, x^{-1}, \frac{1}{x^2 - x + 1}].$$

Consider

$$A_m = A \otimes_{\mathbb{Z}Q^{p^m}} \mathbb{F}_p \cong \mathbb{F}_pQ/(I, Q^{p^m} - 1).$$

Since $(x^2-x+1)^{p^m}-1 = x^{2p^m} - x^{p^m} + 1 - 1 = x^{p^m}(x^{p^m}-1)$, we have

$$A_m = \mathbb{F}_p[x, x^{-1}, \frac{1}{x^2-x+1}]/(x^{p^m}-1, (x^2-x+1)^{p^m}-1) =$$
$$\mathbb{F}_p[x, x^{-1}, \frac{1}{x^2-x+1}]/(x^{p^m}-1)$$

is the localisation

$$B_m S^{-1}$$

where $B_m = \mathbb{F}_p[x, x^{-1}]/(x^{p^m}-1)$ and S is the image of $\{(x^2-x+1)^j\}_{j\geq 1}$ in B_m. Note that $x^{p^m}-1$ and x^2-x+1 do not have a common root in any finite field extension of $\mathbb{F}_p$, for if z were a common root we would have $1 = z^{2p^m} = (z-1)^{p^m} = z^{p^m} - 1 = 0$, which is a contradiction. Thus the polynomials $x^{p^m}-1$ and $(x^2-x+1)^j$ are coprime in $\mathbb{F}_p[x, x^{-1}]$ i.e. generate the whole ring as an ideal, and so the elements of S are invertible in B_m. Therefore $B_m S^{-1} = B_m$ and

$$\dim_{\mathbb{F}_p} A_m = \dim_{\mathbb{F}_p} B_m S^{-1} = \dim_{\mathbb{F}_p} B_m = p^m.$$

Now define

$$\Gamma = A \rtimes Q \text{ and } \Gamma_m = A \rtimes Q^{p^m}.$$

Then, as the split case of Lemma 4.2(b) (with $\mathbb{F}_p$ coefficients in place of rational ones)

$$\dim_{\mathbb{F}_p} H_1(\Gamma_m, \mathbb{F}_p) = \dim_{\mathbb{F}_p} A_m + 2 = p^m + 2,$$

which tends to infinity with m.

By the calculation of $\Sigma_A(Q)$ for $A = \mathbb{F}_p Q/I$, where the ideal I is 1-generated, [9, Thm. 5.2] or by the link between $\Sigma_A^c(Q)$ and valuation theory (as described in subsection 2.2), we have that

$$\Sigma_A^c(Q) = \{[\chi_1], [\chi_2], [\chi_3]\}$$
$$\chi_1(x) = 0, \chi_1(y) = 1, \chi_2(x) = 1, \chi_2(y) = 0 \text{ and } \chi_3(x) = -1, \chi_3(y) = -2.$$

Thus A is 2-tame as $\mathbb{Z}Q$-module, and by the classification of finitely presented metabelian groups in [8], Γ is finitely presented. □

Corollary 5.2. *There exists a finitely presented metabelian group G such that for the class $\mathcal{A}$ of all subgroups of finite index in G*

$$\sup_{M\in\mathcal{A}} d(M) = \infty,$$

where $d(M)$ is the minimal number of generators of M.

Proof. Immediate, since

$$d(M) \geq \dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p).$$

□

It is natural to wonder if the lack of finiteness exhibited in the preceding proposition might be avoided by restricting to subgroups whose index is coprime to p. The following refinement shows that this is not the case.

Proposition 5.3. *Let p be a prime. There exist finitely presented metabelian groups G such that*

$$\sup\{\dim_{\mathbb{F}_p} H_1(S, \mathbb{F}_p) \mid S \in \mathcal{A}_p\} = \infty,$$

where

$$\mathcal{A}_p = \{S \leq G \mid [G:S] \text{ is finite and coprime to } p\}.$$

Proof. Let $A = \mathbb{F}_p[x, \frac{1}{x}, \frac{1}{x+1}]$ and let Q be a free abelian group of rank 2 whose generators x_1, x_2 act on A as multiplication by x and $x+1$ respectively. We consider the group $G = A \rtimes Q$. As an $\mathbb{F}_p[Q]$-module, $A \cong \mathbb{F}_p[Q]/I$ where I is the ideal generated by $x_2 - x_1 - 1$, and the argument given in the preceding proposition shows that $\Sigma_A(Q)^c$ consists of precisely 3 points, no pair of which is antipodal. Therefore G is finitely presented.

Let F be a finite field with p^r elements, $r \geq 2$. Let w be a generator of the multiplicative group $F^* = F \smallsetminus \{0\}$. Let Q_r be the kernel of the homomorphism $Q \to F^*$ defined by $x_1 \mapsto w$ and $x_2 \mapsto w+1$. Let $G_r = A \rtimes Q_r$ and note that $|G/G_r| = |Q/Q_r| = p^r - 1$ is coprime to p.

The ring epimorphism $A \to F$ sending x to w provides an epimorphism of the underlying additive groups which extends to a group-epimorphism $A \rtimes Q_r \to F \times \mathbb{Z}^2$. Since $\dim_{\mathbb{F}_p} F = r$, it follows that $\dim_{\mathbb{F}_p} H_1(G_r, \mathbb{F}_p) \geq r + 2$. And $r \geq 2$ was arbitrary. □

6. Beyond the metabelian case

In this section we shall prove Theorem A, but first we present a consequence of Theorem 4.3 that describes what one can deduce about towers of finite-index subgroups above the commutator subgroup in amenable and related groups.

Proposition 6.1. *Let G be a group of type* FP_2 *that does not contain a non-abelian free group and let $\mathcal{C}$ be the set of finite-index subgroups in G that contain the commutator subgroup G'. Then* $\sup_{M \in \mathcal{C}} \dim_{\mathbb{Q}} H_1(M, \mathbb{Q}) < \infty$.

Proof. By [8, Thm. 5.5] G/G'' is finitely presented. Since $M \supseteq G'$ we have $M' \supseteq G''$, hence we can replace G by G/G'' and M by MG''/G'' without changing $H_1(M, \mathbb{Q})$. Then we can apply Theorem 4.3. □

Our proof of Theorem A relies on the following proposition, which is of interest in its own right.

Proposition 6.2. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups, where N is nilpotent, Q is abelian and G is finitely generated. Let G_n be a subgroup of finite index in G and let $\overline{G}_n$ be the image of G_n in the metabelian group G/N'. Then*

$$\dim_{\mathbb{Q}} H_1(G_n, \mathbb{Q}) = \dim_{\mathbb{Q}} H_1(\overline{G}_n, \mathbb{Q}).$$

Proof. We argue using the Malcev completion $j : N \to N^*$, as defined in [18]. According to [20, Appendix A, Cor. 3.8], for any nilpotent group N the homomorphism $j_N : N \to N^*$ is characterized up to isomorphism by the following properties:

(a) N^* is nilpotent and uniquely divisible;
(b) $\ker j_N$ is the torsion subgroup of N;
(c) for every $x \in N^*$ there is a positive integer n such that $x^n \in N$.

In any nilpotent group, the set $\sqrt{S}$ of elements that have powers in a fixed subgroup S is a subgroup. It follows that, for every subgroup $M < N$, the map $M \to \sqrt{j_N(M)}$ satisfies properties (a) to (c). Thus we may identify M^* with $\sqrt{j_N(M)} < N^*$. If $M < N$ has finite index then $M^* = \sqrt{j_N(M)} = N^*$. And $(N^*)' = (N')^*$.

With these facts in hand, for all subgroups of finite index $G_n < G$ we have $(G_n')^* \supseteq ((G_n \cap N)')^* = ((G_n \cap N)^*)' = (N^*)' = (N')^*$. Thus $(G_n'N')^* = (G_n')^*$, and from (c) we deduce that $G_n'N'/G_n'$ is torsion. As $G_n'N'/G_n'$ is the kernel of the canonical epimorphism $G_n/G'_n \to G_nN'/G'_nN'$, we have $H_1(G_n,\mathbb{Q}) \cong (G_n/G'_n) \otimes \mathbb{Q} \cong (G_nN'/G'_nN') \otimes \mathbb{Q} \cong H_1(\overline{G}_n,\mathbb{Q})$ as required. $\square$

Theorem 6.3. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. If N is nilpotent, Q is abelian and G is of type* FP_2*, then the virtual first betti number* $\mathrm{vb}_1(G)$ *is finite.*

Proof. In the light of the preceding proposition, this follows directly from Theorem 4.3 and the fact, proved in [8, Thm. 5.5], that G/N' is a finitely presented metabelian group. $\square$

Corollary 6.4 (=Theorem A)**.** *If a group G is nilpotent-by-abelian-by-finite group and of type* FP_2*, then* $\mathrm{vb}_1(G)$ *is finite.*

Proof. Let G_0 be a subgroup of finite index in G such that G_0 is nilpotent-by-abelian. Then G_0 has type FP_2, so $\mathrm{vb}_1(G_0)$ is finite, by Theorem 6.3, and hence so is G, by Lemma 4.1. $\square$

Remark 6.5. We did not use the full force of finite presentability in establishing Theorem A: in fact, it is enough to assume that G has a subgroup of finite index G_0 in which there is a nilpotent subgroup $N \lhd G_0$ such that $Q = G_0/N$ is free abelian and, writing $A = N/N'$, the $\mathbb{Q}Q$-module $A \otimes A \otimes \mathbb{Q}$, with diagonal action, should be finitely generated. These requirements follow from the finite presentability of G_0/N' but are strictly weaker.

Corollary 6.6. *Every soluble group of type* FP_∞ *has finite virtual first betti number.*

Proof. Soluble groups S of type FP_∞ are constructible and hence nilpotent-by-abelian-by-finite [16]. $\square$

Corollary 6.7. *Every abelian-by-polycyclic group of type* FP_3 *has finite virtual first betti number.*

Proof. By the main result of [14], abelian-by-polycyclic groups of type FP_3 are nilpotent-by-abelian-by-finite. $\square$

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