ABELIAN COVERS OF GRAPHS AND MAPS BETWEEN OUTER AUTOMORPHISM GROUPS OF FREE GROUPS

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Abstract. We explore the existence of homomorphisms between outer automorphism groups of free groups $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$. We prove that if $n > 8$ is even and $n \neq m \leq 2n$, or n is odd and $n \neq m \leq 2n-2$, then all such homomorphisms have finite image; in fact they factor through $\det\colon \mathrm{Out}(F_n) \to \mathbb{Z}/2$. In contrast, if $m = r^n(n-1)+1$ with r coprime to $(n-1)$, then there exists an embedding $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$. In order to prove this last statement, we determine when the action of $\mathrm{Out}(F_n)$ by homotopy equivalences on a graph of genus n can be lifted to an action on a normal covering with abelian Galois group.

1. Introduction

The contemporary study of mapping class group and outer automophism groups of free groups is heavily influenced by the analogy between these groups and lattices in semisimple Lie groups. In previous papers [4, 5, 6] we have explored rigidity properties of $\mathrm{Out}(F_n)$ in this light, proving in particular that if $m < n$ then any homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has image at most $\mathbb{Z}_2$, and that the only monomorphisms $\mathrm{Out}(F_n) \to \mathrm{Out}(F_n)$ are the inner automorphisms. In this paper we turn our attention to the case $m > n$.

There are two obvious ways in which one might embed $\mathrm{Aut}(F_n)$ in $\mathrm{Aut}(F_m)$ when $m > n$: most obviously, the inclusion $F_n \subset F_m$ of any free factor induces a monomorphism $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_m)$; secondly, if $N \subset F_n$ is a characteristic subgroup of finite index, then the restriction map $\mathrm{Aut}(F_n) \to \mathrm{Aut}(N) \cong \mathrm{Aut}(F_m)$ is injective. However, neither of these constructions sends the group of inner automorphisms $\mathrm{Inn}(F_n) \subset \mathrm{Aut}(F_n)$ into $\mathrm{Inn}(F_m)$, so there is no induced map $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$. In the second case one can fix this by passing to a subgroup of finite index: if $N < F_n$ is any subgroup of finite index, then some subgroup of finite index in $\mathrm{Out}(F_n)$ injects into $\mathrm{Out}(N)$; see Proposition 2.5. Thus $\mathrm{Out}(F_n)$ is commensurable with a subgroup

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of $\mathrm{Out}(F_m)$ if $m = d(n-1)+1$ for some $d \geq 1$; for example $\mathrm{Out}(F_n)$ is commensurable with a subgroup of $\mathrm{Out}(F_{2n-1})$. But if we demand that our homomorphisms be defined on the whole of $\mathrm{Out}(F_n)$, then it is far from obvious that there are *any* maps $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ with infinite image when $m > n \geq 3$.

As usual, the case $n = 2$ is exceptional: $\mathrm{Out}(F_2) = \mathrm{GL}(2,\mathbb{Z})$ maps to $(\mathbb{Z}/2)*(\mathbb{Z}/3)$ with finite kernel, so to obtain a map with infinite image one need only choose elements of order 2 and 3 that generate an infinite subgroup of $\mathrm{Out}(F_m)$. Khramtsov [15] gives an explicit monomomorphism $\mathrm{Out}(F_2) \to \mathrm{Out}(F_4)$. More interestingly, he proved that there are no injective maps from $\mathrm{Out}(F_n)$ to $\mathrm{Out}(F_{n+1})$. *So, for given n, for which values of m is there a monomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$, and for which values of m do all maps $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ have finite image?* These are the questions that we address in this article. In the first part of the paper we give explicit constructions of embeddings, and in the second half we prove, among other things, that no homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ can have image bigger than $\mathbb{Z}/2$ if n is even and $8 < n < m \leq 2n$. This last result disproves a conjecture of Bogopolski and Puga [2].

In order to construct embeddings, we consider characteristic subgroups $N < F_n$, identify F_n with the subgroup of $\mathrm{Aut}(F_n)$ consisting of inner automorphisms, and examine the short exact sequence

$$1 \to F_n/N \to \mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n) \to 1.$$

We want to understand when this sequence splits. When it does split, one can compose the splitting map $\mathrm{Out}(F_n) \to \mathrm{Aut}(F_n)/N$ with the map $\mathrm{Aut}(F_n)/N \to \mathrm{Out}(N)$ induced by restriction, $\phi \to [\phi|N]$, to obtain an embedding of $\mathrm{Out}(F_n)$ into $\mathrm{Out}(N)$.

Bogopolski and Puga [2] used algebraic methods to obtain a splitting in the case where $F_n/N \cong (\mathbb{Z}/r)^n$ with r odd and coprime to $(n-1)$, yielding embeddings $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$ when $m = r^n(n-1)+1$. We adopt a more geometric approach, which begins with a translation of the above splitting problem into a lifting problem for groups of homotopy equivalences of graphs. Proposition 2.1 provides a precise formulation of this translation. (The topological background to it is difficult to pin down in the literature, so we explain it in detail in an appendix.)

The following theorem is the main result in the first half of this paper.

Theorem A. *Let $\widehat{X} \to X$ be a normal covering of a connected graph of genus $n \geq 2$ with abelian Galois group A. The action of $\mathrm{Out}(F_n)$ by homotopy equivalences on X lifts to an action by fiber-preserving homotopy equivalences on $\widehat{X}$ if and only if $A \cong (\mathbb{Z}/r)^n$ with r coprime to $n-1$.*

When translated back into algebra, this theorem is equivalent to the statement that if a characteristic subgroup $N < F_n$ contains the commutator subgroup $F_n' = [F_n, F_n]$, then the short exact sequence $1 \to F_n/N \to \mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n) \to 1$ splits if and only if $N = F_n' F_n^r$, where F_n^r is the subgroup generated by r-th powers and $r > 1$ is coprime to $n-1$. The sufficiency of this condition extends Bogopolski and Puga's theorem to cover the case where r is even.

Corollary A. *There exists an embedding* $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$ *for any* m *of the form* $m = r^n(n-1)+1$ *with* $r > 1$ *coprime to* $n-1$.

The negative part of Theorem A also has an intriguing application. It tells us that $1 \to F_n/F_n' \to \mathrm{Aut}(F_n)/F_n' \to \mathrm{Out}(F_n) \to 1$ does not split. Thus this sequence defines a non-zero class in the second cohomology group of $\mathrm{Out}(F_n)$ with coefficients in the module $M := F_n/F_n'$ (which is the standard left $\mathrm{Out}(F_n)$-module structure on $H_1(F_n)$). The theorem also assures us that this class remains non-trivial when we take coefficients in M/rM, provided that r is coprime to $(n-1)$. The non-triviality of these classes provides a striking counterpoint to what happens when one takes coefficients in the dual module $M^* = H^1(F_n)$, as we shall explain in Section 5.

Theorem B. *Let* $M = H_1(F_n)$ *be the standard* $\mathrm{Out}(F_n)$*-module and let* M^* *be its dual. Then* $H^2(\mathrm{Out}(F_n), M) \neq 0$, *but* $H^2(\mathrm{Out}(F_n), M^*) = 0$ *if* $n \geq 8$.

Theorem A exhausts the ways in which one might obtain embeddings $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ by lifting the action of $\mathrm{Out}(F_n)$ to covering spaces with an abelian Galois group, but one might hope to construct many other embeddings using non-abelian covers. Indeed the construction developed by Aramayona, Leininger and Souto in the context of surface automorphisms [1] proceeds along exactly these lines and, as they remark, it can be adapted to the setting of $\mathrm{Out}(F_n)$. However, in the embeddings $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ obtained by their method, m is bounded below by a doubly exponential function of n, whereas in our construction we can take $m = 2^n(n-1)+1$ if n is odd. If n is even, then the smallest value we obtain is $m = p^n(n-1)+1$ where p is the smallest prime that does not divide $(n-1)$; in section 2.1 we describe how quickly p grows as a function of n.

In the second part of the paper we set about the task of providing lower bounds on the value of m such that there is a monomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$, or even a map with infinite image.

Theorem C. *Suppose* $n > 8$. *If* n *is even and* $n < m \leq 2n$, *or* n *is odd and* $n < m \leq 2n-2$, *then every homomorphism* $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ *factors through* $\det\colon \mathrm{Out}(F_n) \to \mathbb{Z}/2$.

Note how this result contrasts with our earlier observation that $\mathrm{Out}(F_n)$ has a subgroup of finite index that embeds in $\mathrm{Out}(F_m)$ when $m = 2n-1$. The key point here is that subgroups of finite index can avoid certain of the finite subgroups in $\mathrm{Out}(F_n)$ (indeed they may be torsion-free), whereas our proof of Theorem C relies on a detailed understanding of how the finite subgroups of $\mathrm{Out}(F_n)$ can map to $\mathrm{Out}(F_m)$ under putative maps $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$. Two subgroups play a particularly important role, namely $W_n \cong (\mathbb{Z}/2)^n \rtimes S_n$, the group of symmetries of the n-rose R_n, and $G_n \cong \Sigma_{n+1} \times \mathbb{Z}/2$, the group of symmetries of the $(n+1)$-*cage*, i.e. the graph with 2 vertices and $(n+1)$-edges. Indeed the key idea in the proof is to show that no homomorphism can restrict to an injection on both of these subgroups. In order to establish this, we have to analyze in detail all of the ways in which these groups can act by automorphisms on graphs of genus at most $2n$. In the light of the realization theorem for finite subgroups of $\mathrm{Out}(F_m)$, this analysis amounts to a complete description of the conjugacy classes of the finite subgroups in $\mathrm{Out}(F_m)$ that are isomorphic to A_n, W_n and G_n (cf. Propositions 6.7 , 6.10 and 6.12). We believe that these results are of independent interest.

Beyond $m = 2n$, the analysis of $\mathrm{Hom}(W_n, \mathrm{Out}(F_m))$ and $\mathrm{Hom}(G_n, \mathrm{Out}(F_m))$ becomes more complex, but several crucial facts extend well beyond this range (e.g. Lemma 6.5 and Proposition 7.1). Moreover, Dawid Kielak [13] has recently extended our methods to improve the bound $m \leq 2n$. Thus, at the time of writing, we have no good reason to suppose that the lower bound that Theorem C imposes on the least $m > n$ with $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$ is any closer to the truth that the exponential upper bound provided by Theorem A.

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2. Theorem A: Restatement and Discussion

In the appendix to this paper we explain in detail the equivalence of various short exact sequences arising in group theory and topology. In the case of graphs, the basic equivalence can be expressed as follows.

Let N be a characteristic subgroup of a free group F, let X be a connected graph with fundamental group F, let $p : \widehat{X} \to X$ be the covering space corresponding to N, let $\mathrm{HE}(X)$ be the group of free homotopy classes of homotopy equivalences of X, and let $\mathrm{FHE}(\widehat{X})$ be the group of fiber-preserving homotopy classes of fiber-preserving homotopy equivalences of $\widehat{X}$. Note that the deck transformations of $\widehat{X}$ lie in the kernel of the natural map $\mathrm{FHE}(\widehat{X}) \to \mathrm{HE}(X)$.

Proposition 2.1. *The following diagram of groups is commutative and the vertical maps are isomorphisms:*

$$\begin{array}{ccccccc} 1 & \to & F_n/N & \to & \mathrm{Aut}(F_n)/N & \to & \mathrm{Out}(F_n) \to 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & \mathrm{Deck} & \to & \mathrm{FHE}(\widehat{X}) & \to & \mathrm{HE}(X) \to 1 \end{array}$$

The characteristic subgroups $N < F_n$ with F_n/N abelian are the commutator subgroup $F_n' = [F_n, F_n]$ and $F_n'F_n^r$, the subgroup generated by F_n' and all rth powers in F_n. By combining this observation with the preceding proposition, we see that Theorem A is equivalent to the following statement.

Theorem 2.2. *Let F_n be a free group of rank n and let $N < F_n$ be a characteristic subgroup with F_n/N abelian. Then the short exact sequence*

$$1 \to F_n/N \to \mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n) \to 1$$

splits if and only $N = [F_n, F_n]F_n^r$ with r coprime to $n-1$.

The existence of splittings is proved in Section 3 below, and the non-existence in Section 4.

Any splitting of the sequence in Theorem 2.2 gives a monomorphism $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Aut}(F_n)/N$, which we can compose with the restriction map

$$\mathrm{Aut}(F_n)/N \to \mathrm{Aut}(N)/N = \mathrm{Out}(N).$$

To complete the proof of Corollary A we need to know that this last map is injective. This follows from the observation below.

Lemma 2.3. *If F is a finitely generated free group and $N < F$ is a characteristic subgroup of finite index, then the restriction map $\mathrm{Aut}(F) \to \mathrm{Aut}(N)$ is injective.*

Proof. If k is the index of N in F and w is an arbitrary element of F, then $w^k \in N$. If ϕ is in the kernel of the restriction map $\mathrm{Aut}(F) \to \mathrm{Aut}(N)$, then $w^k = \phi(w^k) = (\phi(w))^k$. But elements in F have unique roots, so $w = \phi(w)$, and ϕ is the identity. □

2.1. **Estimates on the growth of m.** The subgroup $N = F_n'F_n^r$ has index r^n in F_n so is free of rank $m = r^n(n-1)+1$. Thus the smallest m for which we obtain an embedding $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ from Theorem 2.2 is $m = p^n(n-1)+1$, where p is the smallest prime which does not divide $n-1$. If n is odd we can take $p = 2$ but for n even the size of p as a function of n is not obvious. However, it turns out that the expected value of p is a constant (which is approximately equal to 3). We are indebted to Roger Heath-Brown for the following argument.

For any natural number $k > 1$, let $f(k)$ denote the smallest prime number which does not divide k and let $Q(k)$ be the product of all prime numbers strictly less than k (with $Q(2) = 1$). An easy consequence of the Prime Number Theorem is that $\log(Q(k))$ is asymptotically equal to k. This implies in particular that the infinite series used to define C in the following proposition is convergent.

Proposition 2.4. *The expected value*

$$E(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{k=1}^{x} f(k),$$

exists and is equal to the constant

$$C := \sum_p \frac{p-1}{Q(p)},$$

where the sum is over all primes p.

Proof. Note that $f(k) = p$ if and only if $Q(p)$ divides k and p does not divide k. The first statement implies, taking logs, that $\log(Q(p)) \leq \log k$, so p can be of order at most $\log k$.

By definition,

$$\sum_{k \leq x} f(k) = \sum_p p \cdot \#\{k \leq x : f(k) = p\}$$

and

$$\begin{aligned} \#\{k \leq x : f(k) = p\} &= \#\{k \leq x : Q(p)|k\} - \#\{k \leq x : pQ(p)|k\} \\ &= \lfloor x/Q(p) \rfloor - \lfloor x/pQ(p) \rfloor \\ &= x\,\frac{p-1}{pQ(p)} + O(1). \end{aligned}$$

As we just observed, the primes that contribute to the above sum have order at most $\log(x)$, so

$$\begin{aligned} \frac{1}{x}\sum_{k \leq x} f(k) &= \sum_{p = O(\log x)} \frac{p-1}{Q(p)} + \frac{1}{x}\,O\Big(\sum_{p=O(\log x)} p\Big) \\ &= \sum_{p=O(\log x)} \frac{p-1}{Q(p)} + \frac{1}{x} O(\log^2 x). \end{aligned}$$

Letting $x \to \infty$, we get $E(f) = C$. $\square$

Given n, the smallest value of m for which Corollary A yields an embedding is $m = f(n-1)^n(n-1) + 1$, and the preceding proposition tells us that "on average" this is no greater than an exponential function of n. In the worst case, m can be larger but still only on the order of $e^{n \log\log n}$. Indeed the worst case arises when $(n-1) = Q(k)$ for some k, in which case $k \leq f(n-1) < 2k$, and since $\log(Q(k)) \sim k$ we see that $f(n-1)$ grows like $\log n$.

2.2. **Embedding a subgroup of finite index.** Corollary A gives conditions under which the entire group $\mathrm{Out}(F_n)$ embeds in $\mathrm{Out}(F_m)$. If we relax this to require only that a subgroup of finite index of $\mathrm{Out}(F_n)$ should embed in $\mathrm{Out}(F_m)$, we can obtain many more embeddings as follows.

Proposition 2.5. *For all positive integers n and d, there exists a subgroup of finite index $\Gamma \subset \mathrm{Out}(F_n)$ and a monomorphism $\Gamma \hookrightarrow \mathrm{Out}(F_m)$, where $m = d(n-1)+1$.*

Proof. For $n = 1$ the proposition is trivial, and for $n = 2$ it follows immediately from the fact that $\mathrm{Out}(F_2)$ has a free subgroup of finite index. So we assume that $n \geq 3$ and fix an epimorphism from F_n to a wreath product $W = G \wr \mathbb{Z}/d$, where G is any finite 2-generator centerless group (S_3 for example). Let N be the kernel of this epimorphism and let $H \supset N$ be the kernel of the composition $F_n \to W \to \mathbb{Z}/d$.

The set of subgroups in F_n that have the same index as N is finite, as is the set that have the same index as H. The action of $\mathrm{Aut}(F_n)$ on each of these sets defines a homomorphism to a finite symmetric group; define Γ_0 to be the intersection of the two kernels. Note that Γ_0 leaves invariant both H and N. Let $\Gamma_1 \subset \Gamma_0$ be the kernel of the natural map $\Gamma_0 \to \mathrm{Aut}(F_n/N)$ and note that since the center of $W = F_n/N$ is trivial, the intersection of Γ_1 with $\mathrm{Inn}(F_n) = F_n$ is contained in N, and hence in H.

H is free of rank $d(n-1)+1$ and the restriction map $\Gamma_1 \to \mathrm{Aut}(H)$, which is injective as in Lemma 2.3, induces an injection $\Gamma_1/(\Gamma_1 \cap H) \hookrightarrow \mathrm{Out}(H)$. To complete the proof, it suffices to note that $\Gamma := \Gamma_1/(\Gamma_1 \cap H)$ is the image of Γ_1 in $\mathrm{Out}(F_n)$, since $(\Gamma_1 \cap H) = (\Gamma_1 \cap F_n)$. □

3. Proof of Theorem A: The existence of lifts

In order to prove the existence of lifts as asserted in Theorem A (equivalently the existence of splittings in Theorem 2.2), we work with the sequence

$$1 \to \mathrm{Deck} \to \mathrm{FHE}(\widehat{X}) \to \mathrm{HE}(X) \to 1$$

where $X = R$ is a 1-vertex graph with n loops (a "rose") and $\widehat{X} \to X$ is the covering space $L_r \to R$ corresponding to $N < \pi_1 X = F_n$, where $N = F_n' F_n^r$ with r coprime to $n-1$. We work with an explicit presentation of $\mathrm{Out}(F_n) = \mathrm{HE}(R)$. We take explicit homotopy equivalences of R that generate $\mathrm{HE}(R)$, lift each to a homotopy equivalence of the universal abelian covering L of R, project down to L_r, and prove that the resulting elements of $\mathrm{FHE}(L_r)$ satisfy the defining relations of our presentation. The case $n = 2$ is special: for $n = 2$ one can split $\mathrm{HE}(R) \to \mathrm{FHE}(L)$.

The generators and relations we will use for $\mathrm{Out}(F_n)$ are based on those given by Gersten in [9] for $\mathrm{SAut}(F_n)$. We fix a generating set $A = \{a_1, \ldots, a_n\}$ for F_n. Gersten gives an elegant and succinct presentation using generators ϕ_{ab} with $a, b \in A \cup A^{-1}, b \neq a, a^{-1}$; here ϕ_{ab} corresponds to the automorphism which sends $a \mapsto ab$ and fixes all elements of $A \cup A^{-1}$ other than a and a^{-1}. In Gersten's paper automorphisms act on F_n on the right and the symbol $[\alpha, \beta]$ means $\alpha\beta\alpha^{-1}\beta^{-1}$. In the current paper we want automorphisms to act on the left to be consistent with composition of functions in $\mathrm{HE}(R)$, but we would like to use the same commutator convention. Thus for us a Gersten relation of the form $[\alpha, \beta] = \gamma$ becomes $[\beta^{-1}, \alpha^{-1}] = \gamma$ or, equivalently, $[\alpha^{-1}, \beta^{-1}] = \gamma^{-1}$, so that his relations are the following:

(1) $\phi_{ab^{-1}} = \phi_{ab}^{-1}$

(2) $[\phi_{ab}^{-1}, \phi_{cd}^{-1}] = 1$ if $a \neq c, d, d^{-1}$ and $b \neq c, c^{-1}$
(3) $[\phi_{ab}^{-1}, \phi_{bc}^{-1}] = \phi_{ac}^{-1}$ for $a \neq c, c^{-1}$
(4) $\phi_{ba}\phi_{ab^{-1}}\phi_{b^{-1}a^{-1}} = \phi_{b^{-1}a^{-1}}\phi_{a^{-1}b}\phi_{ba}$
(5) $(\phi_{b^{-1}a^{-1}}\phi_{a^{-1}b}\phi_{ba})^4 = 1$

We will need to distinguish between right transvections $\rho_{ij} \colon a_i \mapsto a_i a_j$ and left transvections $\lambda_{ij} \colon a_i \mapsto a_j a_i$, for $i \neq j$, so we rewrite Gersten's relations using the translation $\phi_{a_i a_j} = \rho_{ij}$, $\phi_{a_i^{-1}a_j^{-1}} = \lambda_{ij}$, $\phi_{a_i a_j^{-1}} = \rho_{ij}^{-1}$ and $\phi_{a_i^{-1}a_j} = \lambda_{ij}^{-1}$.

In terms of the ρ_{ij} and λ_{ij}, Gersten's first relation is unnecessary and the rest of the presentation for $\mathrm{SAut}(F_n)$ becomes

(1) $[\rho_{ij}, \rho_{kl}] = [\rho_{ij}, \lambda_{kl}] = [\lambda_{ij}, \lambda_{kl}] = 1$ if $i \neq k, l$ and $j \neq k$
(2) $[\rho_{ij}, \lambda_{ik}] = 1$ for all i, j, k
(3) $[\rho_{ij}^{-1}, \rho_{jk}^{-1}] = [\rho_{ij}, \lambda_{jk}] = [\rho_{ij}^{-1}, \rho_{jk}]^{-1} = [\rho_{ij}, \lambda_{jk}^{-1}]^{-1} = \rho_{ik}^{-1}$
(4) $[\lambda_{ij}^{-1}, \lambda_{jk}^{-1}] = [\lambda_{ij}, \rho_{jk}] = [\lambda_{ij}^{-1}, \lambda_{jk}]^{-1} = [\lambda_{ij}, \rho_{jk}^{-1}]^{-1} = \lambda_{ik}^{-1}$
(5) $\lambda_{ij}\lambda_{ji}^{-1}\rho_{ij} = \rho_{ij}\rho_{ji}^{-1}\lambda_{ij}$
(6) $(\rho_{ij}\rho_{ji}^{-1}\lambda_{ij})^4 = 1$

To get a presentation for $\mathrm{Aut}(F_n)$ we must add a generator τ, corresponding to the automorphism $a_1 \mapsto a_1^{-1}$, and relations

(7) $\tau^2 = 1$
(8) $\tau\rho_{1j}\tau = \lambda_{1j}^{-1}, \tau\lambda_{1j}\tau = \rho_{1j}^{-1}$
(9) $\tau\rho_{i1}\tau = \rho_{i1}^{-1}, \tau\lambda_{i1}\tau = \lambda_{i1}^{-1}$
(10) $[\tau, \rho_{ij}] = [\tau, \lambda_{ij}] = 1$ for $i, j \neq 1$.

Finally, to get a presentation for $\mathrm{Out}(F_n)$ we kill the inner automorphisms by adding the relation

(11) $\displaystyle\prod_{i=2}^{n} \rho_{i1}\lambda_{i1}^{-1} = 1.$

We orient and label the petals of R with the generators a_i. If we fix a base vertex $\mathbf{0}$ of L, we may think of L as the 1-skeleton of the standard hypercubulation of $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$. The lift starting at $\mathbf{0}$ of the edge labeled a_i is identified with the standard i-th basis vector $\mathbf{e}_i$.

Any automorphism ϕ of F_n is realized on R by a homotopy equivalence sending the petal labeled a_i to the (oriented) path which traces out the reduced word $\phi(a_i)$. This has a *standard lift* $\widehat{\phi}$ to a $\mathbb{Z}^n$-equivariant homotopy equivalence of L, which sends $\mathbf{e}_i$ to the lift starting at $\mathbf{0}$ of the path labeled by the reduced word $\phi(a_i)$. (Since the homotopy equivalence is $\mathbb{Z}^n$-equivariant, it suffices to describe its effect on the edges $\mathbf{e}_i$.) This in turn induces a lift $\widehat{\phi_r}$ to the quotient $L_r = L/\mathbb{Z}^r$ for each r, which is trivial in $\mathrm{FHE}(L_r)$ if and only if $\widehat{\phi}$ is fiberwise-homotopic to a deck transformation by an element of $r\mathbb{Z}^n$.

Lifting automorphisms to L and L_r by these standard lifts does not give a well-defined homomorphism on $\mathrm{Out}(F_n)$. This is because the standard lift of the inner automorphism $\alpha_1 = \prod_{i>1} \rho_{i1}\lambda_{i1}^{-1}$ sends $\mathbf{e}_i$ to a $\sqcup$-shaped path labeled $a_1^{-1}a_ia_1$. The extension to all of L is freely homotopic to the deck transformation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{e}_1$ of L. Since this deck transformation is not freely homotopic to the identity (even mod r for any $r > 1$), the assignment $\alpha_1 \mapsto \widehat{\alpha}_1$ does not give well-defined map from $\mathrm{Out}(F_n) = \mathrm{HE}(R)$ to $\mathrm{HE}(L_r)$ (much less to $\mathrm{FHE}(L_r)$).

We rectify this situation by choosing lifts which are shifted from the standard lifts by appropriate translations of L. Since $n-1$ is coprime to r, there are integers s and t with $s(n-1)+tr = 1$. We use the standard lift $P_{ij} = \widehat{\rho}_{ij}$ for ρ_{ij}, but for λ_{ij} we choose the lift Λ_{ij} which shifts the standard lift by $-s\mathbf{e}_j$, and for τ_i we choose the lift T_i which shifts the standard lift by $s\mathbf{e}_i$. Thus on the vertices $\mathbf{v} = (x_1, \dots, x_n)$ of L, P_{ij} acts as a shear parallel to the $\mathbf{e}_j$ direction, Λ_{ij} is a shear composed with a shift, and T_i is reflection across the hyperplane $x_i = s/2$. In particular, each of our lifts induces an affine map $\mathbf{v} \mapsto A\mathbf{v} + \mathbf{b}$, with $A \in \mathrm{GL}(n, \mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^n$. Each edge beginning at a vertex $\mathbf{v}$ in the direction $\mathbf{e}_i$ is sent to the path beginning at $A\mathbf{v} + \mathbf{b}$, labeled by $\phi(a_i)$.

We represent an affine map $\mathbf{v} \mapsto A\mathbf{v} + \mathbf{b}$ by the $(n+1) \times (n+1)$ matrix $\begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$, acting on the vector $\begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix}$. Let E_{pq} denote the $n \times n$ elementary matrix with one non-zero entry equal to 1 in the (p, q) position. Thus the action of P_{ij} on the 0-skeleton of L is represented by the matrix with $A = I_n + E_{ji}$ and $\mathbf{b} = \mathbf{0}$, for Λ_{ij} we have the matrix with $A = I_n + E_{ji}$ and $\mathbf{b} = -s\mathbf{e}_j$ and for T_i the matrix with $A = I_n - 2E_{ii}$ and $\mathbf{b} = s\mathbf{e}_i$.

For example, for $n = 2$ we have $s = 1$ and

$$P_{12} \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad \Lambda_{12} \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad T_1 \sim \left(\begin{array}{cc|c} -1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Remark 3.1. *An important point to note is that since the relations (1) to (10) hold in* $\mathrm{Aut}(F_n)$ *and not just* $\mathrm{Out}(F_n)$*, in order to verify that the above assignments respect these relations we need only verify that the appropriate product of matrices is the identity: such a verification tells us that the appropriate product of our chosen lifts acts trivially on the vertices of L, and the action on edges (which is defined in terms of the action on labels) is automatically satisfied. This remark does not apply to relation (11), which requires special attention.*

Proposition 3.2. *For every integer r coprime to* $(n-1)$*, the lifts* P_{ij} *of* ρ_{ij}*,* Λ_{ij} *of* λ_{ij} *and* T_i *of* τ_i *define a splitting of the natural map* $\mathrm{FHE}(L_r) \to \mathrm{HE}(R) = \mathrm{Out}(F_n)$*.*

Proof. We first claim that the maps Λ_{ij}, P_{ij} and T_i (and hence the maps they induce on L_r) satisfy relations (1) to (10). In each case, the verification is a straightforward calculation, which we illustrate with several examples using $i=1, j=2$ and $k=3$. (In the light of remark 3.1, each verification simply requires a matrix calculation.)

An example of a relation of type (4) is $[\lambda_{12}^{-1}, \lambda_{23}^{-1}] = \lambda_{13}^{-1}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & s \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -s \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -s \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & s \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

To verify relation (6), we first compute the action of $P_{12}P_{21}^{-1}\Lambda_{12}$ (we only need 2 indices):

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & -s \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & s \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

then check

$$\left(\begin{array}{cc|c} 0 & -1 & s \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)^4 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

As an example of relation (8) we verify $T_1P_{12}T_1 = \Lambda_{12}^{-1}$:

$$\left(\begin{array}{cc|c} -1 & 0 & s \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} -1 & 0 & s \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & s \\ \hline 0 & 0 & 1 \end{array}\right)$$

Relation (11) is the only relation which requires some thought. For example, the matrix corresponding to the product $\prod_{i>1} P_{i1}\Lambda_{i1}^{-1}$, which lifts conjugation by a_1, is:

$$\left(\begin{array}{cc|c} 1 & 0 & (n-1)s \\ 0 & I_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Thus for all $i > 1$, the map on L sends the edge starting at $\mathbf{v}$ in the direction $\mathbf{e}_i$ to the $\sqcup$-shaped path labeled $a_1^{-1}a_ia_1$ and starting at $\mathbf{v} + s(n-1)\mathbf{e}_1 = \mathbf{v} + \mathbf{e}_1 -$

$tr\mathbf{e}_1$. Dragging all vertices of L one unit along the edge parallel to $\mathbf{e_1}$ gives a fiber-preserving homotopy of this map to the deck transformation $\mathbf{v} \to \mathbf{v} + tr\mathbf{e}_1$. This deck transformation induces the identity on L_r. □

Remark 3.3. *For $r = n = 2$ the above construction gives an embedding $\mathrm{Out}(F_2) \hookrightarrow \mathrm{Out}(F_5)$. Here is an explicit description of the images of the ρ_{ij}, λ_{ij} and τ_i under this embedding, where $F_5 = \langle a, b, c, d, e\rangle$.*

$$\rho_{12} = \begin{cases} a \mapsto db \\ b \mapsto b \\ c \mapsto c \\ d \mapsto ea \\ e \mapsto e \end{cases} \qquad \rho_{21} = \begin{cases} a \mapsto a \\ b \mapsto cea \\ c \mapsto c \\ d \mapsto d \\ e \mapsto db \end{cases} \qquad \tau = \begin{cases} a \mapsto a^{-1} \\ b \mapsto e \\ c \mapsto c^{-1} \\ d \mapsto d^{-1} \\ e \mapsto b \end{cases}$$

$\lambda_{12}(x) = b\rho_{12}(x)b^{-1}$ and $\lambda_{21}(x) = a\rho_{12}(x)a^{-1}$.

4. Proof of Theorem A: The non-existence of lifts

We begin by proving that for $n > 2$ the map $\mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n)$ does not split when $N = F_n'$; this is equivalent to the case $A \cong \mathbb{Z}^n$ in Theorem A. To do this, we consider the cyclic group $\Theta_n < \mathrm{Out}(F_n)$ of order $(n-1)$ that corresponds to the group of rotations of the marked graph shown in Figure 1.

Proposition 4.1. *For $n > 2$. The inverse image of Θ_n in $\mathrm{Aut}(F_n)/F_n'$ is torsion-free, and therefore $\mathrm{Aut}(F_n)/F_n' \to \mathrm{Out}(F_n)$ does not split.*

In this section we present three proofs of this fact. The first is a geometric proof that we feel gives the most insight into the non-splitting phenomenon; this is how we discovered Proposition 4.1. The second proof draws attention to a topological criterion illustrated by the first proof; like the first proof, it is executed using the lower sequence in Proposition 2.1. The third proof is purely algebraic. The first and third proofs also lead to a proof of the following proposition, which completes the proof of Theorem 2.2 (and theorefore of Theorem A).

Proposition 4.2. *Let $N = F_n'F_n^r$ and let p_r denote the natural map $\mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n)$. Then the short exact sequence $1 \to F_n/N \to p_r^{-1}\Theta_n \to \Theta_n \to 1$ splits if and only if r is coprime to $n-1$.*

4.1. A direct geometric proof.

At several points in the following argument we use the elementary fact that if a connected metric graph is a union of (at least two) circuits, then an isometry that is homotopic to the identity is actually equal to the identity.

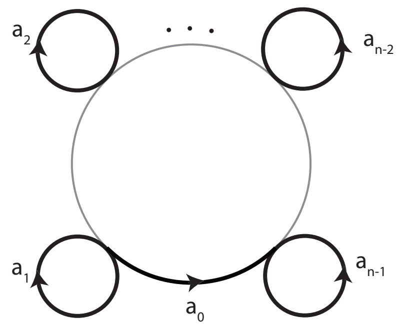

FIGURE 1. The graph X

Let $n \geq 3$ be an integer and let $X = X_n$ be the graph that has $(n-1)$ vertices, contains a simple loop of length $(n-1)$ and has a loop of length 1 at each of its vertices. (See Figure 1)

We fix a maximal tree in the graph, label the remaining edge on the long circuit a_0, and label the loops of length 1 in cyclic order, proceeding around the long cycle: $a_1, \ldots, a_{n-1}$. This provides an identification of F_n with $\pi_1 X$.

Consider the maximal abelian cover of X, that is this graph $\widehat{X} = \widetilde{X}/F_n'$. The Galois group of this covering is $F_n/F_n' \cong \mathbb{Z}^n$, and it is helpful to visualise the following embedding of $\widehat{X}$ in $\mathbb{R}^n$ (see Figure 2):

Fix rectangular coordinates $x_0, \ldots, x_{n-1}$ on $\mathbb{R}^n$ and define $\widehat{X}$ to be the union of the following n families of lines: family Λ_0 consists of all lines parallel to the x_0-axis that have integer x_i-coordinates for all $i > 0$, while Λ_i consists of all lines parallel to the x_i axis that have integer coordinates for all $j \neq i$ with $j > 0$ and which have x_0-coordinate an integer that is congruent to $i \mod (n-1)$.

The action of the Galois group $\mathbb{Z}^n = F_n/F_n'$ is by translations in the coordinate directions, with a_i acting as translation by a distance 1 in the x_i direction for $i = 1, \ldots, n-1$, and with x_0 acting as translation by a distance $(n-1)$ in the x_0 direction.

Now consider the isometry θ of X that rotates the long cycle through a distance 1, carrying the oriented loop labelled a_i to that labelled a_{i+1} for $i = 1, \ldots, n-2$ and taking a_{n-1} to a_1. This isometry has order $n-1$.

A lift $\hat{\theta}$ of θ to $\widehat{X}$ is obtained as follows:

$$\hat{\theta}(y_0, \ldots, y_{n-1}) = (y_0 + 1, y_{n-1}, y_1, y_2, \ldots, y_{n-2}).$$

In other words, $\hat{\theta}$ shifts by 1 unit in the x_0-direction and permutes the positive axes of the other generators cyclically. In particular, $\hat{\theta}^{n-1}$ is the deck transformation corresponding to $(1, 0, \ldots, 0) \in \mathbb{Z}^n$, so is not homotopic to the identity. Any power of $\hat{\theta}$ which is not a multiple of $n-1$ sends the axis for a_1 to a translate of the axis

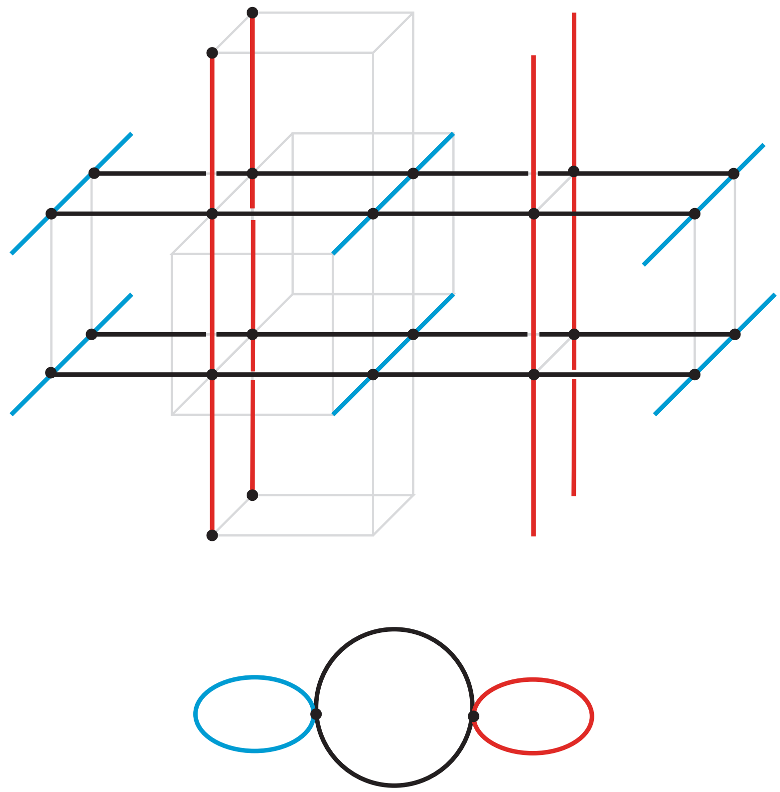

FIGURE 2. Maximal abelian cover of $X = X_n$, $n = 3$.

for a_k, for some $k \neq 1$ so is again not homotopic to the identity. This shows that $\hat{\theta}$ has infinite order in $\mathrm{FHE}(\widehat{X})$.

If we choose a different lift $\hat{\theta}'$ of θ, it differs from $\hat{\theta}$ by a deck transformation $(s, t_1, \ldots, t_{n-1})$. Then $(\hat{\theta}')^{n-1}$ is a deck transformation by $(1 + s(n-1), t_1(n-1), \ldots, t_{n-1}(n-1))$, which is not equal to 0 (so not homotopic to the identity) for any s if $n > 2$, so $\hat{\theta}'$ has infinite order in $\mathrm{FHE}(\widehat{X})$ as before. This proves Proposition 4.1.

However, if we look mod r (i.e. work modulo the action of $r\mathbb{Z}^n$), then the above deck transformation can become trivial: the equation

$$1 + s(n-1) \equiv 0 \mod r$$

has a solution if and only if $(r, n-1) = 1$, so a lift $\hat{\theta}_{(r)}$ of θ to a fiber-homotopy equivalence of $\widehat{X}_{(r)} = \widehat{X}/r\mathbb{Z}^n$ can be chosen so that $\hat{\theta}_{(r)}^{n-1}$ is homotopic (in fact equal) to the identity if and only if $(r, n-1) = 1$. This proves Proposition 4.2. $\square$

4.2. **A topological obstruction to splitting.** The finite cyclic group generated by θ acts freely on the graph X_n, and X_n can be embedded into the torus T^n in such a way that the action extends. The kernel of the map induced on fundamental groups by this embedding is exactly the commutator subgroup F_n'. Both X_n and T^n are aspherical spaces, and in this section we show that the non-splitting of the short exact sequence of Proposition 4.1 is an example of a more general phenomenon associated to this type of situation.

Let G be a group acting freely by homeomorphisms on a connected CW-complex X, and let $\widetilde{X}$ denote the universal cover. Let $\widehat{G} \subset \mathrm{Homeo}(\widetilde{X})$ be the subgroup of $\mathrm{Homeo}(\widetilde{X})$ generated by all lifts of elements of G. (If the action of G is properly discontinuous, then $\widehat{G}$ is isomorphic to the fundamental group of X/G.) There is an obvious short exact sequence

$$1 \to \pi_1 X \to \widehat{G} \to G \to 1.$$

More generally, if the action of G leaves invariant a normal subgroup $N \subset \pi_1 X$ then we write $\widehat{G}_N$ for the group of all lifts of the elements G to $\widetilde{X}/N$. There is short exact sequence

$$1 \to \pi_1 X/N \to \widehat{G}_N \to G \to 1,$$

where $\pi_1 X/N$ is the Galois group of the covering $\widetilde{X}/N \to X$.

Lemma 4.3. *The action of $\widehat{G}_N$ on $\widetilde{X}/N$ is free.*

Proof. If an element $\gamma \in \widehat{G}_N$ had a fixed point in $\widetilde{X}$ then its image in G would fix a point of X. Since the action of G is free, γ would have to lie in the kernel of $\widehat{G}_N \to G$. But this is the group of deck transformations, which acts freely. □

Lemma 4.4. *If X is finite dimensional and aspherical then $\widehat{G}_N$ is torsion free.*

Proof. If $\widehat{G}_N$ had a non-trivial element of finite order, say γ, then by the previous lemma we would have a free action of the finite group $C = \langle \gamma \rangle$ on the contractible finite dimensional space $\widetilde{X}$, contradicting the fact that C has cohomology in infinitely many dimensions. □

Example 1. *If X is a graph, and the action of G is properly discontinuous (e.g. by graph isometries) then $\widehat{G}_N$ is the fundamental group of a graph and hence is free.*

Proposition 4.5. *Let X be a connected space, let Y be an aspherical space, and let $f\colon X \hookrightarrow Y$ be an embedding. Let $N \subset \pi_1 X$ be the*[1] *kernel of the induced map on fundamental groups. Let G be a group that acts freely on X and consider*

$$1 \to \pi_1 X/N \to \widehat{G}_N \to G \to 1.$$

[1]This is well-defined as it is normal and a change of basepoint isomorphism produces no ambiguity mod conjugacy.

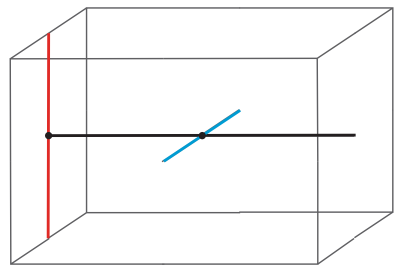

FIGURE 3. X_3 embedded in a torus

If there is a free action of G on Y making f equivariant, then $\widehat{G}_N$ is torsion-free.

Proof. Let $\widehat{G}^Y$ be the group of all lifts to $\widetilde{Y}$ for the action of G on Y. The embedding $f : X \to Y$ lifts to an embedding $\widetilde{X}/N \to \widetilde{Y}$ that induces an isomorphism from $\widehat{G}_N$ to the subgroup of $\widehat{G}^Y$ that preserves the image of $\widetilde{X}/N$. (This will be the whole of $\widehat{G}^Y$ if and only if $f_* : \pi_1 X \to \pi_1 Y$ is surjective.)

We proved in Lemma 4.4 that $\widehat{G}^Y$ is torsion-free. □

Proof of Proposition 4.1. We consider the graph X_n shown in figure 1 and the cyclic group Θ_n of order $(n-1)$ that acts freely on the graph, permuting the vertices in cyclic order. We embed X_n in an n-dimensional torus T by quotienting the embedding $\widehat{X}_n \to \mathbb{R}^n$ of the previous section by the action of $\mathbb{Z}^n$ (see Figure 3).

We make the generator of Θ_n act on T by translation in the x_0 direction through a distance 1 followed by the rotation that leaves invariant the x_0 direction and permutes the coordinates $(x_1, x_2, \ldots, x_{n-1})$ cyclically.

This action in free, and the embedding $X_n \to T$ is equivariant. Thus we are in the situation of Proposition 4.5, and Proposition 4.1 is proved.

4.3. **A proof using presentations.** We are interested in the short exact sequence

$$1 \to F_n/F_n' \to \mathrm{Aut}(F_n)/F_n' \xrightarrow{\pi} \mathrm{Out}(F_n) \to 1.$$

Let $\Theta_n \subset \mathrm{Out}(F_n)$ be the subgroup generated by the class of the automorphism

$$\theta : (a_0, a_1, \ldots, a_{n-2}, a_{n-1}) \mapsto (a_0, a_2, \ldots, a_{n-1}, a_0^{-1} a_1 a_0).$$

Note that Θ_n is cyclic of order $(n-1)$ but that θ has infinite order in $\mathrm{Aut}(F_n)$, since it is a root of the inner automorphism by a_0.

Let $\widehat{\Theta}_n = \pi^{-1}\Theta_n \subset \mathrm{Aut}(F_n)/F_n'$, so the above short exact sequence restricts to:

$$1 \to F_n/F_n' \to \widehat{\Theta}_n \xrightarrow{\pi} \Theta_n \to 1.$$

We produce a presentation for $\widehat{\Theta}_n$ using a standard procedure for constructing presentations of group extensions; this is explained, e.g., in [12], Theorem 1, p. 139.

We fix a basis $\{a_0, \ldots, a_{n-1}\}$ for the free group F_n and write α_i for the image in $\mathrm{Aut}(F_n)/F_n'$ of the inner automorphism $w \mapsto a_i w a_i^{-1}$. Then F_n/F_n' is generated by the α_i subject to the relations $[\alpha_i, \alpha_j] = 1$, and Θ_n is generated by the image of θ subject to the relation that this image has order $n-1$. The automorphisms α_i and θ satisfy the following relations:

(1) $\theta\alpha_0\theta^{-1} = \alpha_0$
(2) $\theta\alpha_i\theta^{-1} = \alpha_{i+1}$ for $i = 1, \ldots, n-2$
(3) $\theta\alpha_{n-1}\theta^{-1} = \alpha_0^{-1}\alpha_1\alpha_0$
(4) $\theta^{n-1} = \alpha_0$

and the theorem cited above assures us that (introducing a generator x to represent θ) these relations suffice to present $\widehat{\Theta}_n$:

$$\widehat{\Theta}_n \cong \langle \alpha_0, \ldots, \alpha_{n-1}, x \mid [\alpha_i, \alpha_j] = 1 \text{ for } i, j = 0, \ldots, n-1, \\ x\alpha_0 x^{-1} = \alpha_0,\ x\alpha_i x^{-1} = \alpha_{i+1} \text{ for } i = 1, \ldots, n-2, \\ x\alpha_{n-1}x^{-1} = \alpha_0^{-1}\alpha_1\alpha_0,\ x^{n-1} = \alpha_0 \rangle$$

Proposition 4.6. $\widehat{\Theta}_n \cong \mathbb{Z}^{n-1} \rtimes_\psi \mathbb{Z}$ *where ψ is the automorphism that permutes a free basis $\{\alpha_1, \ldots, \alpha_{n-1}\}$ cyclically. In particular, $\widehat{\Theta}_n$ is torsion-free.*

Proof. We use Tietze moves to simplify our presentation of $\widehat{\Theta}_n$. First we use $[\alpha_0, \alpha_1] = 1$ to replace $x\alpha_{n-1}x^{-1} = \alpha_0^{-1}\alpha_1\alpha_0$ by $x\alpha_{n-1}x^{-1} = \alpha_1$. Next we use the last relation to remove the superfluous generator α_0, replacing it by x^{n-1} in the other relations where it appears. But in fact, all of the relations where α_0 appeared become redundant when we substitute x^{n-1}: this is obvious for $x\alpha_0x^{-1} = \alpha_0$, and in the remaining cases one can deduce $[x^{n-1}, \alpha_i] = 1$ by combining the relations $x\alpha_ix^{-1} = \alpha_{i+1}$ and $x\alpha_{n-1}x^{-1} = \alpha_1$.

At the end of these moves we are left with the presentation

$$\widehat{\Theta}_n \cong \langle \alpha_1, \ldots, \alpha_{n-1}, x \mid [\alpha_i, \alpha_j] \text{ for } i, j = 1, \ldots, n-1, \\ x\alpha_i x^{-1} = \alpha_{i+1} \text{ for } i = 1, \ldots, n-2,\ x\alpha_{n-1}x^{-1} = \alpha_1 \rangle,$$

which is the natural presentation of $\mathbb{Z}^{n-1} \rtimes_\psi \mathbb{Z}$. □

Corollary 4.7. $\widehat{\Theta}_n$ *is the fundamental group of a closed, flat n-manifold that fibres over the circle with holonomy $\mathbb{Z}/(n-1)$.*

Proof of Propositions 4.1 and 4.2

From our original presentation of $\widehat{\Theta}_n$ we readily deduce the following presentation for the preimage $\Theta_n(n, r)$ of Θ_n in $\mathrm{Aut}(F_n)/F_n'F_n^r$

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, x \mid \alpha_i^r = 1 = [\alpha_i, \alpha_j] \text{ for } i = 0, \ldots, n-1, \\ x\alpha_i x^{-1} = \alpha_{i+1} \text{ for } i = 1, \ldots, n-2,$$

$$x\alpha_{n-1}x^{-1} = \alpha_1,\ \alpha_0 = x^{n-1}\rangle,$$

and making Tietze moves as above we see that $\widehat{\Theta_n}(n,r)$ is a semidirect product $\widehat{\Theta_n}(n,r) \cong (\mathbb{Z}/r)^{n-1} \rtimes_\psi \mathbb{Z}/r(n-1)$; in particular x has order $r(n-1)$.

Let $N = F_n/F_n'F_n^r \cong (\mathbb{Z}/r)^n$ denote the subgroup generated by the α_i. We are interested in when we can split

$$1 \to N \to \widehat{\Theta_n}(n,r) \to \Theta_n \to 1.$$

If r is coprime to $n-1$, then there exists an integer t such that $tr \equiv 1 \mod (n-1)$, so $[\theta]^{tr} = [\theta]$ in Θ_n, and we can split the above sequence by sending the generator $[\theta] \in \Theta_n$ to x^{tr}, noting that

$$(x^{tr})^{n-1} = (x^{r(n-1)})^t = 1.$$

It remains to prove that if r is not coprime to $n-1$ then there is no splitting. To establish this, we consider an arbitrary element in the preimage of $[\theta]$ and examine whether it can have order $n-1$. Such an element has the form vx, where $v = \alpha_0^{m_0}\alpha_1^{m_1}\dots\alpha_{n-1}^{m_{n-1}}$. From our presentation of $\Theta(n,r)$ we see that

$$(vx)^{n-1} = v.(xvx^{-1}).(x^2vx^{-2}).(x^3vx^{-3}).\dots.(x^{n-2}vx^{2-n}).x^{n-1}$$

can be simplified to

$$(vx)^{n-1} = v.\psi(v).\psi^2(v).\psi^3(v).\dots.\psi^{n-2}(v).a_0.$$

And since

$$\alpha_i.\psi(\alpha_i).\psi^2(\alpha_i).\psi^3(\alpha_i).\dots.\psi^{n-2}(\alpha_i) = \mu := \alpha_1\alpha_2\dots\alpha_{n-1}$$

for $i = 1,\dots,n-1$, while $x\alpha_0x^{-1} = \alpha_0$, we have

$$(vx)^{n-1} = \alpha_0^{m_0(n-1)+1}\mu^{m_1+\dots+m_{n-1}}.$$

In order for this to equal the identity in $\Theta_n(n,r)$ the exponent of α_0 has to be zero mod r. But this is impossible, because $n-1$ is not coprime to r and hence there is no integer m_0 such that $m_0(n-1)+1 \equiv 0 \mod r$. $\square$

5. Theorem B: A cohomological remark

Let $M = H_1(F_n) \cong \mathbb{Z}^n$ be the standard module with left action by $\mathrm{Out}(F_n)$. In the previous section we exhibited an extension $1 \to M \to \mathrm{Aut}(F_n)/F_n' \to \mathrm{Out}(F_n) \to 1$ which does not split and therefore determines a non-trivial cohomology class in $H^2(\mathrm{Out}(F_n); M)$; this proves the first statement of Theorem B. For the second statement, we consider the dual $M^* = H^1(F_n)$ of the standard module instead.

Proposition 5.1. *$H^2(\mathrm{Out}(F_n), M^*) = 0$ for $n \geq 8$.*

Proof. To compute $H^2(\mathrm{Out}(F_n), M^*)$, we use the Hochschild-Lyndon-Serre spectral sequence in cohomology for the short exact sequence

$$1 \to F_n \to \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n) \to 1,$$

with trivial $\mathbb{Z}$ coefficients. This has $E_2^{p,q} = H^p(\mathrm{Out}(F_n); H^q(F_n)) \Rightarrow H^{p+q}(\mathrm{Aut}(F_n))$. Since $H^q(F_n) = 0$ for $q > 1$, the E_2-term has exactly two non-zero rows, for $q = 0$ and $q = 1$:

$$\begin{tikzcd}
0 \arrow[drr, "d_2"] & 0 \arrow[drr, "d_2"] & 0 & 0 \\
H^0(\mathrm{Out}(F_n); M^*) \arrow[drr, "d_2"] & H^1(\mathrm{Out}(F_n); M^*) \arrow[drr, "d_2"] & H^2(\mathrm{Out}(F_n); M^*) & H^3(\mathrm{Out}(F_n); M^*) \\
H^0(\mathrm{Out}(F_n); \mathbb{Z}) & H^1(\mathrm{Out}(F_n); \mathbb{Z}) & H^2(\mathrm{Out}(F_n); \mathbb{Z}) & H^3(\mathrm{Out}(F_n); \mathbb{Z})
\end{tikzcd}$$

The $E_2^{p,0}$ terms are $H^i(\mathrm{Out}(F_n); \mathbb{Z})$ with trivial $\mathbb{Z}$-coefficients, and the $E_2^{p,1}$ terms are $H^i(\mathrm{Out}(F_n); M^*)$. Now

$$E_\infty^{p,0} = E_3^{p,0} = H^p(\mathrm{Out}(F_n); \mathbb{Z})/im(d_2).$$

Since the spectral sequence converges to the cohomology of $\mathrm{Aut}(F_n)$, we have a two-stage filtration

$$0 \subset E_\infty^{p,0} \subset H^p(\mathrm{Aut}(F_n); \mathbb{Z}) \quad \text{with} \quad E_\infty^{p,1} = H^p(\mathrm{Aut}(F_n); \mathbb{Z})/E_\infty^{p,0}.$$

The map on cohomology induced by $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$ factors through the edge homomorphism $e\colon E_\infty^{p,0} \to H^p(\mathrm{Aut}(F_n); \mathbb{Z})$:

$$\begin{tikzcd}
H^p(\mathrm{Out}(F_n); \mathbb{Z}) \arrow[rr] \arrow[dr, two heads] & & H^p(\mathrm{Aut}(F_n); \mathbb{Z}) \\
& H^p(\mathrm{Out}(F_n); \mathbb{Z})/im(d_2) \arrow[ur, hook, "e"] &
\end{tikzcd}$$

But the top arrow is an isomorphism for $n >> p$ ([10, 11]), so in this range all of these maps are isomorphisms. In particular $d_2 = 0$ and $E_3^{2,1} = E_\infty^{2,1} = H^2(\mathrm{Out}(F_n); M^*)$ must be zero.

The exact stable range for $H^p(\mathrm{Out}(F_n))$ is still unknown. A lower bound, from [10], is $n \geq 2p + 4$, which gives $n \geq 8$ when $p = 2$. □

The form of the cohomology argument above may be abstracted as follows.

Lemma 5.2. *Let* $1 \to F \to \Gamma \xrightarrow{\pi} Q \to 1$ *be a short exact sequence with* F *a free group, and let* $M \cong H^1(F, \mathbb{Z})$ *be the associated* $\mathbb{Z}Q$*-module. If* π *induces a* *isomorphism* $H^{p-1}(Q;\mathbb{Z}) \to H^{p-1}(\Gamma;\mathbb{Z})$ *and an injection* $H^p(Q;\mathbb{Z}) \to H^p(\Gamma;\mathbb{Z})$, *then* $H^{p-2}(Q;M) = 0$.

6. Theorem C: Classification of graphs realizing finite subgroups

In the course of this section and the next, we shall prove that if n is even and $n < m \leq 2n$, or n is odd and $n < m \leq 2n-2$, then every homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has image of order at most two. We do this by examining the possible images in $\mathrm{Out}(F_m)$ of the finite subgroups of $\mathrm{Out}(F_n)$. We show that the possible embeddings of the largest finite subgroups of $\mathrm{Out}(F_n)$ are so constrained that none can be extended to a homomorphism defined on the whole of $\mathrm{Out}(F_n)$. Arguing in this manner, we deduce that no homomorphism from $\mathrm{Out}(F_n)$ to $\mathrm{Out}(F_m)$ can restrict to an injection on the largest finite subgroup $W_n \subset \mathrm{Out}(F_n)$. This enables us to apply results from our previous work [6], in which we described the homomorphic images of $\mathrm{Out}(F_n)$ into which W_n does not inject.

6.1. Admissible graphs.

Definition 6.1. *A graph is* admissible *if it is finite, connected, has no vertices of valence 1 or 2, and has no non-trivial forests that are invariant under the full automorphism group of the graph. An admissible graph on which a group G acts is said to be G-minimal if the action is faithful and there are no forests which are invariant under the G-action; thus every admissible graph is minimal for its full automorphism group.*

The following theorem explains our interest in admissible graphs.

Theorem 6.2 ([7, 14])**.** *Every finite subgroup of* $\mathrm{Out}(F_n)$ *can be realized as a subgroup of the automorphism group of an admissible graph with fundamental group* F_n.

An easy exercise using Euler characteristic yields:

Lemma 6.3. *An admissible graph of genus m has at most $2m-2$ vertices and $3m-3$ edges.*

The *genus* of a graph X is the rank of $H_1(X)$. It can be computed as $e - v + c$, where e is the number of edges of X, v is the number of vertices and c is the number of components.

Lemma 6.4. *A proper subgraph of an admissible graph has strictly smaller genus.*

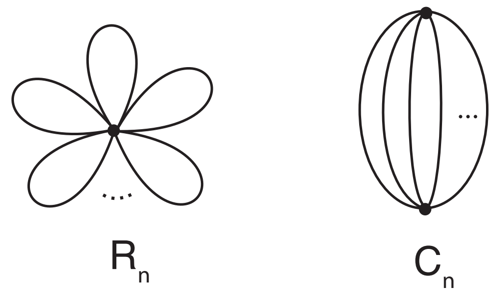

FIGURE 4. An n-rose R_n and and n-cage C_n

6.2. **Classification of admissible A_n-graphs.** We are interested in finite subgroups of $\mathrm{Out}(F_m)$ that contain alternating groups, so we begin by classifying admissible graphs of genus $m \leq 2n$ which realize the alternating group A_n.

Two graphs which admit obvious A_n-actions are the n-cage C_n, which has two vertices and n edges joining them, and the n-rose R_n which has one vertex and n loops (see Figure 4). These will appear frequently in our discussion of A_n-graphs.

If X is a graph with an A_n-action, we denote the orbit of a vertex v by $[v]$. In the next lemma we consider orbits of cardinality n. We use the fact that the action of A_n on a set of size n is either trivial or standard, provided $n \neq 4$. (For $n = 4$, however, A_4 has the Klein 4-group as a normal subgroup, with quotient $\mathbb{Z}/3$, which acts on four points by fixing one of them.)

Lemma 6.5. *Suppose $n \geq 5$, and let X be a graph of genus $m < (n-1)(n-2)/2$ which realizes A_n. If all vertex-orbits $[v]$ have size n, then X is the disjoint union of n subgraphs that are permuted by the action of A_n in the standard way.*

Proof. Since $n \geq 5$, the action of A_n on each orbit $[v]$ is the standard permutation action. In particular, the stabilizer of each vertex v is isomorphic to A_{n-1}, and acts transitively on the other vertices in $[v]$. Moreover these point stabilizers account for all of the subgroups of A_n that are isomorphic to A_{n-1}.

Fix a vertex v_0. For each vertex-orbit $[w]$, the stabilizer of v_0 fixes a unique vertex $w_0 \in [w]$. Let X_0 be the subgraph spanned by all of the w_0, including v_0. We clam that X is the disjoint union of copies of X_0 permuted by the action of A_n.

If a vertex $w_0 \in X_0$ is connected to a vertex u outside of X_0 by an edge, then the orbit of u under the stabilizer of w_0 has $n-1$ elements, so w_0 has valence at least $n-1$; similarly u has valence at least $n-1$. Let X_1 be the subgraph spanned by $[w]$ and $[u]$. If $[u] = [w]$, then X_1 contains the complete graph on n vertices (plus possibly some loops of length one); but this graph has genus $(n-1)(n-2)/2 > m$,

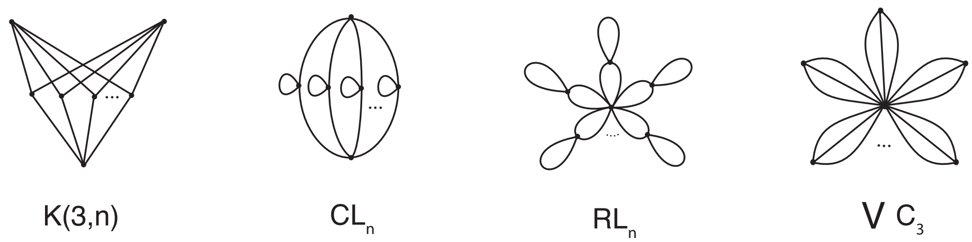

FIGURE 5. Admissible graphs of rank $\leq 2n$ realizing A_n

so this is impossible. If $[u] \neq [w]$, the genus of X_1 is at least $n(n-1) - 2n + 1$, but again this genus is strictly bigger than m so this is impossible.

It follows that X is the disjoint union of n copies of X_0, one for each $v \in [v_0]$, and that these are permuted by the action of A_n in the standard way.

□

If X realizes A_n, then A_n acts on the set of vertices and on the set of edges of X. Our analysis of A_n-graphs depends on the following result of M. Liebeck.

Proposition 6.6 ([17], Prop. 1.1)**.** *If $n > 8$ then the orbits of the action of A_n on a finite set S have size 1, n, $\binom{n}{2}$ or larger. If $n = 7$ or 8 there may also be an orbit of size 15. If $n = 6$ there may be an orbit of size 10.*

In the following proposition, the names of graphs refer to Figure 5. Note that in each case, the automorphism group of the graph contains a unique copy of A_n, up to conjugacy. Thus, for the most part, we need not specify how A_n is acting each time such a graph appears.

We use the following standard notation: if X_1 and X_2 are graphs, each with a distinguished vertex, then we write $X_1 \vee X_2$ for the graph obtained from the disjoint union $X_1 \amalg X_2$ by identifying these vertices; if each of X_1 and X_2 is equipped with an action by a group G, we refer to the induced action on $X_1 \amalg X_2$ and $X_1 \vee X_2$ as the diagonal action.

Proposition 6.7 (Classification of admissible A_n-graphs)**.** *Suppose $n > 8$, and let X be an admissible graph of genus $m \leq 2n$ which realizes $A = A_n$. Let X_A be the subgraph of X spanned by edges with non-trivial A-orbits.*

1. *If $m < n - 1$ there are no admissible graphs realizing A.*
2. *If $n - 1 \leq m < 2n - 2$ then $X_A = R_n$ or C_n.*
3. *If $m = 2n - 2$ then X_A is R_n, C_n, $C_n \vee C_n$ or $K(3, n)$.*
4. *If $m = 2n - 1$ then X_A is one of the above or CL_n, or is C_{2n}, $C_n \vee R_n$ or $C_n \amalg C_n$ with diagonal action.*

(5) *If $m = 2n$ then X_A is one of the above, $R_{2n} = R_n \vee R_n$ with diagonal action, $R_n \amalg C_n$, RL_n or $\bigvee_n C_3$.*

X is obtained from X_A by adding additional edges and vertices, fixed by A, in an arbitrary manner subject to the requirement that X must be connected and must not contain a non-trivial forest that is invariant under the action of $\mathrm{Aut}(X)$*.*

Proof. Since X is admissible of genus $m \leq 2n$, it has at most $2m-2 \leq 4n-2$ vertices, and since $n > 8$ all vertex orbits have size 1 or n. Therefore there are at most three non-trivial vertex orbits, and we divide the classification into cases according to the number of these.

Case 1: All vertices of X are fixed. In this case the subgraph X_A is a union of cages and roses. Since $n > 8$ and the genus of X is at most $2n < \binom{n}{2}$, these cages and roses must have exactly n edges. The genus of X_A gives a lower bound on the genus of X, so the genus of X is at least $n-1$. If $n - 1 \leq m < 2n - 2$, the graph X contains exactly one cage or one rose.

If $m = 2n - 2$ then the only new possibility is $X_A = C_n \vee C_n = X$, with diagonal action of A_n.

If $m = 2n - 1$ we must consider the new possibilities $X_A = R_n \vee C_n$, $X_A = C_{2n}$ and $X_A = C_n \amalg C_n$. If $X_A = C_n \amalg C_n$ then X is obtained from X_A by adding two extra (fixed) edges, which must both have the same endpoints, since otherwise the full group of isometries of X would have an invariant forest. If $X_A = C_n \vee C_n$ then X has one fixed edge and this cannot join the vertices of the C_n which lie opposite the wedge vertex, for the same reason.

If $m = 2n$ the only new possibility for X_A is $R_n \vee R_n = X$, where the action of A_n is diagonal.

If there are non-trivial vertex orbits, consider the (invariant) subgraph of X_A obtained by deleting all fixed vertices (and adjacent edges) of X. By Lemma 6.5, this subgraph is a disjoint union of subgraphs $X_1, \ldots, X_n$ which are permuted by A_n. Since $m \leq 2n$, each of these subgraphs can have genus at most 1.

Case 2: X has one non-trivial vertex orbit. In this case X_1 has only a single vertex v, possibly with one loop attached.

If v has a loop attached, there must be at least two other edges of X adjacent to v (since X has no separating edges), and each of these edges has its other end at a fixed vertex. If these vertices are the same, then $X = X_A$ is the "rose with loops" RL_n which has genus $2n$. If they are different, then X contains the "cage with loops" CL_n, and has genus $2n - 1$ or $2n$.

If there is no loop at v, there must be at least three edges e_1, e_2 and e_3 of X adjacent to v, terminating at fixed vertices u_1, u_2 and u_3. If $u_1 = u_2 = u_3$ then $X = \bigvee_n C_3$, which has genus $2n$. If u_1, u_2 and u_3 are distinct, then $X_A = K(3, n)$, which has genus $2n-2$. The case $u_1 = u_2 \neq u_3$ cannot occur, since the resulting graph has a forest invariant under its full isometry group.

Case 3: X has two non-trivial vertex orbits. In this case X_1 has two vertices v and w, and we claim this can never give an admissible graph of rank $\leq 2n$. If X_1 is a 2-cage C_2, then there must be another edge starting at v and another edge at w, since X is admissible. These edges may terminate at the same or at different fixed points. In either case, their orbits form a forest invariant under the full isometry group of X. Other possibilities for X_1 are eliminated by using the fact that X is admissible to count the minimal number of orbits of edges terminating in X_1, then estimating the genus of the subgraph spanned by these edge-orbits; in all cases, this genus is bigger than $2n$. For example, if X_1 is a single edge, there must be at least 4 more edges adjacent to X_1, and all must be in different edge-orbits since there are no orbits of size $2n$. The subgraph spanned by the orbit of X_1 and these additional edge-orbits has $5n$ edges and at most $2n+4$ vertices, so its genus is at least $3n-3 > 2n$.

Case 4: X has three non-trivial vertex orbits. This case also cannot occur. Let u, v and w be the vertices of X_1. In all cases, the fact that X is admissible allows us to find a subgraph of genus greater than $2n$. For example, if X_1 is a triangle, there are at least 3 additional edges terminating in X_1. The subgraph spanned by the orbits of X_1 and these additional edges has $6n$ edges and at most $3n+3$ vertices, so has genus at least $3n-2 > 2n$.

□

6.3. **Classification of minimal admissible W_n-graphs.** Let $W_n \cong (\mathbb{Z}/2)^n \rtimes S_n$ be the full group of automorphisms of R_n. If we identify R_n with the standard rose with petals labelled by the generators of F_n, the subgroup S_n is generated by permutations of the generators and the subgroup $(\mathbb{Z}/2)^n$ is generated by the automorphisms ε_i, where ε_i inverts the i-th generator. In this section we classify all minimal admissible W_n-graphs X of genus $m \leq 2n$.

Lemma 6.8. *Suppose S_n acts on a finite set Ω. Then S_n permutes the A_n-orbits in Ω, and the action on this set of orbits factors through the determinant map $S_n \to S_n/A_n \cong \mathbb{Z}/2$.*

Proof. For all $\sigma \in S_n$ and $\omega \in \Omega$ we have $\sigma(A_n\omega) = A_n(\sigma\omega)$. □

Lemma 6.9. *Suppose W_n acts on a finite set Ω, and A_n has a single non-trivial orbit Ω_A of size n. Then Ω_A is invariant under the full group W_n. Each ε_i acts as the identity on Ω_A, all the ε_i act by the same involution*[2] *on the fixed set Ω^A of the A_n-action, and every transposition in S_n acts by the same involution on Ω^A.*

[2]for brevity, we use the term "involution" to mean a symmetry that either has order 2 or is the identity

Proof. By Lemma 6.8 the action of $S_n < W_n$ preserves Ω^A and Ω_A, and all permutations of determinant -1 act by the same involution of Ω^A. Thus it only remains to check the action of the ε_i.

Let $\omega_1, \dots, \omega_n$ be the elements of Ω_A, with the standard A_n action on the subscripts. The centralizer of ε_1 contains a copy of A_{n-1}, so ε_1 acts on the fixed point set of this A_{n-1}, which is $\Omega^A \cup \omega_1$. Suppose ε_1 sends ω_1 to $t \in \Omega^A$. Set $\sigma = (12)(ij)$ for some $i \neq j > 2$. Then $\sigma \in A_n$ and $\varepsilon_2 = \sigma\varepsilon_1\sigma$. Applying this to ω_2 shows that $\varepsilon_2(\omega_2) = t$. Thus $\varepsilon_1\varepsilon_2(\omega_2) = \varepsilon_1(t) = \omega_1$. Since ε_1 and ε_2 commute, this gives $\varepsilon_2\varepsilon_1(\omega_2) = \omega_1$, i.e. $\varepsilon_1(\omega_2) = \varepsilon_2(\omega_1)$, which implies that $\varepsilon_1(\omega_2) \neq \omega_1, \omega_2$ or t. Thus $\varepsilon_1(\omega_2) = \omega_i$ for some $i > 2$. Now $\varepsilon_1\varepsilon_2 = \varepsilon_1\sigma\varepsilon_1\sigma$ and $\varepsilon_2\varepsilon_1 = \sigma\varepsilon_1\sigma\varepsilon_1$; applying the first expression to ω_2 gives ω_1, but the second expression sends ω_2 to $\sigma\varepsilon_1(\omega_i) \neq \omega_1$. We conclude that $\varepsilon_1(\omega_1) = \omega_1$ and $\varepsilon_1(\omega_i) \in \{\omega_2, \dots, \omega_n\}$ for all $i > 1$, i.e. ε_1 preserves Ω_A and Ω^A.

In fact, we must have $\varepsilon_1(\omega_i) = \omega_i$ for all i. To see this, suppose, e.g., that $\varepsilon_1(\omega_2) = \omega_3$. Then

$$\begin{aligned}\omega_3 = \varepsilon_1(\omega_2) &= \varepsilon_1\varepsilon_2(\omega_2) = \varepsilon_2\varepsilon_1(\omega_2)\\ &= \varepsilon_2(\omega_3) = (12)\varepsilon_1(12)(\omega_3) = (12)\varepsilon_1(\omega_3) = (12)(\omega_2) = \omega_1,\end{aligned}$$

giving a contradiction.

Since all ε_i are conjugate by elements of A_n, they all act in the same way on Ω^A. $\square$

Now let X be a minimal admissible W_n graph. As in the previous section, we denote by X_A the subgraph of X spanned by edges in non-trivial A_n-orbits, and by X^A the subgraph fixed by the A_n-action. Note that $X = X^A \cup X_A$. Let $\Delta = \varepsilon_1 \dots \varepsilon_n \in W_n$ and let $\alpha : W_n \to W_n$ be the homomorphism that is the identity on $S_n < W_n$ and sends each ε_i to $\varepsilon_i\Delta$. Note that α is an automorphism if n is even but has kernel $\langle\Delta\rangle$ if n is odd.

In light of Theorem 6.2, the following proposition provides a complete description, up to conjugacy, of the subgroups of $\mathrm{Out}(F_m)$ isomorphic to W_n with $n > 8$ and $m \leq 2n$.

Proposition 6.10. *Suppose $n > 8$ and let X be a W_n-minimal, admissible graph of genus $m \leq 2n$. Then X_A is invariant under the whole group W_n, and all of the ε_i have the same restriction to X^A. The possibilities for X_A are:*

(1) *If $m \leq 2n-2$ then $X_A = R_n$ and the action of W_n on R_n is either the standard one or else the standard one twisted by $\alpha : W_n \to W_n$.*
(2) *If $m = 2n-1$, the only additional possibilities for X_A are C_{2n}, $R_n \vee C_n$, and CL_n. In all cases, $X = X_A$. In the action on C_{2n}, the edges are grouped in pairs $\{e_i, e_i'\}$ so that the action of $\sigma \in S_n$ sends e_i to $e_{\sigma(i)}$ and e_i' to $e'_{\sigma(i)}$,*

and either the action of ε_i is standard (i.e. it exchanges e_i and e_i' only) or else it is the standard action twisted by $\alpha : W_n \to W_n$. In addition, the ε_i and the transpositions in S_n may exchange the vertices of C_{2n}. The action of W_n on $R_n \subset R_n \vee C_n$ is as in (1) and the ε_i act trivially on $C_n \subset R_n \vee C_n$. In a standard action of W_n on CL_n, each ε_i flips the ith loop and leaves all others other fixed, and W_n interchanges the vertices of C_{2n} via a non-trivial homomorphism $W_n \to \mathbb{Z}/2$; any action of W_n on CL_n is either standard or else a standard one twisted by $\alpha : W_n \to W_n$.

(3) *If $m = 2n$, the only additional possibilities for X_A graphs are $R_{2n} = R_n \vee R_n'$, RL_n and $R_n \amalg C_n$. In the first two cases $X = X_A$ and in the last case X is obtained by connecting R_n to C_n with two edges that have the same endpoints. The action of W_n on each factor of $R_{2n} = R_n \vee R_n'$ will be as described in (1), except that on at most one factor the ε_i might act trivially. In the action of W_n on CL_n, either each ε_i is supported on the ith figure-8 graph in the wedge, or else the action is obtained from one with this property by twisting with $\alpha : W_n \to W_n$.*

Proof. We divide the proof into cases according to the classification in Proposition 6.7 of A_n-graphs.

Case 1: $X_A = R_n$ or C_n. In this case we can apply Lemma 6.9 to the action of W_n on the set of (unoriented) edges of X to conclude that X_A is invariant and that each ε_i acts as the identity on the set of edges of X_A, and as a fixed involution τ on X^A. If $X_A = C_n$ and ε_1 inverts an edge, then it must interchange the vertices of C_n and thus invert all of the edges; furthermore, since the ε_i are all conjugate by the action of A_n, they must all do this. But then $\varepsilon_1\varepsilon_2$ acts as the identity on X, so X does not realize W_n. Thus $X_A = R_n$, and we label the edges so that the action is standard. Each ε_i acts by flipping some of the petals. Since all ε_i are conjugate by elements of A_n, they all flip the same number of petals. If ε_i flips a_j for some $j \neq i$, it must flip all a_j for $j \neq i$, because ε_i commutes with a copy of A_{n-1} which acts transitively on these a_j. It can't flip all (or none) of the petals, since then $\varepsilon_i\varepsilon_j$ would act as the identity. Therefore ε_i must flip e_i alone, or else all edges except e_i.

If $m < 2n-2$ this takes care of all possibilities for X_A, by Proposition 6.7.

Case 2: $X_A = K(3,n)$. The full group of isometries of $K(3,n)$ is isomorphic to $S_3 \times S_n$, which has order only $6n!$. This is less than the order of W_n, so $K(3,n)$ cannot realize W_n. Adding one or two extra edges to $K(3,n)$ can only reduce the size of the isometry group, so in fact X_A cannot be isomorphic to $K(3,n)$ for any $m \leq 2n$.

Case 3: $X_A = C_n \vee C_n$. Write $X_A = C_n \vee C_n'$ where C_n' is another copy of C_n, and A_n acts diagonally. Applying Lemma 6.9 to the set Ω consisting of pairs $\{e_i, e_i'\}$ and single edges $\{f_i\} \in X^A$, we conclude that ε_i either fixes all e_j or sends each e_j to

e'_j. But we know that all ε_i act by the same involution of X^A, so this would imply that $\varepsilon_1\varepsilon_2$ acts as the identity on $X = X_A \cup X^A$, contradicting the assumption that the action of W_n is minimal, hence faithful.

Case 4: $X_A = C_n \amalg C_n$**.** This cannot be a minimal W_n-graph; the proof is identical to Case 3.

Case 5: $X_A = \bigvee C_3$**.** Since no edge of X_A can be inverted by an isometry, W_n acts on the set of A_n-orbits of edges. Since A_n acts trivially on this set, the action factors through $W_n/A_n \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. But any action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ on a set of 3 elements has a fixed element. This means that some A_n-orbit is invariant under W_n, so X has an invariant forest and is not minimal for W_n.

All other cases support a W_n action. Specifically, we have:

Case 6: $X_A = R_n \vee C_n$ **or** $R_n \amalg C_n$. Here, R_n and C_n are each invariant under the full isometry group of X_A. Apply Lemma 6.9 separately to the set of edges in R_n and in C_n to conclude that ε_i acts as in Case 1 on R_n and trivially on C_n.

Case 7: $X_A = C_{2n}$**.** If we write $C_{2n} = C_n \cup C'_n$ with diagonal A_n-action, then Lemma 6.9 applied to the set Ω of pairs $\{e_i, e'_i\}$ shows that each ε_j acts trivially on the set of such pairs. Arguing as in case 1, we see that ε_i interchanges only e_i and e'_i, or else interchanges e_j and e'_j for all j except $j = i$. In addition, all of the ε_i interchange the vertices of C_{2n}, or else fix them.

Case 8: $X_A = CL_n$**.** Arguing as in case 1, we see that ε_i acts by flipping the i-th loop or else flipping all loops except the ith. It may also interchange the top and bottom vertices. (If ε_i did not flip any loops, then $\varepsilon_1\varepsilon_2$ would act as the identity.) If the ε_i do not interchange the vertices, then the transpositions in S_n must, since otherwise there is an invariant forest.

Case 9: $X_A = RL_n$**.** Again, an argument akin to case 1 shows that (twisting with $\alpha : W_n \to W_n$ if necessary) we may assume that ε_i is supported on the ith figure-8 in the wedge.

Case 10: $X_A = R_{2n}$. We have $R_{2n} = R_n \vee R'_n$ with A_n acting diagonally. ε_i acts by flipping e_i and e'_i or flipping all other petals and/or interchanging e_i with e'_i.

□

Remark 6.11. *When n is odd, certain of the actions in the preceding proposition may fail to be faithful because of the twisting by α: when the action of W_n on X_A factors through $\alpha : W_n \to W_n$, the action of Δ on X^A must be non-trivial if the action of W_n on X is to be faithful.*

6.4. **Classification of minimal admissible G_n-graphs.** Let $G_n \cong S_{n+1} \times \mathbb{Z}/2$ be the subgroup of $\mathrm{Out}(F_n)$ which is realized as the full automorphism group of the n-cage C_{n+1} with the first n edges labelled by the generators $a_1, \dots, a_n$ of F_n. The $\mathbb{Z}/2$ factor of G_n is generated by $\Delta = \varepsilon_1\varepsilon_2 \dots \varepsilon_n$, which interchanges the two vertices of C_{n+1}, leaving each unoriented edge invariant.

Let Y be a minimal admissible G_n-graph of genus at most $m \leq 2n = 2(n+1) - 2$. Assume $n > 7$. Since G_n contains $B = A_{n+1}$, Proposition 6.7 tells us that Y_B is isomorphic to either C_{n+1} or R_{n+1} if $m < 2n$, with the additional possibilities $Y = Y_B = C_{n+1} \vee C_{n+1}$ and $Y = K(3, n+1)$ if $m = 2n$. In fact, this last possibility does not occur, because in any faithful action of G_n on $K(3, n+1)$, the central $\mathbb{Z}/2$ leaves the set of 3 cone points invariant and hence fixes one of them, so the star of this fixed point is a G_n-invariant forest, which shows that the G_n-action is not minimal.

Proposition 6.12. *If Y is a G_n-minimal admissible graph and $n > 7$, then the subgraph Y_B is invariant under all of G_n.*

If the central element $\Delta \in G_n$ acts on Y_B non-trivially, then it flips all edges (if $Y_B = R_{n+1}$), interchanges the two vertices (if $Y_B = C_{n+1}$) or interchanges the two copies of C_{n+1} (if $Y = C_{n+1} \vee C_{n+1}$) without permuting the edges. The odd permutations of S_{n+1} act on Y_B by permuting the edges in the standard way or else each acts by the permutation composed with the action of Δ.

Proof. Since $B = A_{n+1}$ is normal in G_n, the action of G preserves the fixed subgraph Y^B of B, and hence also preserves the complementary subgraph Y_B. In particular, Δ acts on Y_B by an automorphism that commutes with the B-action. If $Y_B = R_{n+1}$, the only non-trivial graph automorphism which commutes with the B-action is the one which flips each petal of the rose without permuting the petals. If $Y_B = C_{n+1}$, the only non-trivial graph automorphism which commutes with the B-action is the one which interchanges the vertices of C_{n+1} without permuting the edges. If $Y = Y_B = C_{n+1} \vee C_{n+1}$ the only non-trivial graph automorphism which commutes with the (diagonal) B-action is the one which interchanges the two copies of C_{n+1}.

The statement about the action of odd permutations on Y_B follows from Lemma 6.8. □

7. Proof of Theorem C

We are now in a position to prove that for $n > 8$, any homomorphism from $\mathrm{Out}(F_n)$ to $\mathrm{Out}(F_m)$ has image of order at most two for $n < m \leq 2n - 2$. If n is even, this can be improved to $m \leq 2n$. We first reduce our problem using the following:

Proposition 7.1. *If $n \geq 3$ and $m < 2^{n-1} - 1$, then any homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ which is not injective on both W_n and on G_n has image of order at most two.*

Proof. If a homomorphism is not injective on G_n then the kernel either consists of the central involution Δ or contains A_{n+1}. In either case, it follows that the homomorphism is not injective on W_n. In ([6], Proposition C) we proved that any homomorphism from $\mathrm{Out}(F_n)$ that is not injective on W_n must factor through

$\mathrm{Out}(F_n) \to \mathrm{PGL}(n,\mathbb{Z})$, and by [3] all homomorphisms $\mathrm{PGL}(n,\mathbb{Z}) \to \mathrm{Out}(F_m)$ have finite image.

The kernel of the natural map $\mathrm{Out}(F_m) \to \mathrm{GL}(m,\mathbb{Z})$ is torsion-free, so the image of $\mathrm{PGL}(n,\mathbb{Z})$ in $\mathrm{Out}(F_m)$ maps injectively to $\mathrm{GL}(m,\mathbb{Z})$. A non-trivial finite image of $\mathrm{PGL}(n,\mathbb{Z})$ is either just $\mathbb{Z}/2$ (and the map factors through the determinant) or else it contains a copy of $\mathrm{PSL}(n,\mathbb{Z}/p)$ for some prime p. The minimal degree of a complex representation of $\mathrm{PSL}(n,\mathbb{Z}/p)$ occurs when $p=2$ and is equal to $m = 2^{n-1}-1$; see Lanazuri and Seitz [16]. □

Remark 7.2. *For our purposes, it is sufficient to have the above result in the range $8 < n < m \leq 2n$ and one can prove this without recourse to* [16]. *Indeed, since an elementary p-group in $\mathrm{GL}(m,\mathbb{Z})$ can be diagonalised in $\mathrm{GL}(m,\mathbb{C})$, it has rank at most*[3] *m, whereas $\mathrm{PSL}(n,\mathbb{Z}/p)$ contains an elementary p-group of rank $\lfloor n/2 \rfloor^2$, namely the largest unipotent subgroup that one can fit in a square block above the diagonal.*

We need the following observation:

Lemma 7.3. *Suppose $m \geq n \geq 3$ and let $\phi\colon \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ be any homomorphism. If $\det(\alpha) = \det(\beta)$, then $\det(\phi(\alpha)) = \det(\phi(\beta))$*

Proof. For $n \geq 3$, the abelianization of $\mathrm{Out}(F_n)$ is $\mathbb{Z}/2$, given by the determinant map. □

For the remainder of this section we suppose that $8 < n < m$ and that we have a homomorphism

$$\phi\colon \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

which is injective on G_n and on W_n. We fix a minimal admissible graph X of genus m realizing $\phi(W_n)$ and a minimal admissible graph Y of genus m realizing $\phi(G_n)$.

Note that the intersection $G_n \cap W_n$ is isomorphic to $S_n \times \mathbb{Z}/2$, where S_n permutes the generators of F_n and $\mathbb{Z}/2$ is generated by the automorphism Δ which inverts all of the generators. The image of each element in this intersection is realized both as an automorphism of X and as an automorphism of Y.

Proposition 7.4. *For $m \leq 2n$, the only possibilities for X and Y are those with $X_A = R_n$ and $Y_B = C_{n+1}$ or R_{n+1}.*

Proof. We first consider the induced action of $\phi(\sigma)$ on $H_1(F_m)$, where $\sigma = (12)(34) \in A_n$. We calculate the dimension of the (-1)-eigenspace $V_{-1}(\sigma)$ using the action of σ on both X and Y. If $Y = R_{n+1} \vee Y^B$ or $C_{n+1} \cup Y^B$, this calculation gives

[3] using Smith theory one can improve this to $\lfloor m/2 \rfloor$ if p is odd

$\dim(V_{-1}(\sigma)) = 2$, and if $Y = C_{n+1} \vee C_{n+1}$ we have $\dim(V_{-1}(\sigma)) = 4$. This covers all possibilities for Y by Proposition 6.12.

Proposition 6.10 lists all possibilities for W_n-graphs, for $m \leq 2n$. Using these, we calculate $\dim(V_{-1}(\sigma)) = 4$ if $X_A = C_{2n}, R_n \vee C_n, R_n \amalg CL_n, R_{2n}$ or RL_n, and $\dim(V_{-1}(\sigma)) = 2$ if $X = R_n \vee X^A$.

Therefore to prove the proposition we need only eliminate the possibilities that $Y = C_{n+1} \vee C_{n+1}$ (which has rank $2n$) and

(1) $X = C_{2n} \vee S^1$
(2) $X = R_n \vee C_n \vee S^1$
(3) $X = R_n \amalg C_n \cup S^1$
(4) $X = CL_n \vee S^1$
(5) $X = RL_n$, and
(6) $X = R_{2n}$.

We eliminate these possibilities by considering the action of Δ. We know that Δ acts on $Y = C_{n+1} \vee C_{n+1}$ by interchanging the two copies of C_{n+1} and commuting with the A_{n+1}-action, so in an appropriate basis for $H_1(F_{2n})$ the matrix of the induced action on $H_1(F_{2n})$ is

$$D_Y = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

This has (-1)-eigenspace $V_{-1}(\Delta)$ of dimension exactly n, so this must also be true for the action of Δ on X. If $X = C_{2n} \vee S^1$ (case (1) above), Δ must therefore interchange the two copies of C_n in C_{2n} and flip the extra S^1, so that the matrix of Δ is

$$D_X = \pm \begin{pmatrix} 0 & I_{n-1} & 0 & 0 \\ I_{n-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In cases (2)-(5) the fact that $V_{-1}(\Delta)$ has dimension exactly n implies that Δ acts by flipping exactly n loops in some A_n-orbit, and therefore in an appropriate basis the matrix for Δ must be

$$D_X = \pm \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

In all of these cases, D_X and D_Y are not conjugate in $\mathrm{GL}(2n, \mathbb{Z})$. To see this, note that the sublattice of $\mathbb{Z}^n$ spanned by all eigenvectors has different covolume for D_Y and D_X.

Finally, if $X = R_{2n} = R_n \vee R_n'$ then the same reasoning shows that the action of Δ must exchange the two copies of R_n. The transposition (12) must act either by sending $e_1 \to e_2$ and $e_1' \to e_2'$ or by sending $e_1 \to e_2'$ and $e_1' \to e_2$. In either case (12) acts by a transformation with determinant $+1$. If n is odd, then Δ acts with determinant -1, contradicting Lemma 7.3. If n is even, then ε_i must act by exchanging the i-th edge e_i of R_n with the corresponding edge e_i' of R_n' (possibly flipping them both) and fixing (or flipping) all e_j for $j \neq i$; in any case the induced map on $H_1(F_{2n})$ has determinant -1, again contradicting Lemma 7.3. □

Proposition 7.5. *For $n < m \leq 2n$, the action of Δ on X^A must be non-trivial.*

Proof. Suppose that the action of Δ on X^A is trivial. Then the action of Δ on $R_n \subset X$ cannot be trivial, and must commute with the action of A_n. The only possibility is that Δ acts by inverting all of the petals of R_n, so that the dimension of the (-1)-eigenspace $V_{-1}(\Delta)$ is exactly equal to n.

We now calculate the dimension of $V_{-1}(\Delta)$ using Y. If Δ acts trivially on Y_B, then the dimension of $V_{-1}(\Delta)$ is at most the rank of Y^B, which is strictly less than n. If Δ acts non-trivially on Y_B it must invert all edges, since it commutes with the action of $B = A_{n+1}$ on Y_B. If $Y_B = R_{n+1}$, then it is clear that $V_{-1}(\Delta)$ has dimension at least $n + 1$. This is also true if $Y_B = C_{n+1}$, since Δ interchanges the vertices of the cage, so must also act non-trivially on Y^B. Thus the computation of $\dim(V_{-1}(\Delta))$ made with Y is inconsistent with the computation made with X. □

Corollary 7.6. *If $n > 8$ is even and $n < m \leq 2n$ then every homomorphism $\phi\colon \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has image of order at most two.*

Proof. If the image of ϕ has order larger than 2, then by Proposition 7.1 ϕ is injective on W_n and G_n, and we have minimal admissible X and Y realizing $\phi(W_n)$ and $\phi(G_n)$ as above. By Proposition 6.10, each ε_i acts by the same involution of X^A; since n is even, this means that $\Delta = \prod \varepsilon_i$ acts trivially on X^A, contradicting Proposition 7.5. □

Corollary 7.7. *If $n > 8$ is odd, then every homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_{n+1})$ has image of order at most two.*

Proof. In this case $X = R_n \vee S^1$, and by Proposition 7.5 we may assume Δ acts non-trivially on S^1. Thus Δ acts as $-I$ on $H_1(X)$, which has determinant $+1$.

According to Proposition 7.4 the possibilities for Y are $Y = R_{n+1}$ or $Y = C_{n+1} \cup e$. However, the second cannot occur, since the only way to symmetrically add an edge to C_{n+1} is to embed C_{n+1} in C_{n+2}; but then e is an invariant forest, so this graph is not minimal for the G_n-action. Therefore $Y = R_{n+1}$, and by Proposition 6.12 the transposition (12) acts on Y with determinant -1 (since $n+1$ is even), contradicting Lemma 7.3. □

If n is odd and $m > n + 1$ the above arguments do not work so we employ a different approach.

Proposition 7.8. *If n is odd, $n > 8$ and $n < m \leq 2n - 2$ then the image of any homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has order at most two.*

Proof. A theorem of Potapchik and Rapinchuk ([18], Theorem 3.1) implies that for $m \leq 2n-2$ any representation $\mathrm{Out}(F_n) \to \mathrm{GL}(m,\mathbb{Z})$ factors through the standard representation $\mathrm{Out}(F_n) \to \mathrm{GL}(n,\mathbb{Z})$. We apply this fact to the map $\mathrm{Out}(F_n) \to \mathrm{GL}(m,\mathbb{Z})$ obtained by composing an arbitrary homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ with the natural map $\mathrm{Out}(F_m) \to \mathrm{GL}(m,\mathbb{Z})$.

Either the image of $\mathrm{SL}(n,\mathbb{Z})$ in $\mathrm{GL}(m,\mathbb{Z})$ under the induced map is finite or else, by super-rigidity, it extends to a representation $\mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(m,\mathbb{R}) \subset \mathrm{GL}(m,\mathbb{C})$. If the image is finite then as in the proof of Proposition 7.1 it is at most $\mathbb{Z}/2$. In particular, the map from W_n to $\mathrm{Out}(F_m)$ cannot be injective because the kernel of $\mathrm{Out}(F_m) \to \mathrm{GL}(m,\mathbb{Z})$ is torsion-free. It then follows from the statement of Proposition 7.1 that the image of our original homomorphism $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ has order at most 2.

Suppose now that the image of $\mathrm{SL}(n,\mathbb{Z})$ is infinite and consider its extension $\rho : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(m,\mathbb{R}) \subset \mathrm{GL}(m,\mathbb{C})$. Complete reducibility for $\mathrm{SL}(n,\mathbb{R})$ implies that ρ is a sum of irreducible representations (see [8] page 130). A calculation with the hook formula shows that the only irreducible representations below dimension $2n$ are the trivial one, the standard n-dimensional representation and its contragradient. Since we are assuming that the image of $\mathrm{SL}(n,\mathbb{Z})$ is infinite, we must have exactly one copy of the standard representation or its contragradient.

Let $\tau \in S_n \subset \mathrm{Out}(F_n)$ be a transposition. Since n is odd, $\tau\Delta$ has determinant 1. It follows from the above that the -1 eigenspace of $\tau\Delta$ in $H_1(F_m,\mathbb{C})$ has the same dimension as in the standard representation of $\mathrm{SL}(n,\mathbb{Z})$, that is $\dim(V_{-1}(\tau\Delta)) = n-1$.

We prove that this last equality is impossible by considering the action of τ on the homology of the graphs X and Y. Proposition 7.4 limits the possibilities for X and Y, and Propositions 6.10 and 7.5 describe the action in each case.

If τ acts by both permuting the edges of $R_n = X_A \subset X$ and flipping them, then

$$\dim(V_{-1}(\tau\Delta)) \leq 1 + rank(X^A) < 1 + (n-2) = n-1.$$

Since this is impossible, τ must act without flipping the edges of R_n. But then the same calculation shows that $\dim(V_{-1}(\tau)) < n-1$. It follows that τ must also act without flipping the edges of Y_B, since otherwise the dimensions of the (-1)-eigenspaces of τ, as calculated with X and Y, would not agree. But then using Y to compute $V_{-1}(\tau\Delta)$ gives $\dim(V_{-1}(\tau\Delta)) \geq n$, a contradiction. $\square$

This completes the proof of Theorem C.

In the proof of Corollary 7.6 we invoked Proposition 7.1 to promote the fact that $\Delta \in X^A$ was acting trivially on X to the fact that the image of $\mathrm{Out}(F_n)$ was finite. Up to that point, we had not used the ambient structure of $\mathrm{Out}(F_n)$ and thus our arguments prove the following more abstract theorem.

Theorem 7.9. *Let $W_n = (\mathbb{Z}/2)^n \rtimes S_n$, let $G_n = S_{n+1} \times \mathbb{Z}/2$, and consider the amalgamated free product*

$$P_n = W_n *_{(S_n \times \mathbb{Z}/2)} G_n$$

where the amalgamation identifies the visible $S_n < W_n$ with $S_n < S_{n+1}$ and identifies the $\mathbb{Z}/2$ factor of G_2 with the centre of W_n (which therefore is central in P_n).

If $n > 8$ is odd and $n < m \leq 2n$, then the centre of P_n lies in the kernel of every homomorphism $P_n \to \mathrm{Out}(F_m)$.

8. Appendix: Characteristic covers

The method that we used in the first part of this paper to construct monomorphisms $\mathrm{Out}(F_n) \hookrightarrow \mathrm{Out}(F_m)$ was this: we took a characteristic subgroup of finite index in F_n that contains the commutator subgroup and split the short exact sequence $1 \to F_n/N \to \mathrm{Aut}(F_n)/N \to \mathrm{Out}(F_n) \to 1$. But in truth it was not this sequence *per se* that we split, but rather an isomorphic sequence involving groups of homotopy equivalences of graphs. The purpose of this appendix is to prove the following theorem, which explains why these two splitting problems are equivalent.

Notation. Let X be a connected CW complex X with basepoint x_0, let $\mathrm{he}(X)$ be the set of homotopy equivalences $X \to X$, with the compact-open topology, and let $\mathrm{he}_0(X) \subset \mathrm{he}(X)$ be those that fix x_0. Define

$$\mathrm{HE}(X) = \pi_0(\mathrm{he}(X)) \quad \text{and} \quad \mathrm{HE}_\bullet(X) = \pi_0(\mathrm{he}_0(X)).$$

Thus $\mathrm{HE}(X)$ is the group of homotopy classes of self-homotopy equivalences of X, and $\mathrm{HE}_\bullet(X)$ is the group homotopy classes rel x_0 of basepoint-preserving self-homotopy equivalences of X. Let $\iota : \mathrm{HE}_\bullet(X) \to \mathrm{HE}(X)$ be the map induced by $\mathrm{he}_0(X) \hookrightarrow \mathrm{he}(X)$.

Given a connected covering space $p\colon \widehat{X} \to X$, we define $\mathrm{fhe}(\widehat{X})$ to be the set of self-homotopy equivalences $\hat{h} : \widehat{X} \to \widehat{X}$ that are fibre-preserving, i.e. if $p(\widehat{x}) = p(\widehat{y})$, then $p\widehat{h}(\widehat{x}) = p\widehat{h}(\widehat{y})$. Consider the group

$$\mathrm{FHE}(\widehat{X}) = \pi_0(\mathrm{fhe}(\widehat{X})).$$

Theorem 8.1. *Let X be a connected CW complex with basepoint $x_0 \in X$. Let $N < \pi = \pi_1(X, x_0)$ be a characteristic subgroup and suppose that the centralizer $Z_\pi(N)$ is trivial. Let $p\colon \widehat{X} \to X$ be the covering corresponding to N and fix $\widehat{x}_0 \in p^{-1}(x_0)$.*

Then there is a commutative diagram of groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow \delta & & & & \\
1 & \to & \pi & \xrightarrow{\delta} & \mathrm{HE}_\bullet(X) & \xrightarrow{\iota} & \mathrm{HE}(X) & \to & 1 \\
 & & \downarrow & & \downarrow \lambda & & \| & & \\
1 & \to & \mathrm{Deck} & \to & \mathrm{FHE}(\widehat{X}) & \xrightarrow{p_*} & \mathrm{HE}(X) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $\mathrm{Deck} \cong \pi/N$ *is the group of deck transformations of* $p : \widehat{X} \to X$ *and where* $\lambda([h])$ *is defined to be the class of the lift of* h *that fixes* $\widehat{x}_0$.

The proof of the above theorem involves little more than the homotopy extension property, the homotopy lifting property, and some thought about the role of base-points. But we found it hard to track down precise references for the relevant facts (although much of what we need is in [19]). We therefore provide a complete proof. We require three lemmas, the first of which involves the map $\delta : \pi_1(X, x_0) \to \mathrm{HE}_\bullet(X)$ that is defined as follows.

Let $I = [0,1]$. Given any continuous map $h : X \to X$ and any path $\sigma : I \to X$ from x_0 to $h(x_0)$, we apply the homotopy extension principle to obtain a map $H : X \times [0,1] \to X$ with $H|_{X\times\{0\}} = h$ and $H(x_0, t) = \sigma(1-t)$. (Here σ is viewed as a homotopy of a point.) Define $d(h,\sigma) : X \to X$ to be the restriction of H to $X \times \{1\}$; it is thought of as the map obtained by "dragging $h(x_0)$ back to x_0 along σ". Note that h is freely homotopic to $d(h,\sigma)$. Note too that a further application of homotopy extension shows that a different choice of homotopy H' would lead to a map $d'(h,\sigma)$ that is homotopic to $d(h,\sigma)$ rel x_0.

If $\sigma \simeq \sigma'$ rel endpoints, then by a further application of homotopy extension we see that $d(h,\sigma) \simeq d(h,\sigma')$. In particular, if σ is a loop based at x_0, then the homotopy class of $d(\mathrm{id}_X, \sigma)$ depends only on $[\sigma] \in \pi_1(X, x_0)$. Thus we obtain a well-defined map $\delta : \pi_1(X, x_0) \to \mathrm{HE}_\bullet(X)$ by defining $\delta([\sigma]) := [d(\mathrm{id}_X, \sigma)]$. And because we dragged backwards along σ in the definition of $d(h,\sigma)$, this is a homomorphism.

Lemma 8.2. *Let* $\pi = \pi_1(X, x_0)$ *and suppose the center* $Z(\pi)$ *is trivial. Then the following sequence is exact:*

$$1 \to \pi \xrightarrow{\delta} \mathrm{HE}_\bullet(X) \xrightarrow{\iota} \mathrm{HE}(X) \to 1.$$

Proof. Given $h \in \mathrm{HE}(X)$, we choose a path σ from x_0 to $h(x_0)$. By construction, $d(h,\sigma)$ fixes x_0 and is freely homotopic to h. Thus ι is surjective.

To see that $\mathrm{im}(\delta) \subset \ker(\iota)$, note that the homotopy used to define $\delta([\sigma])$ gives a (free) homotopy from $\delta([\sigma])$ to id_X. To establish the opposite inclusion, we fix $h \in \ker(\iota)$ and choose a homotopy G of h to the identity. Let $\sigma(t) = G(x_0, 1-t)$. Then, by definition, $\delta([\sigma]) = h$.

To see that δ is injective, fix a loop γ and suppose that $\delta([\gamma])$ is trivial, i.e. that there is a basepoint preserving homotopy from $d(\mathrm{id}_X, \gamma)$ to id_X. By combining this homotopy with the homotopy H used to define $d(\mathrm{id}_X, \gamma)$, we get a homotopy $F : X \times [-1, 1] \to X$ from id_X to itself with $F|_{\{x_0\}\times[0,1]} = \gamma$ and $F|_{\{x_0\}\times[-1,0]}$ a constant path at x_0; let $\gamma' : [-1, 1] \to X$ be this reparamerisation of γ. Given any loop τ based at x_0, the map $I \times [-1, 1] \to X$ defined by $(s, t) \mapsto F(\tau(s), t)$ restricts on the top and bottom of the square to τ and on the two sides to γ'. Thus $[\tau]$ and $[\gamma'] = [\gamma]$ commute in $\pi_1(X, x_0)$. Since τ is arbitrary, we conclude that $[\gamma]$ is in the centre of π, which is trivial by hypothesis. □

Now let $p\colon \widehat{X} \to X$ be a connected normal covering space, fix $\widehat{x}_0$ with $p(\widehat{x}_0) = x_0$, and let $N = p_*\pi_1(\widehat{X}, \widehat{x}_0)$. If N is a characteristic subgroup of $\pi_1(X, x_0)$, then we say that the covering is *characteristic.*

Lemma 8.3. *Let $p\colon \widehat{X} \to X$ be a characteristic covering space, with group of deck transformations* $\mathrm{Deck} = \pi_1(X, x_0)/N$*, and assume that the centralizer of N in $\pi_1(X, x_0)$ is trivial. Then the following sequence is exact:*

$$1 \to \mathrm{Deck} \to \mathrm{FHE}(\widehat{X}) \overset{p_*}{\to} \mathrm{HE}(X) \to 1.$$

Proof. Every $h \in \mathrm{he}(X)$ lifts to a self-homotopy equivalence $\widehat{h} \in \mathrm{fhe}(\widehat{X})$, because N is characteristic and therefore $h_*(N) = N$. Thus $p_* : \mathrm{FHE}(\widehat{X}) \to \mathrm{HE}(X)$ is surjective.

Let $\pi = \pi_1(X, x_0)$. The map $\pi \to \mathrm{Aut}(N)$ defined by conjugation is injective, because we have assumed that the centralizer of N in π is trivial. It follows that the induced map $\mathrm{ad} : \pi/N \to \mathrm{Out}(N)$ is also injective. But ad is the natural map from $\mathrm{Deck} = \pi/N$ to $\mathrm{Out}(\pi_1(\widehat{X}, \widehat{x}_0)) = \mathrm{Out}(N)$ (where the identifications are given by path-lifting). And $\mathrm{Deck} \to \mathrm{Out}(\pi_1(\widehat{X}, \widehat{x}_0))$ extends to $\mathrm{HE}(\widehat{X}) \to \mathrm{Out}(N)$. Therefore $\mathrm{Deck} \to \mathrm{FHE}(\widehat{X})$ is injective.

It is clear that $\mathrm{im}(\delta)$ is contained in $\ker(p_*)$; this just says that deck transformations project to the identity. Conversely, if $p_*\widehat{h}$ is homotopic to the identity, the homotopy can be lifted to a fiber-preserving homotopy of $\widehat{h}$ which covers the identity; but the only maps which cover the identity are deck transformations. □

We are studying the normal covering $p : \widehat{X} \to X$. Path-lifting at the basepoint $\widehat{x}_0 \in p^{-1}(x_0)$ gives the standard identification $\mathrm{Deck} \cong \pi_1(X, x_0)/N$; we write $\mathrm{deck}(\gamma)$ for the deck transformation determined by γ, and we write $[\mathrm{deck}(\gamma)]$ for its image in

$\mathrm{FHE}(\widehat{X})$. The homomorphism $\lambda : \mathrm{HE}_\bullet(X) \to \mathrm{FHE}(\widehat{X})$ was defined in the statement of Theorem 8.1: it sends $[h]$ to the fibre-preserving homotopy class of the lift of h that fixes $\widehat{x}_0 \in \widehat{X}$. The homomorphism $\delta : \pi_1(X, x_0) \to \mathrm{HE}_\bullet(X)$ was defined prior to Lemma 8.2.

Lemma 8.4. *For all $\gamma \in \pi_1(X, x_0)$ we have $\lambda(\delta(\gamma)) = [\mathrm{deck}(\gamma)]$.*

Proof. Fix a loop $\sigma[0,1] \to X$ with $[\sigma] = \gamma$ in $\pi_1(X, x_0)$. The construction of $\delta(\gamma)$ involves a homotopy $H : X \times [0,1] \to X$ with $H(x_0, t) = \sigma(1-t)$ and $H|_{X\times\{0\}} = \mathrm{id}_X$ while $h_1 := H|_{X\times\{1\}} \in \delta(\gamma)$. By definition, $\lambda(\delta(\gamma))$ is the fibre-preserving homotopy class of the lift $\widehat{h}_1$ of h_1 that fixes $\widehat{x}_0$. Now H lifts to a fibre-preserving homotopy $\widehat{H} : \widehat{X} \times [0,1] \to \widehat{X}$ with $\widehat{H}_{\widehat{X}\times\{1\}} = \widehat{h}_1$ and $\widehat{H}(\widehat{x}_0, t) = \widehat{\sigma}(1-t)$, where $\widehat{\sigma} : [0,1] \to \widehat{X}$ is the lift of σ with $\widehat{\sigma}(0) = \widehat{x}_0$. Thus $\widehat{H}|_{\widehat{X}\times\{0\}}$ is the lift of id_X that sends $\widehat{x}_0$ to $\widehat{\sigma}(1)$. By definition, this lift of id_X is $\mathrm{deck}([\sigma]) = \mathrm{deck}(\gamma)$. Therefore $\widehat{H}$ is a fibre-preserving homotopy from $\widehat{h}_1$ to $\mathrm{deck}(\gamma)$, showing that $\lambda(\delta(\gamma)) = [\mathrm{deck}(\gamma)]$. □

Proof of Theorem 8.1 Lemmas 8.2 and 8.3 tell us that the rows of the diagram are exact. It is clear from the definitions that $p_*\lambda = \iota$, and Lemma 8.4 tells us that the square beneath $\pi \xrightarrow{\delta} \mathrm{HE}_\bullet(X)$ commutes. Thus the diagram is commutative. With commutativity in hand, an elementary diagram chase proves that second column is exact. □

Corollary 8.5. *Let X be a $K(\pi, 1)$ space and let $p\colon \widehat{X} \to X$ be a covering space with $N = p_*\pi_1(\widehat{X})$ characteristic in π. If $Z_\pi(N)$ is trivial, then the following diagram of groups is commutative and the vertical maps are isomorphisms:*

$$\begin{array}{ccccccc}
1 & \to & \pi/N & \to & \mathrm{Aut}(\pi)/N & \to & \mathrm{Out}(\pi) \to 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathrm{Deck} & \to & \mathrm{FHE}(\widehat{X}) & \to & \mathrm{HE}(X) \to 1
\end{array}$$

where $\pi/N \to \mathrm{Aut}(\pi)/N$ is the map induced by the action of π on itself by inner automorphisms.

Proof. When X is a $K(\pi, 1)$, the natural maps $\mathrm{HE}_\bullet(X) \to \mathrm{Aut}(\pi)$ and $\mathrm{HE}(X) \to \mathrm{Out}(\pi)$ are isomorphisms, which we use to identify these groups. By definition $\delta(\gamma)$ is the class of the homotopy equivalence that drags the basepoint of X backwards around the loop γ, and therefore the map that it induces on $\pi = \pi_1(X, x_0)$ is the inner automorphism by γ. Thus, with the above identifications, by factoring out N from the top row of the diagram in Theorem 8.1 we obtain the top row of the diagram displayed in the statement of the corollary. □

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