HOMOMORPHISMS FROM AUTOMORPHISM GROUPS OF FREE GROUPS

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Abstract. The automorphism group of a finitely generated free group is the normal closure of a single element of order 2.

If $m < n$ then a homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_m)$ can have image of cardinality at most 2. More generally, this is true of homomorphisms from $\mathrm{Aut}(F_n)$ to any group that does not contain an isomorphic image of the symmetric group S_{n+1}. Strong restrictions are also obtained on maps to groups which do not contain a copy of $W_n = (\mathbb{Z}/2)^n \rtimes S_n$, or of $\mathbb{Z}^{n-1}$.

These results place constraints on how $\mathrm{Aut}(F_n)$ can act. For example, if $n \geq 3$ any action of $\mathrm{Aut}(F_n)$ on the circle (by homeomorphisms) factors through $\det : \mathrm{Aut}(F_n) \to \mathbb{Z}_2$.

1. Introduction

In recent articles we began to explore the extent to which the well-known analogies between lattices in semisimple Lie groups and automorphism groups of free groups can be extended to cover various aspects of rigidity.

For example, in [**6**] we proved that all automorphisms of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are inner if $n \geq 3$. In the direction of super-rigidity, it follows from the main theorem of [**4**] that if Γ is an irreducible, non-uniform lattice in a higher rank semisimple group, then any homomorphism from Γ to $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ has finite image — see [**5**]. In the classical setting, Margulis super-rigidity tells one that if there is no homomorphism from one semi-simple group G_1 to another G_2, then any map from a lattice in G_1 to G_2 must

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have finite image. An example of this phenomenon is the fact that if $m < n$ then any map from $\mathrm{GL}(n,\mathbb{Z})$ to $\mathrm{GL}(m,\mathbb{Z})$ has image $\mathbb{Z}_2$ or $\{1\}$. In the present article we establish the analogous result for automorphism groups of free groups.

Theorem A. *If $n \geq 3$ and $n > m$, then every homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_m)$ is either trivial or has image of order 2. Likewise for maps $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_m)$ and $\mathrm{Aut}(F_n) \to \mathrm{GL}(m,\mathbb{Z})$.*

In section 3 we shall see that Theorem A is a special case of:

Theorem B. *Let $n \geq 3$. If a group G does not contain a copy of the symmetric group S_{n+1}, then the image of any homomorphism $\mathrm{Aut}(F_n) \to G$ has cardinality at most 2.*

Note that there is only one surjection $\mathrm{Aut}(F_n) \to \mathbb{Z}_2$, namely the composition of the determinant map $\mathrm{GL}(n,\mathbb{Z}) \to \mathbb{Z}_2$ and the map $\mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z})$ describing the action of $\mathrm{Aut}(F_n)$ on the abelianization of F_n. We shall denote this surjection $\det : \mathrm{Aut}(F_n) \to \mathbb{Z}_2$.

Theorem B is proved by examining the pattern of finite subgroups in $\mathrm{Aut}(F_n)$. Another result in this direction concerns the largest finite subgroup $W_n \subset \mathrm{Aut}(F_n)$ (which is unique up to conjugacy).

Proposition C. *Let $n \geq 3$, and let $\phi\colon \mathrm{Aut}(F_n) \to G$ be a homomorphism to a group G that does not contain a copy of $W_n = (\mathbb{Z}_2)^n \rtimes S_n$. Then either $\mathrm{Im}(\phi) \cong \mathrm{PGL}(n,\mathbb{Z})$ or else $\mathrm{Im}(\phi)$ is finite.*

Our proof of the above results proceeds via the following observation:

Proposition D. *If $n \geq 3$, then $\mathrm{Aut}(F_n)$ (and hence each of its quotients) is the normal closure of a single element of order 2.*

Whenever one is able to control the nature of the quotients that a group admits, one immediately obtains constraints on the type of actions that it admits. For example, because the finite subgroups of the homeomorphism group of the circle are all cyclic or dihedral, Theorem B implies:

Theorem E. *If $n \geq 3$, then any action of* $\mathrm{Aut}(F_n)$ *on the circle (by homeomorphisms) factors through* $\det : \mathrm{Aut}(F_n) \to \mathbb{Z}_2$.

B. Farb and J. Franks [**8**] give a different proof of this fact for C^2-actions in the case $n \geq 6$.

Our results limiting the actions of $\mathrm{Aut}(F_n)$ do not compare favourably with results from the classical setting in that they rely heavily on the nature of the torsion in $\mathrm{Aut}(F_n)$ and therefore do not extend to subgroups of finite index (which may be torsion-free). One might be tempted to conjecture that if $\Gamma \subset \mathrm{Aut}(F_n)$ has finite index and if $m < n$ then every homomorphism from Γ to $\mathrm{Aut}(F_m)$ has finite image. However this fails for $n = 3$, because $\mathrm{Aut}(F_3)$ has a torsion-free subgroup of finite index which maps onto $\mathbb{Z}$. The situtation for $n > 3$ is far from clear.

In the case of $\mathrm{GL}(n, \mathbb{Z})$ it is easy to extend results to finite index subgroups, because one has a very concrete description of these subgroups in terms of congruence subgroups. Part of the difficulty for $\mathrm{Aut}(F_n)$ is that the following naive analogues $Q_{n,m}$ of the congruence quotients $\mathrm{GL}(n, \mathbb{Z}/m\mathbb{Z})$ can be infinite.

Fix a basis $\{a_1, \dots, a_n\}$ for F_n and let λ_{ij} be the automorphism of F_n that sends a_i to $a_j a_i$ and fixes a_k for $k \neq i$. Let $Q_{n,m}$ be the quotient of $\mathrm{Aut}(F_n)$ defined by setting $\lambda_{ij}^m = 1$ for all i and j.

Proposition F. *$Q_{n,m}$ surjects onto a group that contains a copy of the free Burnside group of exponent m on $n-1$ generators.*

The groups $Q_{n,m}$ enjoy the following universal property with respect to maps from $\mathrm{Aut}(F_n)$ to groups without large-rank abelian subgroups (see section 5).

Proposition G. *If G is a group that does not contain $\mathbb{Z}^{n-1}$, then any homomorphism* $\mathrm{Aut}(F_n) \to G$ *factors through $Q_{n,m}$ for some m.*

2. Some elements and relations in $\mathrm{Aut}(F_n)$

We assume that $n \geq 3$ and we fix a basis $\{a_1, \dots, a_n\}$ for the free group F_n. Associated to this choice of basis one has the finite subgroup isomorphic to $\mathbb{Z}_2^n$ generated by the involutions ε_i, where $\varepsilon_i \colon a_i \mapsto a_i^{-1}$ and $a_j \mapsto a_j$ for $j \neq i$. The permutations of our fixed basis form a copy of the symmetric group $S_n \subset \mathrm{Aut}(F_n)$. We shall write elements of this symmetric group as products of cycles, for example (1 2) will denote the automorphism that interchanges a_1 and a_2 and leaves the remaining a_i fixed.

Note that $\varepsilon_{\sigma(i)} = \sigma\varepsilon_i\sigma^{-1}$ for each permutation $\sigma \in S_n$.

The ε_i and S_n together generate a subgroup $W_n \cong (\mathbb{Z}_2)^n \rtimes S_n$, which is the unique largest finite subgroup of $\mathrm{Aut}(F_n)$ for $n \geq 4$.

Associated to our basis we also have the left Nielsen transformations: λ_{ij} sends a_i to $a_j a_i$ and fixes a_k for $k \neq i$. One also has the corresponding right Nielsen moves $\rho_{ij} : a_i \mapsto a_i a_j$. All Nielsen moves lie in the index 2 subgroup $\mathrm{Aut}^+(F_n)$ which is the inverse image of $\mathrm{SL}(n, \mathbb{Z})$ under the natural map $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, Z)$. It is well-known that they generate $\mathrm{Aut}^+(F_n)$ (see, e.g., [**10**]).

If σ is a permutation that sends (i, j) to (k, l), then σ conjugates λ_{ij} to λ_{kl}. And conjugation by $\varepsilon_i\varepsilon_j$ sends λ_{ij} to ρ_{ij}. Thus all Nielsen moves with respect to a fixed basis are conjugate. And since $\mathrm{Aut}(F_n)$ acts transitively on bases of F_n, the Nielsen moves associated to different bases are also all conjugate in $\mathrm{Aut}(F_n)$. In particular we have:

Lemma 1. *If one Nielsen move lies in the kernel of a homomorphism $\phi : \mathrm{Aut}(F_n) \to G$, then ϕ factors through* $\det : \mathrm{Aut}(F_n) \to \mathbb{Z}_2$.

In order to write relations in $\mathrm{Aut}(F_n)$ in a standard and convenient manner, it is best to think of $\mathrm{Aut}(F_n)$ acting *on the right* on F_n, and we shall adopt this convention (which is standard in texts on combinatorial group theory). Our commutator convention is $[a, b] = aba^{-1}b^{-1}$. Thus $[\alpha, \beta]$ means *apply the automorphism α, then β, then α^{-1}, then β^{-1}.*

We shall make frequent use of the relations

$$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik} \quad \text{and} \quad [\lambda_{ij}, \lambda_{kj}] = 1.$$

for i, j and k distinct.

Proposition 1. *Let $n \geq 3$ and let $\phi : \mathrm{Aut}(F_n) \to G$ be a homomorphism. If ϕ is not injective on S_n, then ϕ is either trivial or has image $\mathbb{Z}_2$.*

Proof. Let K be the kernel of $\phi|_{S_n}$. Since $n \geq 3$ and $K \neq S_n$, we must have $K = A_n$ or, if $n = 4$, possibly $K = \mathbb{Z}/2 \times \mathbb{Z}/2$.

If $K = A_n$, then all 3-cycles $(i\,j\,k)$ are in the kernel, so the relations

$$(i\,j\,k)\lambda_{jk}(i\,j\,k)^{-1} = \lambda_{ij} \text{ and } [\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}$$

show that λ_{ik} maps trivially under ϕ, and hence the whole of $\mathrm{Aut}^+(F_n)$ maps trivially (Lemma 1).

If $n = 4$ and $K = \mathbb{Z}/2 \times \mathbb{Z}/2$, then all products of two disjoint 2-cycles in S_4 are mapped trivially, so the identity

$$(j\,k)(i\,l)\ \lambda_{jk}\ (j\,k)(i\,l) = \lambda_{kj}$$

gives $\phi(\lambda_{jk}) = \phi(\lambda_{kj})$. Applying ϕ to the relation

$$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}$$

gives

$$\phi(\lambda_{ik}) = \phi([\lambda_{ij}, \lambda_{jk}]) = \phi([\lambda_{ij}, \lambda_{kj}]) = 1,$$

and as before all of $\mathrm{Aut}^+(F_n)$ is mapped trivially. □

Let $\iota = (1\,2)$.

Proposition 2. *Let G be a group and let $\phi : \mathrm{Aut}(F_n) \to G$ be a homomorphism. The image of ϕ is trivial if and only if $\phi(\iota) = 1$.*

Proof. If $\phi(\iota) = 1$, then since S_n is generated by conjugates of ι, the whole of S_n has trivial image. The previous proposition then shows that $\phi(\mathrm{Aut}^+(F_n))$ is trivial, and since $\iota \notin \mathrm{Aut}^+(F_n)$, in fact $\phi(\mathrm{Aut}(F_n))$ must be trivial. □

Corollary 1. $\mathrm{Aut}(F_n)$ *is the normal closure of the transposition* $(1\,2)$.

3. Maps to groups without large symmetric groups

We shall exploit the following well-known facts.

Lemma 2. $\mathrm{GL}(n-1, \mathbb{Z})$ *does not contain a copy of S_{n+1}.*

Proof. See, e.g., [**9**], Exercise 4.14. □

Lemma 3. $\mathrm{Aut}(F_n)$ *contains a symmetric group* $\Sigma \cong S_{n+1}$ *that intersects* $W_n = (\mathbb{Z}_2)^n \rtimes S_n$ *in the visible copy of* S_n.

Proof. Finite subgroups of $\mathrm{Aut}(F_n)$ correspond to vertex stabilizers in Outer Space [**7**]. The subgroup Σ fixes the two vertices of the marked graph that has $(n+1)$ directed edges with common source and sink, n of which are labelled $a_1, \ldots, a_n$. $\square$

Theorem B. *Let G be a group, and suppose $n \geq 3$. If G does not contain a copy of S_{n+1} then every homomorphism $\phi : \mathrm{Aut}(F_n) \to G$ has image of cardinality at most* 2.

Proof. Let Σ be as in the previous lemma, and let K be the kernel of $\phi|_\Sigma$. If $K = A_{n+1}$, then $K \cap S_n = A_n$, so ϕ is not injective on S_n and we are done by Proposition 1. There is an additional possibility when $n = 3$, namely $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, generated by the order 2 automorphisms α and β:

$$\alpha\colon \begin{cases} a_1 \mapsto a_1^{-1} \\ a_2 \mapsto a_3 a_1^{-1} \\ a_3 \mapsto a_2 a_1^{-1} \end{cases} \qquad \beta\colon \begin{cases} a_1 \mapsto a_3 a_2^{-1} \\ a_2 \mapsto a_2^{-1} \\ a_3 \mapsto a_1 a_2^{-1} \end{cases}$$

The relation

$$\beta\lambda_{12}\beta = \lambda_{32}^{-1}$$

gives $\phi(\lambda_{12}) = \phi(\lambda_{32}^{-1})$. But expanding the relation

$$\lambda_{12} = [\lambda_{13}, \lambda_{32}]$$

gives

$$\lambda_{12} = (\lambda_{13}\lambda_{32}\lambda_{13}^{-1})\, \lambda_{32}^{-1},$$

so applying ϕ we deduce $\phi(\lambda_{13}\lambda_{32}\lambda_{13}^{-1}) = 1$, and $\phi(\lambda_{32}) = 1$. Thus, in the light of Lemma 1, the image of ϕ is at most $\mathbb{Z}_2$. $\square$

Theorem A. *If $n \geq 3$ and $m < n$, then every homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_m)$ has image of cardinality at most* 2.

Proof. The kernel of the natural map $\mathrm{Aut}(F_m) \to \mathrm{GL}(m, \mathbb{Z})$ is torsion-free [**3**], so the first of the above lemmas tells us that $\mathrm{Aut}(F_m)$ does not contain a copy of S_{n+1}. $\square$

Exactly the same argument applies to maps from $\mathrm{Aut}(F_n)$ to $\mathrm{Out}(F_m)$ or $\mathrm{GL}(m,\mathbb{Z})$.

4. Maps to groups without copies of W_n

Lemma 4. *Consider a semi-direct product $N \rtimes S_n$ where N is abelian and one of the following conditions holds: $n \geq 5$, or $n = 4$ and N does not contain an S_n-invariant subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $n = 3$ and N does not contain an S_n-invariant subgroup of order 3. If $\phi\colon N \rtimes S_n \to Q$ is a homomorphism with non-trivial kernel K, then either $K \cap S_n$ or $N \cap K$ is non-trivial.*

Proof. If both $K \cap S_n$ and $N \cap K$ are trivial, then since ϕ is assumed to be non-injective, $\phi(N) \cong N$ and $\phi(S_n) \cong S_n$ must intersect non-trivially. Since N is normal in $N \rtimes S_n$, this intersection is normal in $\phi(S_n)$. If $n \geq 5$, then S_n has no non-trivial normal abelian subgroups, so we have a contradiction. In S_4 one has the possibility that the intersection could be the Klein 4-group, and in S_3 it could be $A_3 \cong \mathbb{Z}_3$.

But the intersection is of the form $\phi(N_0) \cong N_0$, where $N_0 \subset N$ is an S_n-invariant subgroup of N. And we have imposed hypotheses to exclude the possible existence of non-trivial N_0 in the cases $n = 3$ and $n = 4$. (We leave the reader to consider the easy case $n = 2$.) □

Proposition C. *Let $n \geq 3$. If $\phi\colon \mathrm{Aut}(F_n) \to G$ is a homomorphism to a group G that does not contain a copy of $(\mathbb{Z}_2)^n \rtimes S_n$, then the image of ϕ is either isomorphic to* $\mathrm{PGL}(n,\mathbb{Z})$ *or else it is finite.*

Proof. Since G does not contain a subgroup isomorphic to $W_n = (\mathbb{Z}_2)^n \rtimes S_n$, the map $\phi|_{W_n}$ has non-trivial kernel K. If $\phi|_{S_n}$ is not injective, then the image of ϕ has order at most 2, by Proposition 1.

Now assume ϕ is injective on S_n. The only non-trivial subgroups of $N := (\mathbb{Z}_2)^n \subset W_n$ that are S_n-invariant are N, its centre $\langle z \rangle$ ($z = \varepsilon_1\varepsilon_2 \ldots \varepsilon_n$) and $H = \langle \varepsilon_i\varepsilon_j \mid i \neq j \rangle$. These are all 2-groups, of course, but in the case $n = 4$ none of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus we are in the situation of the lemma, and deduce that $K := \ker \phi|_N$ is non-trivial. Since K is normal in W_n, it must be one of the subgroups of N listed above.

If $K = N$ or $\langle z \rangle$, or if n is even and $K = H$, then $\phi(z) = 1$, and the identity $z\lambda_{ij}z = \rho_{ij}$ implies that $\phi(\lambda_{ij}) = \phi(\rho_{ij})$ for all i, j. Since adding the relations $\lambda_{ij} = \rho_{ij}$

to the standard presentation for $Aut(F_n)$ gives a presentation for $\mathrm{GL}(n,\mathbb{Z})$, this implies that ϕ factors through $\mathrm{GL}(n,\mathbb{Z})$. Since $\phi(z)$ must be contained in the centre of $\mathrm{GL}(n,\mathbb{Z})$, in fact ϕ factors through $P\mathrm{GL}(n,\mathbb{Z})$. And since all normal subgroups of $P\mathrm{GL}(n,\mathbb{Z})$ have finite index [**2**], the image of ϕ is either finite or isomorphic to $P\mathrm{GL}(n,\mathbb{Z})$.

If n is odd and $K=H$, then K does not contain z. But $\phi(\varepsilon_i\varepsilon_j)=1$ for all i,j, so the relations $\varepsilon_i\varepsilon_j\lambda_{ij}\varepsilon_j\varepsilon_i=\rho_{ij}$ again imply that $\phi(\lambda_{ij})=\phi(\rho_{ij})$ for all i,j. Therefore ϕ factors through $\mathrm{GL}(n,\mathbb{Z})$. Let $\bar{\phi}\colon \mathrm{GL}(n,\mathbb{Z})\to G$ be the induced map. Since $\phi(\varepsilon_1\varepsilon_2)=1$, the commutator $[\lambda_{12},\varepsilon_1\varepsilon_2]$ is in the kernel of ϕ. The image of $[\lambda_{12},\varepsilon_1\varepsilon_2]$ in $\mathrm{GL}(n,\mathbb{Z})$ is the matrix $I+2E_{21}$, which has infinite order in $\mathrm{GL}(n,\mathbb{Z})$. Since $\mathrm{GL}(n,\mathbb{Z})$ has no infinite normal subgroups of infinite index, the image of ϕ must be finite. □

5. Maps to groups without copies of $\mathbb{Z}^{n-1}$

We now consider maps of $\mathrm{Aut}(F_n)$ to groups G which do not contain large free abelian subgroups. We shall see that the following family of quotients of $\mathrm{Aut}(F_n)$ play a distinguished role in the study of such maps.

Definition . *Let $Q_{n,m}$ be the quotient of $\mathrm{Aut}(F_n)$ by the relations $\lambda_{ij}^m=1$ for all $i\neq j$.*

Proposition 3. *Let $n\geq 3$. If G is a group that does not contain a free abelian group of rank $n-1$, then every homomorphism $\phi:\mathrm{Aut}(F_n)\to G$ factors through $Q_{n,m}$ for some $m\geq 1$.*

Proof. Let S_{n-1} be the symmetric group which permutes the basis elements $\{a_2,\dots,a_n\}$ of F_n, let $N\cong\mathbb{Z}^{n-1}$ be the subgroup of $\mathrm{Aut}(F_n)$ generated by the elements λ_{i1}, $i=2,\dots,n$, and let $L=N\rtimes S_{n-1}$.

Both N and the kernel of ϕ are normalized by S_{n-1}, and hence so is their intersection, which we denote K. (This is non-trivial by hypothesis.) The only S_{n-1}-invariant subgroups of $N\cong\mathbb{Z}^{n-1}$ are the lattices mN, for $m\in\mathbb{Z}$, the diagonal subgroup $D=\langle(\lambda_{21},\dots,\lambda_{n1})\rangle\cong\mathbb{Z}$, the complementary hyperplane $H=\{(\lambda_{21}^{m_1},\dots,\lambda_{n1}^{m_n})\mid\sum m_i=0\}\cong\mathbb{Z}^{n-2}$, and the intersections $mN\cap D$ and $mN\cap H$; thus these are the only possibilities for K.

Case 1. $K=mN$.

This case is trivial; ϕ sends all λ_{ij}^m to 1, so factors through $Q_{m,n}$.

Case 2. $K = mN \cap H$.

Since $mN \cap H$ contains the elements $\lambda_{i1}^m \lambda_{j1}^{-m}$ for $i \neq j$, we have $\phi(\lambda_{i1}^m) = \phi(\lambda_{j1}^m)$ for all i, j.

A simple calculation shows that $[\lambda_{12}, \lambda_{23}^m] = \lambda_{13}^m$. By taking the image of this relation under ϕ and making the substitution $\phi(\lambda_{13}^m) = \phi(\lambda_{23}^m)$ we get $\phi(\lambda_{12})\phi(\lambda_{13})^m\phi(\lambda_{12})^{-1} = \phi(\lambda_{13})^{2m}$. Similarly, from $[\lambda_{13}, \lambda_{32}^m] = \lambda_{12}^m$ and $\phi(\lambda_{12}^m) = \phi(\lambda_{32}^m)$, we get $\phi(\lambda_{13})\phi(\lambda_{12})^m\phi(\lambda_{13})^{-1} = \phi(\lambda_{12})^{2m}$.

Set $a = \phi(\lambda_{12})$ and $b = \phi(\lambda_{13})^m$. Then $aba^{-1} = b^2$ and $ba^mb^{-1} = a^{2^m m}$. These relations imply that $a = \phi(\lambda_{12})$ has finite order. Indeed, in any group, if $aba^{-1} = b^2$ and $ba^rb^{-1} = a^s$ with $r \neq s$, then b (and hence a) has finite order, because if we conjugate b by the left-hand side of the second relation then the first relation tells us that the result is b^{2^r}, whereas if we conjugate by the right-hand side of the second relation we obtain b^{2^s}.

Since all λ_{ij} are conjugate in $\mathrm{Aut}(F_n)$, we conclude that $\phi(\lambda_{ij}^{m'}) = 1$ for all i and j and some m', i.e. ϕ factors through $Q_{n,m'}$.

Case 3. $K = mD = mN \cap D$.

The diagonal D is generated by $\lambda_{21} \dots \lambda_{n1}$, which is conjugated to λ_{21} by the automorphism that fixes a_1 and a_2 and sends a_i to $a_2 a_i$ for $i > 2$. Thus if $K = mD$, then $\lambda_{21}^m \in K$ and hence all λ_{ij}^m are in K. Thus ϕ factors through $Q_{n,m}$. □

If $m = 1$, then $Q_{n,1} = \mathbb{Z}/2$, but in general the groups $Q_{n,m}$ are infinite:

Proposition 4. *For large m, the groups $Q_{n,m}$ are infinite.*

Proof. There is a natural homomorphism from $\mathrm{Aut}(F_n)$ to the automorphism group of the free object in the variety of n-generator groups of exponent m, which is written $B_{n,m}$, "the free Burnside group". The image of this map contains isomorphic copies of $B_{n-1,m}$, for example the subgroup consisting of the automorphisms obtained by composing left Nielsen moves on the first generator: $a_1 \mapsto wa_1$, $a_i \mapsto a_i$ for $i = 2, \dots, n$, where w is a word in the letters $a_2, \dots, a_n$ and their inverses. If m is sufficiently large then this group is infinite [**1**], [**11**].

Since the image of each λ_{ij}^m in $\mathrm{Aut}(B_{m,n})$ is trivial, the map $\mathrm{Aut}(F_n) \to \mathrm{Aut}(B_{m,n})$ factors through $Q_{n,m}$. $\square$

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