THE ISOMORPHISM PROBLEM FOR PROFINITE COMPLETIONS OF RESIDUALLY FINITE GROUPS

MARTIN R. BRIDSON AND HENRY WILTON

ABSTRACT. We consider pairs of finitely presented, residually finite groups $u : P \hookrightarrow \Gamma$. We prove that there is no algorithm that, given an arbitrary such pair, can determine whether or not the associated map of profinite completions $\hat{u} : \hat{P} \to \hat{\Gamma}$ is an isomorphism. Nor do there exist algorithms that can decide whether $\hat{u}$ is surjective, or whether $\hat{P}$ is isomorphic to $\hat{\Gamma}$.

For Pierre de la Harpe on his 70th birthday, with respect and affection

1. INTRODUCTION

The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ; it is denoted $\hat{\Gamma}$. The natural map $\Gamma \to \hat{\Gamma}$ is injective if and only if Γ is residually finite. A *Grothendieck pair* is a monomorphism $u : P \hookrightarrow \Gamma$ of finitely presented, residually finite groups such that $\hat{u} : \hat{P} \to \hat{\Gamma}$ is an isomorphism but P is not isomorphic to Γ. The existence of Grothendieck pairs was established in [5], raising the following recognition problem: *is there an algorithm that, given a monomorphism of finitely presented, residually finite groups $u : P \hookrightarrow \Gamma$, can determine whether or not $\hat{u}$ is an isomorphism?*

We shall resolve this question by proving the following theorem.

Theorem A. *There are recursive sequences of finite presentations for residually finite groups $P_n = \langle A_n \mid R_n \rangle$ and $\Gamma_n = \langle B_n \mid S_n \rangle$ together with explicit monomorphisms $u_n : P_n \hookrightarrow \Gamma_n$, such that:*

(1) *$\hat{P}_n \cong \hat{\Gamma}_n$ if and only if the induced map of profinite completions $\hat{u}_n$ is an isomorphism;*

(2) *$\hat{u}_n$ is an isomorphism if and only if $\hat{u}_n$ is surjective; and*

(3) *the set $\{n \in \mathbb{N} \mid \hat{P}_n \not\cong \hat{\Gamma}_n\}$ is recursively enumerable but not recursive.*

The groups Γ_n that we construct are of the form $H_n \times H_n$, where H_n is a residually finite hyperbolic group with a 2-dimensional classifying space.

Theorem A is an outgrowth of our recent work on the profinite triviality problem for finitely presented groups [9]: we shall prove it by applying the main construction of [5] to a sequence of finitely presented groups satisfying the following strengthening of [9, Theorem A].

Theorem B. *There is a recursive sequence of finite combinatorial[1] CW-complexes K_n so that*

(1) *each K_n is aspherical;*
(2) $H_1(K_n, \mathbb{Z}) \cong H_2(K_n, \mathbb{Z}) \cong 0$ *for all $n \in \mathbb{N}$; and*
(3) *the set of natural numbers*

$$\{n \in \mathbb{N} \mid \widehat{\pi_1 K_n} \not\cong 1\}$$

is recursively enumerable but not recursive.

In [2, Theorem B] the first author proved a less satisfactory version of Theorem A. He too constructed a recursive sequence of pairs $\iota_n : P_n \hookrightarrow \Gamma_n$, with P_n and Γ_n residually finite, such that there is no algorithm that can determine whether or not $\hat{\iota}_n$ is an isomorphism, nor whether $\hat{P}_n$ is isomorphic to $\hat{\Gamma}_n$. But in his construction, although the Γ_n are given by explicit finite presentations, the subgroups $P_n \hookrightarrow \Gamma$ are given by specifying a finite generating set $\Sigma_n \subset \Gamma$ (with a guarantee that $P_n = \langle \Sigma_n \rangle$ is finitely presentable). In [8] we explored in detail the question of when such data is sufficient to allow the algorithmic construction of a finite presentation for P_n, and in this situation it is not. Indeed, the lack of an algorithm to present P_n is an essential feature of the proof of [2, Theorem B]: what is actually proved is that one cannot decide if $\hat{\iota}_n$ is *injective* (note the contrast with our Theorem A) because there is no algorithm that can determine if the map $H_1(P, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ induced by ι_n is injective; if one had a finite presentation for both groups, it would be easy to determine if $H_1(P, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ were injective.

Theorem A deals with maps $u : P \to \Gamma$ that are assumed to be *monomorphisms* of *residually finite* groups. If one does not require the groups to be residually finite, then Theorem A of [9] implies immediately that there does not exist an algorithm to recognise if $\hat{u} : \hat{P} \to \hat{\Gamma}$ is an isomorphism: it is enough to consider the case where P is the trivial subgroup. If one also drops the requirement that $u : P \to \Gamma$ is injective, then the result already follows from the work of Slobodskoi that is invoked in [9]: Slobodskoi [16] exhibited a finitely presented group G for which there is no algorithm that can determine which words w have trivial image in $\hat{G}$; and since finitely generated profinite groups are Hopfian, w has trivial image in $\hat{G}$ if and only if the quotient map $G \to G/\langle\!\langle w \rangle\!\rangle$ induces an isomorphism of profinite completions.

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[1]A *combinatorial* map between complexes is one that maps open cells homeomorphically to open cells, and a CW complex is *combinatorial* if the attaching map of each closed k-cell is a combinatorial map from a polyhedral subdivision of the $(k-1)$-sphere.

2. Reducing Theorem A to Theorem B

The reduction of Theorem A to Theorem B follows a template for constructing Grothendieck pairs that was devised by Bridson and Grunewald in [5] and first applied to algorithmic questions in [3]. The key ingredients are adaptations of the Rips construction [14] and the 1-2-3 Theorem [1]. In our case, these will be applied to the groups $\pi_1 K_n$ supplied by Theorem B.

2.1. **Rips and Algorithmic 1-2-3.** The following adaptation of the Rips construction is due to Wise [17].

Theorem 2.1. [17] *There is an algorithm that takes as input a finite group presentation $\mathcal{Q} \equiv \langle \underline{\mathbf{x}} \mid \underline{\mathbf{y}} \rangle$ and will output a finite presentation $\mathcal{H} \equiv \langle \underline{\mathbf{x}}, a, b, c \mid \underline{\mathbf{z}} \rangle$ for a residually finite hyperbolic group H, so that there is a short exact sequence*

$$1 \to N \to H \xrightarrow{p} Q \to 1,$$

where $N = \langle a, b, c \rangle$, the group Q is presented by $\mathcal{Q}$, and $p : H \to Q$ is the map implicit in the labelling of generators. Moreover, $|\underline{\mathbf{z}}| = |\underline{\mathbf{y}}| + 6|\underline{\mathbf{x}}|$.

The short exact sequences obtained by applying this theorem with $Q = \pi_1 K_n$ provide input for the following algorithmic version of the 1-2-3 Theorem. Recall that the *fibre product* associated to a short exact sequence

$$1 \to N \to H \xrightarrow{p} Q \to 1$$

is the pre-image of the diagonal subgroup under the product map $p \times p : H \times H \to Q \times Q$.

Theorem 2.2. [6] *There exists an algorithm that, given a short exact sequence of groups $1 \to N \to H \to Q \to 1$, a finite generating set for N, a finite presentation of H, and a finite combinatorial CW-complex that is the 3-skeleton of a $K(Q, 1)$, will construct a finite presentation for the associated fibre product $P < H \times H$, and express the generators of P as words in the generators of $H \times H$.*

The theorem proved in [6] is actually somewhat stronger than the one we have stated and is cast in different language; we shall explain the translation. The algorithm in [6] requires as input a finite presentation for H, a finite generating set for N, a finite presentation $\mathcal{Q}$ for Q, and a set of elements of $\pi_2 \mathcal{Q}_n$ that generate it as a $\mathbb{Z}Q_n$-module. (Our results in [8] show that this last piece of data is essential.)

In [6], the generators of $\pi_2 \mathcal{Q}$ are given as identity sequences, which can be thought of as formal products $\Theta \equiv \Pi_{i=1}^{N} x_i r_i x^{-1}$, where the x_i are words in the generators of $\mathcal{Q}$, the $r_i^{\pm 1}$ are defining relations in $\mathcal{Q}$, and Θ is freely equal to the empty word. Given a combinatorial map of a 2-sphere to the standard 2-complex $K(\mathcal{Q})$ of the presentation $\mathcal{Q}$,

it is easy to associate such a product to it: this is the first, simple step in the proof of van Kampen's Lemma where, given a diagram with boundary word w, one cuts it into a "lollipop diagram", the boundary of which is labelled by a product of conjugates of defining relations that is freely equal to w; we are discussing the case $w = \varnothing$ (see [11] p.151 or [4] p.49, for example). Thus, instead of giving the generators of $\pi_2 \mathcal{Q}$ as identity sequences, it suffices to give them as combinatorial maps $\mathbb{S}^2 \to K(\mathcal{Q})$.

If one attaches to $K(\mathcal{Q})$ one 3-cell for each element in a set of maps $\mathbb{S}^2 \to K(\mathcal{Q})$, then the resulting 3-complex will be the 3-skeleton of a $K(Q, 1)$ if and only if the homotopy classes of these maps generate $\pi_2 K(\mathcal{Q})$ as a $\mathbb{Z}Q$-module. Changing perspective, if one is given a finite combinatorial model for the 3-skeleton of a $K(Q, 1)$, one can shrink a maximal tree in the 1-skeleton to make the 2-skeleton a presentation 2-complex, and the set of attaching maps of the 3-cells will serve as a set of generators for π_2 of this presentation.

2.2. **Criteria for profinite equivalence.** Together, Theorems 2.1 and 2.2 provide a mechanism that, given a finite combinatorial complex with 2-connected universal cover and fundamental group Q, will output finite presentations for $H \times H$ and P, and will describe the inclusion $u : P \hookrightarrow H \times H$ by expressing the generators of P as words in the generators of $H \times H$. The final step in the reduction of Theorem A to Theorem B involves identifying when $\hat{u}$ is surjective, when $\hat{u}$ is an isomorphism, and when $\hat{P} \cong \widehat{H \times H}$.

Lemma 2.3. *Let* $1 \to N \to H \to Q \to 1$ *be an exact sequence of groups and let* $P < H \times H$ *be the associated fibre product. The maps that* $N \hookrightarrow H$ *and* $P \hookrightarrow H \times H$ *induce on profinite completions are surjective if and only if* $\hat{Q} = 1$.

Proof. If Q has no finite quotients then N is dense in $\hat{H}$, so $\hat{N} \to \hat{H}$, which has compact image, is surjective. Likewise, $N \times N < P$ is dense in $\widehat{H \times H}$, so $\hat{P} \to \widehat{H \times H}$ is surjective.

Conversely, if there is a surjection from Q to a non-trivial finite group F, then the composition $H \to Q \to F$ extends to $\hat{H}$ and the kernel of this extension is a proper closed subgroup containing N. And the image of P in $\widehat{H \times H}$ is contained in the proper closed subgroup obtained by pulling back the diagonal of $F \times F$. $\qquad\square$

The question of when an inclusion induces an isomorphism of profinite completions is more subtle. The following criterion, due to Platonov and Tavgen [13], plays a central role in [13], [5] and [3] — see [5, Theorem 5.1] for a proof.

Proposition 2.4. *Let* $1 \to N \to H \to Q \to 1$ *be an exact sequence of finitely generated groups with fibre product P. If $H_2(Q, \mathbb{Z}) = 0$ and*

$\widehat{Q} = 1$ *then the inclusion* $u : P \hookrightarrow H \times H$ *induces an isomorphism of profinite completions.*

We also need to be sure that $\widehat{P} \not\cong \widehat{H \times H}$ when $\hat{u}$ is not an isomorphism. Our proof of this requires a lemma and some notation.

We write $\mathrm{Epi}(G, S)$ for the set of epimorphisms from a group G to a finite group S. Composition of epimorphisms with the natural map $G \to \widehat{G}$ defines a bijection $\mathrm{Epi}(\widehat{G}, S) \to \mathrm{Epi}(G, S)$, so if $\widehat{G}_1 \cong \widehat{G}_2$ then there is a bijection $\mathrm{Epi}(G_1, S) \to \mathrm{Epi}(G_2, S)$

Lemma 2.5. *Let* G *be a finitely generated group, let* $N_1, N_2 < G$ *be normal subgroups that commute, let* $\Gamma_i = G/N_i$, *let* $Q = G/N_1 N_2$, *and let* S *be a non-abelian finite simple group. Then*

$$|\mathrm{Epi}(G, S)| = |\mathrm{Epi}(\Gamma_1, S)| + |\mathrm{Epi}(\Gamma_2, S)| - |\mathrm{Epi}(Q, S)|.$$

Proof. Since S is simple, any epimorphism $\phi : G \to S$ must either restrict to an epimorphism on N_i or else map N_i trivially. Since S is not abelian and N_1 commutes with N_2, it cannot be that $\phi(N_1) = \phi(N_2) = S$. Thus $\mathrm{Epi}(G, S)$ is the union of the set of epimorphisms that map N_1 trivially and the set of those that map N_2 trivially; the former is bijective with $\mathrm{Epi}(\Gamma_1, S)$ and the latter with $\mathrm{Epi}(\Gamma_2, S)$. The intersection of these sets is the set of epimorphisms where $N_1 N_2$ maps trivially, and this is bijective with $|\mathrm{Epi}(Q, S)|$. $\square$

Proposition 2.6. *Let* $P < H \times H$ *be the fibre product of* $1 \to N \to H \to Q \to 1$, *where* H *is finitely generated and* Q *is finitely presented. Let* S *be a non-abelian finite simple group* S. *There is a bijection from* $\mathrm{Epi}(P, S)$ *to* $\mathrm{Epi}(H \times H, S)$ *if and only if* $\mathrm{Epi}(Q, S) = \varnothing$.

Proof. The 0-1-2 Lemma (see, for instance, [5, Lemma 2.1]) tells us that P is finitely generated, so all of the sets $\mathrm{Epi}(\cdot, S)$ under consideration are finite. Let $N_1 = P \cap (H \times 1)$, let $N_2 = P \cap (1 \times H)$, and note that $P/N_i \cong H$ (this is the image of P under projection to one of the factors of $H \times H$). Also, $P/N_1 N_2 \cong Q$. Lemma 2.5 tells us that $|\mathrm{Epi}(P, S)| = 2|\mathrm{Epi}(H, S)|$ if and only if $\mathrm{Epi}(Q, S) = \varnothing$. A second application of that lemma, with $H \times 1$ in the role of N_1 and $1 \times H$ in the role of N_2, and $H \times H$ in the role of G, implies that $|\mathrm{Epi}(H \times H, S)| = 2|\mathrm{Epi}(H, S)|$. $\square$

2.3. **Summary of the reduction.** Let K_n be the sequence of complexes given by Theorem B and let $Q_n = \pi_1 K_n$. Combining Theorems 2.2 and 2.1, we obtain (by means of an algorithmic procedure) finite presentations for residually finite groups P_n and $\Gamma_n := H_n \times H_n$ and a monomorphism $u : P_n \hookrightarrow \Gamma_n$. Lemma 2.3 tells us that $\hat{u}$ is an epimorphism if and only if $\widehat{Q}_n \cong 1$, and Proposition 2.4 tells us that $\widehat{Q}_n \cong 1$ also implies that $\hat{u}$ is an isomorphism. Finally, since Q_n is perfect, $\widehat{Q}_n \not\cong 1$ implies $\mathrm{Epi}(Q, S) \neq \varnothing$ for some finite non-abelian simple group S, in which case $\widehat{P}_n \not\cong \widehat{\Gamma}_n$, by Proposition 2.6. $\square$

3. Universal Central Extensions

In our proof of Theorem B we will need to construct and present universal central extensions of groups.

A *central extension* of a group Q is a group $\widetilde{Q}$ equipped with a homomorphism $\pi : \widetilde{Q} \to Q$ whose kernel is central in $\widetilde{Q}$. Such an extension is *universal* if given any other central extension $\pi' : E \to Q$ of Q, there is a unique homomorphism $f : \widetilde{Q} \to E$ such that $\pi' \circ f = \pi$.

The standard reference for universal central extensions is [12] pp. 43–47. The properties that we need here are these: Q has a universal central extension $\widetilde{Q}$ if (and only if) $H_1(Q, \mathbb{Z}) = 0$; there is a short exact sequence

$$1 \to H_2(Q, \mathbb{Z}) \to \widetilde{Q} \to Q \to 1 \ ;$$

and if Q has no non-trivial finite quotients, then neither does $\widetilde{Q}$.

The following result is Corollary 3.6 of [3]; the proof relies on an argument due to Chuck Miller.

Proposition 3.1. *There is an algorithm that, given a finite presentation $\langle A \mid R \rangle$ of a perfect group Q, will output a finite presentation $\langle A \mid \widetilde{R} \rangle$ for the universal central extension $\widetilde{Q}$. Furthermore, $|\widetilde{R}| = |A|(1 + |R|)$.*

4. Classifying spaces for universal central extensions

In our earlier work on the triviality problem for profinite completions, we considered a construction that transforms a presentation $\mathcal{P}$ for a group G into an aspherical complex $\mathcal{J}(\mathcal{P})$ with fundamental group $\mathcal{J}_G$, such that $\widehat{\mathcal{J}}_G = \widehat{G}$. (See [9, Definition 7.1], in which $\mathcal{J}(\mathcal{P})$ is denoted by $S(\mathcal{P})$.) In this section, our main task is to algorithmically construct an aspherical complex that represents the universal central extension of $\mathcal{J}_G$. We start by recalling the construction of $\mathcal{J}(\mathcal{P})$.

4.1. **The main construction.** Let X be the standard 2-complex of a finitely presented group $J = \langle B \mid S \rangle$ and let c be a parameterization of the edge corresponding to a generator α of infinite order. Let $\mathcal{P} \equiv \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ be a presentation of a group G.

Definition 4.1. We define $\mathcal{J}(\mathcal{P})$ to be the 2-complex obtained from the standard 2-complex of the presentation $\mathcal{P}$ by replacing each 2-cell with a copy of X that is attached with an annulus along which one can homotop the boundary cycle of the deleted 2-cell to the loop c in X. Let $\mathcal{J}_G$ be the fundamental group of $\mathcal{J}(\mathcal{P})$.

Using subscripts to distinguish generators and relations in the m disjoint copies of J, we get the following presentation of $\mathcal{J}_G$:

$$\langle B_1, \ldots, B_m, a_1, \ldots, a_n \mid S_1, \ldots, S_m, r_1 \alpha_1^{-1}, \ldots, r_m \alpha_m^{-1} \rangle \ .$$

Remark 4.2.   (1) The construction of $\mathcal{J}(\mathcal{P})$ from $\mathcal{P}$ is entirely algorithmic.

(2) The group $\mathcal{J}_G$ is the fundamental group of a tree of groups $\mathcal{T}$; the underlying tree is T_m, as defined in Definition 4.4, the local group at the central vertex is free on $\{a_1, \ldots, a_n\}$, the other vertex groups are copies of J, and the edge groups are infinite cyclic.

(3) There is a canonical map $\mathcal{J}_G \to G$ obtained by quotienting out the generators B_i, for all i.

We shall require the following basic properties of this construction.

Proposition 4.3. *Let X, J, $\mathcal{P}$ and G be as above.*

(1) *If J is perfect, then $H_1(\mathcal{J}_G, \mathbb{Z}) \cong H_1(G, \mathbb{Z})$.*

(2) *If X is aspherical, then $\mathcal{J}(\mathcal{P})$ is aspherical.*

(3) *If $\hat{J} \cong 1$, then the canonical map $\mathcal{J}_G \to G$ induces an isomorphism $\hat{\mathcal{J}}_G \xrightarrow{\cong} \hat{G}$.*

Proof. If J is perfect, then in the abelianisation of $\mathcal{J}_G$, the generators from B_i must all have trivial image for $i = 1, \ldots, m$. It follows that the abelianisation of $\mathcal{J}_G$ is the abelianisation of the group obtained by deleting all occurrences of the letters B_i from the given presentation of $\mathcal{J}_G$, and this group is G. This proves (1).

Assertion (2) is a standard fact about graphs of aspherical spaces. See, for instance [15, Proposition 3.6(ii)].

To prove (3), note that since $J_i \cong J$ for $i = 1, \ldots, m$ and $\hat{J} \cong 1$, each $J_i < \mathcal{J}_G$ has trivial image under every homomorphism $\phi : \mathcal{J}_G \to Q$ to a finite group. Thus ϕ factors through the quotient of $\mathcal{J}_G$ by the normal closure of the J_i, as required. $\square$

4.2. **Tubular bundles.** We now introduce a class of spaces which conveniently describe the universal central extensions of groups of the form $\mathcal{J}_G$.

Definition 4.4. Let d, n and m be positive integers and let T_m be the simplicial tree with vertex set $\{v_0, \ldots, v_m\}$ and edge set $\{\{v_0, v_i\} : i = 1, \ldots m\}$. A *tubular bundle of type* $(d; n, m)$ is defined by the following data:

(1) a family of finite combinatorial 2-complexes $\underline{V} = (V_i \mid i = 1, \ldots, m)$;

(2) a family $\underline{C} = (c_i \mid i = 1, \ldots, m)$, where $c_i : \mathbb{S}^1 \to V_i^{(1)}$ is a locally injective loop in the 1-skeleton of V_i;

(3) a family of non-trivial reduced words $\underline{R} = (\rho(i) \mid i = 1, \ldots, m)$ in the free group $F_n = \text{Free}(a_1, \ldots, a_n)$ of rank n;

(4) a family of elements $\underline{Z} = (z_i \in \mathbb{Z}^d \mid i = 1, \ldots, m)$.

Canonically associated to these data, there is a $(d + 2)$-dimensional space $K(\underline{V}, \underline{C}, \underline{R}, \underline{Z})$ defined as follows.

Let Y_n denote the n-rose (1-vertex graph) whose 1-cells are oriented and labelled $a_1, \ldots, a_n$, and identify words $u \in F_n$ with edge-loops $\gamma_u :$ $\mathbb{S}^1 \to Y_n$ based at the vertex. Let $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$ be the standard d-torus. Let $\theta : [0,1] \to \mathbb{S}^1$ be a parametrisation of $\mathbb{S}^1$. Then $K(\underline{V}, \underline{C}, \underline{R}, \underline{Z})$ is the quotient of

$$(\mathbb{T} \times Y_n) \sqcup \coprod_{i=1}^{m} \left(\mathbb{T} \times \mathbb{S}^1 \times [0,1]\right)_i \sqcup \coprod_{i=1}^{m}(\mathbb{T} \times V_i)$$

by the equivalence relation that attaches $(\mathbb{T} \times \mathbb{S}^1 \times \{0\})_i$ to $\mathbb{T} \times Y_n$ by $(t, \theta, 0) \sim (t, \gamma_{\rho(i)}(\theta))$ and attaches $(\mathbb{T} \times \mathbb{S}^1 \times \{1\})_i$ to $\mathbb{T} \times V_i$ by $(t, \theta, 1) \sim (t + \theta z_i, c_i(\theta))$.

Notation 4.5. When V_i and c_i are independent of i (equal to X and c, say) we adopt the abbreviated notation $K^{X,c}(\underline{R}, \underline{Z})$ and say that the complex is a *tubular bundle of type $(d; n, m)$ over (X, c)*.

A sequence of such complexes $K_n = K^{X,c}(\underline{R}_n, \underline{Z}_n)$ is termed *recursive* if the sequences $\underline{R}_n$ and $\underline{Z}_n$ are recursively enumerable.

Example 4.6. For fixed (X, c) and $(d; n, m)$, the set of all tubular bundles of type $(d; n, m)$ over (X, c) is recursive.

Remark 4.7. (1) Because the attaching maps of the edge spaces are affine, there is a simple algorithmic process that will subdivide $K^{X,c}(\underline{R}, \underline{Z})$ into a combinatorial CW complex, thence a simplicial complex, but it is more natural to work with tubular bundles since they faithfully and naturally encode the geometry of the situation that interests us, with an economical amount of defining data.

(2) A technical advantage of tubular bundles over combinatorial CW-complexes is the flexibility afforded by the fact that the attaching maps of the edge spaces can map k-cells across cells of dimension greater than k. This is equally natural and desirable when trying to cellulate torus bundles over the circle, for example.

(3) In Definition 4.4 we restricted to the case where the underlying graph is a tree, the fibre is a torus, and the vertex spaces are combinatorial 2-complexes. We did so because this is the case that is needed in the proof of Theorem B, but it is easy to imagine other contexts in which one would want to relax these constraints.

Our interest in tubular bundles derives from the following proposition, which is expressed in the notation of Subsection 4.1, in particular G is the group with presentation $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ and X is the standard 2-complex of a presentation for the group J, with $\alpha \in J$ a generator of infinite order.

Proposition 4.8. *If X is aspherical, G is perfect and $H_1(J, \mathbb{Z}) \cong H_2(J, \mathbb{Z}) \cong 0$, then the universal central extension of $\mathcal{J}_G$ is the fundamental group of an aspherical tubular bundle $K^{X,c}(\underline{R}, \underline{Z})$ of type $(d; n, m)$, where $d = m - n$ and $\underline{R} = (r_i \mid i = 1, \ldots, m)$.*

Proof. Proposition 4.3 assures us that $\mathcal{J}_G$ is perfect and therefore has a universal central extension. It also assures us that the 2-dimensional complex $\mathcal{J}(\mathcal{P})$ is aspherical, so $H_2(\mathcal{J}_G, \mathbb{Z})$ is the kernel of the second boundary map in the cellular chain complex of $\mathcal{J}(\mathcal{P})$, hence free abelian. The rank of this group can be calculated by Euler characteristic: on the one hand, since $H_1(\mathcal{J}_G, \mathbb{Z}) \cong 0$, we have $\chi(\mathcal{J}(\mathcal{P})) =$ rk $H_2(\mathcal{J}(\mathcal{P}), \mathbb{Z}) + 1$. On the other hand, counting cells, we have $\chi(\mathcal{J}(\mathcal{P})) = m\chi(X) + (1 - n)$, whence $\chi(\mathcal{J}(\mathcal{P})) = m - n + 1$, since $H_1(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \cong 0$ implies that $\chi(X) = 1$.

Thus the universal central extension of $\mathcal{J}_G$ has the form

$$(1) \qquad\qquad 1 \to \mathbb{Z}^{m-n} \to \tilde{\mathcal{J}}_G \xrightarrow{p} \mathcal{J}_G \to 1.$$

We are assuming that $H_2(J, \mathbb{Z}) = H_2(X, \mathbb{Z}) \cong 0$, so restricting to $J_i < G$ for each $i = 1, \ldots, m$, we get a central extension

$$1 \to \mathbb{Z}^{m-n} \to p^{-1}(J_i) \xrightarrow{p} J_i \to 1$$

that splits. And $1 \to \mathbb{Z}^{m-n} \to p^{-1}(F_n) \xrightarrow{p} F_n \to 1$ also splits, where $F_n = \pi_1 Y_n$. We fix splittings $\sigma_i : J_i \to \mathcal{J}_G$ and $\sigma_0 : F_n \to \mathcal{J}_G$. Since (1) itself does not split, the splittings σ_i and σ_0 cannot all agree on the edge groups $\langle \alpha_i \rangle = J_i \cap F_n$, so at least one of the elements $z_i := \sigma_0(r_i)^{-1}\sigma_i(\alpha_i) \in \mathbb{Z}^{m-n}$ is non-trivial.

Via the projection $\tilde{\mathcal{J}}_G \to \mathcal{J}_G$, we have an action of $\tilde{G}$ on the Bass–Serre tree of the splitting $\mathcal{J}_G = \pi_1 \mathcal{T}$ from Remark 4.2. The kernel of the action on this tree is the centre $\mathbb{Z}^{m-n}$ of $\tilde{\mathcal{J}}_G$. The resulting graph-of-groups decomposition of $\tilde{\mathcal{J}}_G$ has in the middle the vertex group $p^{-1}(F_n) \cong F_n \times \mathbb{Z}^{m-n}$, and the other vertex groups are $p^{-1}(J_i) = J \times \mathbb{Z}^{m-n}$ (where we write $=$ rather than $\cong$ because the splitting is unique); the edge groups are of the form $\mathbb{Z}^{m-n+1} = \mathbb{Z} \times \mathbb{Z}^{m-n}$, where the edge morphisms respect the $\mathbb{Z}^{m-n} = H_2(\mathcal{J}_G, \mathbb{Z})$ factor. We realise this graph of groups as a graph of spaces with vertex spaces $X_i \times \mathbb{T}^{m-n}$ and $Y_n \times \mathbb{T}^{m-n}$, with the edge-space inclusions represented by the unique base-point preserving affine maps that induce the desired maps at the level of π_1. This inclusion map is the obvious one at the extremal vertices, with the first factor of $\mathbb{Z} \times \mathbb{Z}^{m-n}$ mapping to $\sigma_i(\alpha_i)$, the image of α_i under the unique splitting of $p^{-1}(J_i) \xrightarrow{p} J_i$. At the middle vertex, the decompositon $p^{-1}(F_n) \cong F_n \times \mathbb{Z}^{m-n}$ depends on the choice of splitting $\sigma_0 : F_n \to p^{-1}(F_n)$ and, correspondingly, while the central factor $1 \times \mathbb{Z}^{m-n}$ in the edge group maps to the second factor of $p^{-1}(F_n)$, the generator of the first factor $\mathbb{Z}$ maps to $\sigma_0(r_i)$, which is $\sigma_i(\alpha_i)z_i^{-1}$.

Thus we have exhibited $\tilde{\mathcal{J}}_G$ as the fundamental group of a graph of spaces whose total space is $K^{X,c}(\underline{R}, \underline{Z})$ where $\underline{R} = (r_i \mid i = 1, \ldots, m)$ and $\underline{Z} = (z_i \mid i = 1, \ldots, m)$ with $z_i := \sigma_0(r_i)^{-1}\sigma_i(\alpha_i) \in \mathbb{Z}^{m-n}$. $\qquad\square$

Example 4.9. We shall need groups J with $\hat{J} = 1$ that satisfy the hypotheses of Proposition 4.8. The first such group was discovered by Graham Higman [10]; it has the aspherical presentation

$$J = \langle a_1, a_2, a_3, a_4 \mid a_2^{-1} a_1 a_2 a_1^{-2}, a_3^{-1} a_2 a_3 a_2^{-2}, a_4^{-1} a_3 a_4 a_3^{-2}, a_1^{-1} a_4 a_1 a_4^{-2} \rangle.$$

Many more such examples are described in [5].

5. Proving Theorem B with a recursive class of tubular bundles of type $(d; n, m)$

In [9] we constructed a recursive sequence of finite group presentations $\mathcal{P}(k) \equiv \langle A \mid R_k \rangle$ (with A a fixed finite alphabet and S_k of a fixed cardinality) so that for the groups $G(k) = |\mathcal{P}(k)|$,

$$\{k \in \mathbb{N} \mid \widehat{G(k)} \not\cong 1\}$$

is recursively enumerable but not recursive. We may also assume that the groups are perfect. Let $n = |A|$, let $m = |R_k|$, and let $d = m - n$.

We fix a group J as in Example 4.9, take α to be a generator of infinite order and X to be the standard 2-complex of an aspherical presentation of J. We then apply the construction of Subsection 4.1 to the sequence of presentations $\mathcal{P}(k)$ to obtain a recursive sequence of presentations $\mathcal{J}(\mathcal{P}(k))$ for groups $\mathcal{J}_{G(k)}$.

According to Proposition 4.3, the groups $\mathcal{J}_{G(k)}$ are perfect, the presentations $\mathcal{J}(\mathcal{P}(k))$ are aspherical, and $\hat{\mathcal{J}}_{G(k)} \cong 1$ if and only if $\widehat{G(k)} \cong 1$. So Theorem B would be proved at this stage if each $H_2(\mathcal{J}_{G(k)}, \mathbb{Z}) = 0$. But euler characteristic tells us that this is not the case, because $m > n$. We therefore pass to the universal central extension of $\mathcal{J}_{G(k)}$, appealing to Proposition 3.1 to derive a recursive sequence of finite presentations $\widetilde{\mathcal{P}}_k$ for the groups $\widetilde{\mathcal{J}}_{G(k)}$, noting that $\widetilde{\mathcal{J}}_{G(k)}$ has a non-trivial finite quotient if and only if $\mathcal{J}_{G(k)}$ does.

We will be done if we can descibe an algorithm that, taking each $\widetilde{\mathcal{P}}_k$ in turn, produces an explicit model for the classifying space $K(\widetilde{\mathcal{P}}_k, 1)$.

Proposition 4.8 assures us that $\widetilde{\mathcal{J}}_{G(k)}$ is the fundamental group of an aspherical tubular bundle $K^{X,c}(\underline{R_k}, \underline{Z})$ of type $(d; n, m)$. An obvious algorithm enumerates all such bundles as $\underline{Z}$ varies, and the Seifert–van Kampen theorem provides an explicit presentation for the fundamental group of each. Proceeding along this list of presentations with a naive search, we will eventually identify an isomorphism between $\widetilde{\mathcal{J}}_{G(k)}$, as presented by $\widetilde{\mathcal{P}}_k$, and a presentation on the list. When such is found, our algorithm outputs the corresponding complex $K^{X,c}(\underline{R_k}, \underline{Z})$ as the sought-after model for the classifying space $K(\widetilde{\mathcal{P}}_k, 1)$.

This completes the proof of Theorem B. $\square$

Remarks 5.1. (1) The interested reader can verify that the presentations that we constructed to prove Theorem A have the additional

property that the sets A_n, B_n and S_n are of fixed cardinality, while $|R_n|$ is uniformly bounded.

(2) If there exists a hyperbolic group that is not residually finite, then one can prove Theorem A much more directly: if such a group exists, then Theorem 9.6 of [9] will produce a recursive sequence of finite presentations $\mathcal{Q}_n$, each presenting a perfect hyperbolic group Q_n, so that there is no algorithm that can determine which of the groups presented has trivial profinite completion; one can then apply the algorithm of [7] to construct an explicit model for the 3-skeleton of a classifying space for the universal central extension $\tilde{Q}_n$; and one can use this sequence of complexes in place of those yielded by Theorem B.

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MARTIN R. BRIDSON, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, ANDREW WILES BUILDING, OXFORD OX2 6GG, UK
   E-mail address: bridson@maths.ox.ac.uk

HENRY WILTON, DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET, LONDON WC1E 6BT, UK
   E-mail address: hwilton@math.ucl.ac.uk