LENGTH FUNCTIONS, CURVATURE AND THE DIMENSION OF DISCRETE GROUPS

Martin R. Bridson

Abstract. We work with the class of groups that act properly by semisimple isometries on complete CAT(0) spaces. Define $\dim_{ss} \Gamma$ to be the minimal dimension in which Γ admits such an action. By examining the nature of translation length functions we show that there exist finitely-presented, torsion-free groups Γ for which $\dim_{ss} \Gamma$ is greater than the cohomological dimension of Γ. We also show that $\dim_{ss} \Gamma$ can decrease when one passes to a subgroup of finite index.

Introduction

Associated to an isometry γ of any metric space X one has the translation length

$$|\gamma| = \inf\{d(x, \gamma \cdot x) \mid x \in X\} ;$$

the isometry is said to be semisimple if this infimum is attained. If γ' is conjugate to γ in $\mathrm{Isom}(X)$ then $|\gamma'| = |\gamma| = |\gamma^{-1}|$. And if X is a complete CAT(0) space then $|\gamma^n| = n\,|\gamma|$ for all $n \in \mathbb{N}$.

The nature of the functions $\gamma \mapsto |\gamma|$ that an abstract group Γ admits can provide valuable information about the group, in particular the way that it can act on various types of metric spaces. For example, minimal actions of Γ on $\mathbb{R}$-trees are in 1-to-1 correspondence with length functions $\Gamma \to [0, \infty)$ satisfying five simple axioms ([CM], page 579); if the function has discrete image and $|\gamma| \neq 0$ for all $\gamma \neq 1$, then the associated tree is simplicial and the group is free. Length functions also play an important role in the study of geometric structures on surfaces. For example, the conjugacy classes of discrete faithful representations of the group $\Gamma_n = \langle a_1, b_1, \dots , a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1\rangle$ into $\mathrm{Isom}\,\mathbb{H}^2$ can be distinguished from one another by the values that the associated length function takes on a certain set of $9n - 9$ elements in Γ_n (see [Th] and compare with [Ot] and [Ki]).

At a more mundane level, one knows that the length functions associated to actions on length spaces that are proper and cocompact have discrete images. Less trivially, this is also true of discrete, cellular actions on polyhedral complexes with only finitely many isometry types of cells (see [Br]). Length functions are used in [BH] to examine group actions on complete, 1-connected spaces that are of non-positive curvature in the sense of

The author thanks the EPSRC for the Advanced Fellowship that supports his research. He also thanks the FIM at the ETH Zürich for its hospitality and support during the preparation of this manuscript.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

A.D. Alexandrov, i.e. CAT(0) spaces. In this article we shall use length functions as a tool to determine the minimal dimension in which certain groups admit reasonable actions on CAT(0) spaces.

CAT(0) spaces are contractible and therefore give rise to classifying spaces for discrete groups of isometries. Given a CAT(0) space X and a proper action $\Gamma \to \mathrm{Isom}(X)$ by a torsion-free group, as a measure of how good a model X/Γ is for $K(\Gamma, 1)$ one might compare the (cohomological or geometric) dimension $\dim \Gamma$ with the topological covering dimension of X. To this end, we define the invariant

$$\dim_{ss} \Gamma = \min\{\dim X \mid X \text{ complete CAT(0) and } \Gamma \text{ acts properly by s.s. isometries on } X\}$$

with $\dim_{ss} \Gamma := \infty$ if Γ does not admit such an action. (As an alternative invariant one might consider only cocompact actions.)

One obviously has $\dim_{ss} \Gamma \geq \dim \Gamma$. We shall show that in general one does not get equality, even in the case where Γ is the fundamental group of a compact, finite-dimensional complex of non-positive curvature. Also, in contrast to the behaviour of $\dim \Gamma$, we shall see that there exist groups Γ and subgroups of finite index $\hat{\Gamma} \subset \Gamma$ such that $\dim_{ss} \hat{\Gamma} < \dim_{ss} \Gamma < \infty$. Specifically:

Theorem. *There exist compact aspherical 2-complexes X such that:*

i) *X is not homotopy equivalent to any 2-dimensional space of non-positive curvature, indeed $\Gamma := \pi_1 X$ does not act properly by semisimple isometries on any complete, 2-dimensional* CAT(0) *space.*

ii) *X is homotopy equivalent to a compact 3-dimensional cubical complex of non-positive curvature.*

iii) *A certain 2-sheeted covering $\hat{X}$ of X is homotopy equivalent to a compact 2-dimensional piecewise Euclidean complex of non-positive curvature.*

In the setting of this theorem, writing $\hat{\Gamma} = \pi_1 \hat{X}$ we have a "dimension gap" $\dim_{ss} \hat{\Gamma} - \dim_{ss} \Gamma = 1$. One can obtain arbitrarily large dimension gaps between commensurable groups by combining the construction in the above theorem with the Product Decomposition Theorem ([BH], p. 239).

The above result contrasts with the behaviour of groups of isometries of the most classical CAT(0) spaces: as a consequence of various rigidity results one knows that if a group Γ acts properly and cocompactly by isometries on a product of symmetric spaces of non-positive curvature and irreducible Tits buildings that are of affine type and have rank at least 2, then every finite extension of Γ acts properly and cocompactly by isometries on the same space.

We shall reduce the proof of the above theorem to a problem concerning the length functions associated to discrete actions of free groups on $\mathbb{R}$-trees, which is the subject of section 1. The reduction, which is explained in section 3, relies on several facts concerning centralizers in groups that act by isometries on CAT(0) spaces, and on the explicit constructions of spaces and groups given in section 2.

In the course of their work on Artin groups, Noel Brady and John Crisp [BC] have also discovered classes of compact non-positively curved spaces X with $\dim_{ss} \pi_1 X > \dim \pi_1 X$.

I thank the referee for his/her careful reading and helpful comments.

1. Length functions for free groups

In all that follows $|\gamma|$ will denote the translation number of an isometry γ. Note that $|\gamma|$ depends only on the conjugacy class of γ, and $|\gamma^{-1}| = |\gamma|$. Given a length space X and $\gamma \in \pi_1 X$, we shall write $|\gamma|$ to denote the translation number of the corresponding deck transformation of the universal covering $\tilde{X}$, which is assumed to be endowed with the length metric induced from X. If X is compact and non-positively curved, then $|\gamma|$ is the length of any closed geodesic in X that belongs to the free homotopy class of γ. In this section we shall make extensive use of this last remark in order to identify the translation numbers of elements in the fundamental groups of $\mathbb{R}$-graphs.

1.1 Realisations of free groups. There are many ways to realise the free group $F = F(a, b)$ as the fundamental group of a compact non-positively curved space. Perhaps the most natural way is to realise it as the fundamental group of one of the metric graphs shown in figure 1. We shall also need to consider the following realization of F as the fundamental group of a non-positively curved squared 2-complex T: consider the torus obtained by taking a Euclidean square of side length 2 and identifying opposite pairs of edges; label the loops obtained as the images of the sides of the square a and b; divide the square into four unit squares in the obvious way; delete the image of one of the newly-introduced edges from the quotient torus and take the completion of what remains.

Note that the boundary curve of T is a closed geodesic of length 2. Since this curve represents the conjugacy class of $[a, b]$ in F, for this realization of F we have $|a| = |b| = |[a, b]|$.

In contrast to the above 2-dimensional CAT(0) realization of F, we shall show that there is no 1-dimensional realization of F by a proper action with $|a| = |b| = |[a, b]|$.

Recall that an $\mathbb{R}$-tree is a geodesic metric space that does not contain any topologically embedded circles.

1.2 Proposition. *Let $w \mapsto |w|$ be the length function associated to an action of the free group $F = F(a, b)$ by isometries on an $\mathbb{R}$-tree. If the action is proper, then either $|[a, b]| \neq |a|$ or else $|[a, b]| \neq |b|$.*

Proof. First we recall that the union of the conjugacy classes of $c^{\pm 1} := [a, b]^{\pm 1}$ in F is independent of the chosen basis $\{a, b\}$. Indeed, one sees immediately that the images of c under the automorphisms $[a \mapsto a^{-1}, b \mapsto b]$, $[a \mapsto b, b \mapsto a]$, and $[a \mapsto ab, b \mapsto b]$ are all conjugate to $c^{\pm 1}$, and these automorphisms generate $Aut(F_2)$. Thus $|c| = |[\alpha, \beta]|$ for all bases $\{\alpha, \beta\}$ of F.

The idea of the proof is to take an arbitrary proper action of F by isometries on an $\mathbb{R}$-tree, look at a simplicial $\mathbb{R}$-graph G with the same length function, choose a convenient basis $\{\alpha, \beta\}$ for $\pi_1 G \cong F$, and then by examining the lengths of curves in G show that if $g, h \in F$ are such that $|g| = |h| = |[\alpha, \beta]|$ then $\{g, h\}$ is not a basis for F.

Whenever one has a group Γ acting by isometries on an $\mathbb{R}$-tree X so that $|\gamma| \neq 0$ for some $\gamma \in \Gamma$, there exists a unique minimal Γ-invariant subtree $T \subseteq X$ (see [CM] or [Ti]). If the action of Γ is proper, then T is a simplicial tree. The length function associated to the action of Γ on X is the same as that associated to its action on T. In the case $\Gamma = F_2$, the minimality of T means that $G = T/\Gamma$ is equal to its core (i.e., there is no proper subgraph

G' such that $G' \to G$ is a homotopy equivalence). Thus G is an $\mathbb{R}$-graph homeomorphic to one of the three shown in figure 1.

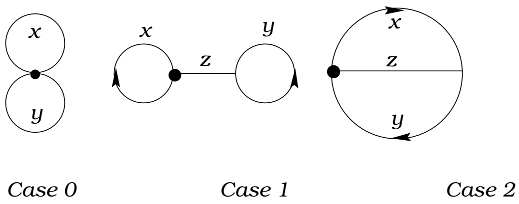

Figure 1: The $\mathbb{R}$-graphs of genus 2

We have labelled the edges by their lengths. We regard the first of these graphs as the degenerate case $z = 0$.

Case 1: We orient the edges as shown and regard F as the fundamental group of T/F with a basepoint at the vertex of x. Let α be the element of F corresponding to the loop x, and let β be the loop that crosses z, loops around y then returns along z. Note that $\{\alpha, \beta\}$ is a basis for F and $|\alpha| = x$, $|\beta| = y$, $|[\alpha, \beta]| = 2(x + y) + 4z =: L$.

We are interested in locally-injective loops of length L that represent the free homotopy classes of primitive[1] elements. Since α^n and β^n are not primitive if $|n| \neq 1$, we need not consider words of the form α^n or β^n. Given any other word $w = \alpha^{n_1}\beta^{m_1} \cdots \alpha^{n_\ell}\beta^{m_\ell}$, replacing it by its inverse and cyclically permuting (i.e. conjugating) it if necessary, we may assume that all of the exponents n_i, m_i are non-zero and $n_1 > 0$. Then,

$$|w| = x \sum_{i=1}^{\ell} |n_i| + y \sum_{i=1}^{\ell} |m_i| + 2\ell z \; .$$

We seek *primitive* words w such that $|w| = L$. There are only three cases to consider, $\ell = 0, 1, 2$; for if $\ell > 2$ then $\Sigma |n_i| > 2$ and $\Sigma |m_i| > 2$, so $|w| > L$.

The case $\ell = 1$ gives rise to the possibilities $w = \alpha^n \beta^m$. We claim that one cannot find a basis $\{a, b\}$ for F so that a is conjugate to $\alpha^n \beta^m$ and b is conjugate to $\alpha^{n'} \beta^{m'}$ where

$$|n|\, x + |m|\, y = |n'|\, x + |m'|\, y.$$

Indeed, without loss of generality we may assume $n > n' > 0$, hence $|m'| > |m|$. But this implies $|nm'| - |mn'| > 2$, whereas if $\{\alpha^n\beta^m, \alpha^{n'}\beta^{m'}\}$ were a basis for $\mathbb{Z}^2$ then the determinant of $\begin{pmatrix} n & n' \\ m & m' \end{pmatrix}$ would be ± 1.

If $\ell = 2$ then $|w| = L$ implies $\sum |n_i| = \sum |m_i| = 2$, so the only possibilities are $w = \alpha\beta^{\epsilon_1}\alpha^{\epsilon_2}\beta^{\epsilon_3}$ with $|\epsilon_i| = 1$. None of these possibilities maps to a primitive element in the abelianization of F.

Case 2: In this case we take α to be the loop that follows x and then returns to the basepoint along z, and we take β to be the loop that runs along z and then returns along y. In this setting we have

$$|[\alpha, \beta]| = 2(x + y + z) \; .$$

[1]An element of a free group is called primitive if it belongs to some free basis of the group.

Arguing as in case 1, we are left to examine the primitivity of words $w = \alpha^{n_1}\beta^{m_1}\cdots\alpha^{n_\ell}\beta^{m_\ell}$ with $n_1 > 0$ and all exponents non-zero. In case 2, the length of the geodesic loop representing w is

$$|w| = x\sum_{i=1}^{\ell}|n_i| + y\sum_{i=1}^{\ell}|m_i| + \sigma z\ ,$$

where σ is the number of subwords in w read cyclically of the form $\alpha^\varepsilon\alpha^\varepsilon, \beta^\varepsilon\beta^\varepsilon$, $\alpha^\varepsilon\beta^{-\varepsilon}$ or $\beta^\varepsilon\alpha^{-\varepsilon}$, where $\varepsilon = \pm 1$ and the subwords may overlap; if $w = \alpha$ or $w = \beta$, then by definition $\sigma = 1$.

The equality $|w| = 2(x + y + z)$ can only arise in the following cases.

- If $\sigma = 0$, then $w = (\alpha\beta)^n$ with $n \geq 2$.
- If $\sigma = 1$, then $w = (\alpha\beta)^n\beta$ or $\alpha(\alpha\beta)^n$, with $n \geq 2$.
- If $\sigma = 2$, then $w = \alpha\beta\alpha^{-1}\beta^{-1}$, $\alpha\beta\alpha^{-1}\beta$, $\alpha^3\beta$, $\alpha\beta^3$, $\alpha^2\beta^2$, $\alpha\beta^{-1}\alpha^{-1}\beta^{-1}$ or $\alpha\beta^{-1}\alpha^{-1}\beta$.
- If $\sigma \geq 3$, then $w = \alpha^m\beta$ or $\alpha\beta^m$, with $m \geq 4$, else $w = \alpha^r\beta^{-1}$ or $\alpha\beta^{-r}$ with $r \geq 2$.

We are looking for primitive elements, so we can ignore those of the above words whose abelianization is trivial or a proper power. This leaves us $(\alpha\beta)^n\beta$ and $\alpha(\alpha\beta)^n$ with $n \geq 2$, and $\alpha^m\beta$, $\alpha\beta^m$ with $m \geq 3$, and $\alpha^r\beta^{-1}$, $\alpha\beta^{-r}$ with $r \geq 2$.

Note that

$$\begin{aligned} |(\alpha\beta)^n\beta| &= nx + (n+1)y + z \ \text{ and} \\ |\alpha(\alpha\beta)^n| &= (n+1)x + ny + z. \end{aligned}$$

And in order for these lengths to equal $2(x+y+z)$, it must be that $z > \max\{x, y\}$, except that if $n = 2$ then $x = y = z$.

On the other hand,

$$\begin{aligned} |\alpha^m\beta| &= mx + y + (m-1)z \ \text{ and} \\ |\alpha^r\beta^{-1}| &= rx + y + (r+1)z \end{aligned}$$

can equal $2(x + y + z)$ only if $y > \max\{x, z\}$, except if $r = 2$ or $m = 3$, when $x = y = z$.

Similarly,

$$\begin{aligned} |\alpha\beta^m| &= x + my + (m-1)z \ \text{ and} \\ |\alpha\beta^{-r}| &= x + ry + (r+1)z \end{aligned}$$

can equal $2(x + y + z)$ only if $x > \max\{y, z\}$, except if $r = 2$ or $m = 3$, when $x = y = z$.

We are trying to prove that $F(\alpha, \beta)$ does not have a basis consisting of two elements of translation length $2(x + y + z)$. By comparing the compatibility of the above conditions we see that the only pairs of our words that can simultaneously have translation length $2(x + y + z)$ are

$$\{(\alpha\beta)^n\beta,\ \alpha(\alpha\beta)^n\},\ \{\alpha^m\beta^{-1},\ \alpha^{m+1}\beta\},\ \{\alpha\beta^{-m},\ \alpha\beta^{m+1}\},$$

with $n \geq 2$ and $m \geq 3$, and those pairs arising from the exceptional setting $x = y = z$. The words arising in the exceptional setting have abelianization $\alpha^p\beta^q$ where $\{p, q\} = \{1, 3\}$, $\{2, 3\}$ or $\pm\{1, -2\}$.

Calculating determinants, we find that the areas of the parallelogram in $\mathbb{R}^2$ spanned by any pair of these vectors is 0, 3, 5, 7 or 8. And the determinants for the other pairs of words displayed above are, respectively $(2n + 1)$, $(2m + 1)$ and $(2m + 1)$. Since none of these expression is equal to 1, we conclude that in case 2 there does not exist a basis for the free group in which both elements have translation length $2(x + y + z)$.

This completes the proof that there does not exist an $\mathbb{R}$-graph with fundamental group $F(a, b)$ in which the basis $\{a, b\}$ satisfies $|a| = |b| = |[a, b]|$. □

2. The groups Γ and $\hat{\Gamma} \leq \Gamma$, and the covering space $\hat{X} \to X$

2.1. Consider the group

$$\Gamma = \langle a, b, \gamma, s, t \mid \gamma a \gamma^{-1} = a^{-1}, \gamma b \gamma^{-1} = b^{-1}, sas^{-1} = [a, b] = tbt^{-1} \rangle \ .$$

This is the fundamental group of the compact aspherical 2-complex that one obtains as follows. First join two copies of the Klein bottle along an embedded circle as shown in figure 2 (the labelled loops in this figure correspond to the generators in the presentation given above). Fix a loop C in the complex that represents the homotopy class of $[a, b]$ and attach two cylinders to the complex as shown in figure 3. Let X be the resulting 2-complex. (Note that in our description of X there is an ambiguity in the choice of how we attach the cylinders — for example C may follow the edge path labelled $a^{-1}b^{-1}ab$, or it may be the loop shown in figure 4 — to be definite we shall assume the latter choice.)

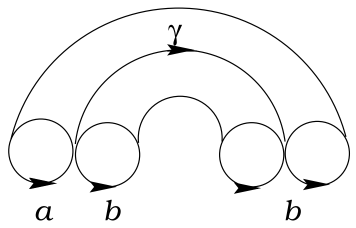

Figure 2: Two Klein bottles joined along a circle

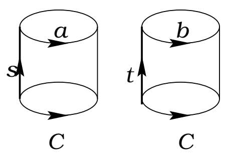

Figure 3: The two cylinders

2.2. One can map Γ onto $\mathbb{Z}_2$ by sending γ to the generator of $\mathbb{Z}_2$ and killing all other generators of Γ. The covering space $\hat{X} \to X$ corresponding to the kernel of this map is shown in figure 4 — it is obtained by first taking two tori joined along a meridian (the loop representing γ^2); the deck-group $\langle I \rangle$ preserves each torus; the two lifts of C that are

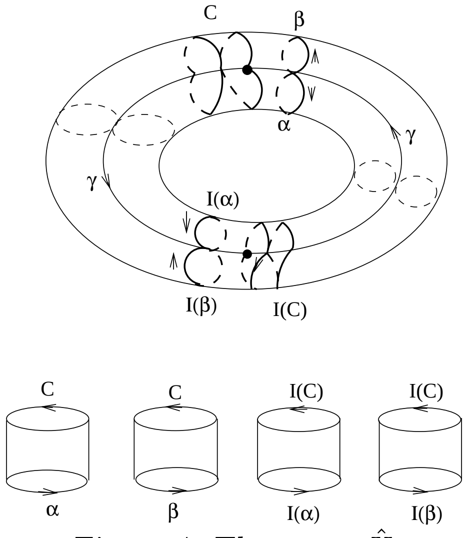

Figure 4: The space $\hat{X}$

drawn are interchanged by I, as are the loops $\{\alpha, I(\alpha)\}$, $\{\beta, I(\beta)\}$ drawn to represent the homotopy classes of $a^{\pm1}$ and $b^{\pm1}$; the involution I extends naturally to the cylinders that are drawn.

By the van Kampen theorem (equivalently, the Reidemeister–Schreier rewriting process [LS]) we see that $\hat{\Gamma} = \pi_1 \hat{X}$ has presentation:

$$\hat{\Gamma} = \langle a, b, y, s_1, s_2, t_1, t_2 \mid [a, y] = [b, y] = 1,\ s_1 a s_1^{-1} = t_1 b t_1^{-1} = [a, b],\ s_2 a s_2^{-1} = t_2 b t_2^{-1} = [b, a] \ \rangle$$

Here, $y = \gamma^2$ in Γ. Also, $s_1 = s$, $t_1 = t$, and $s_2 = (\gamma ab)^{-1} s\gamma$, $t_2 = (\gamma ab)^{-1} t\gamma$ in Γ.

2.3. A non-positively curved space with fundamental group $\hat{\Gamma}$

Consider the skew tori obtained by identifying the sides of the parallelograms shown in figure 5, and let Y be the piecewise-Euclidean 2-complex obtained by gluing these tori along the circles labelled y. (The length of the side labelled y is not important.)

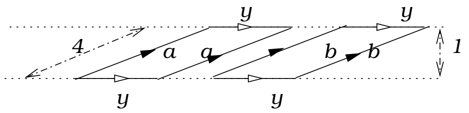

Figure 5: The skew tori

Figure 6 shows part of the universal cover of Y with a lift of a geodesic loop representing the conjugacy class of the commutator $[a, b] = a^{-1}b^{-1}ab$.

The important point to note is that $[a, b]$ translates its axes (which are orthogonal to the axes for y) a distance $4 = |a| = |b|$. Thus, noting that $[a, b] = [b, a]^{-1}$, we have the following equalities for translation lengths in $\pi_1 Y$:

$$|a| = |b| = |[a, b]| = |[b, a]| \, .$$

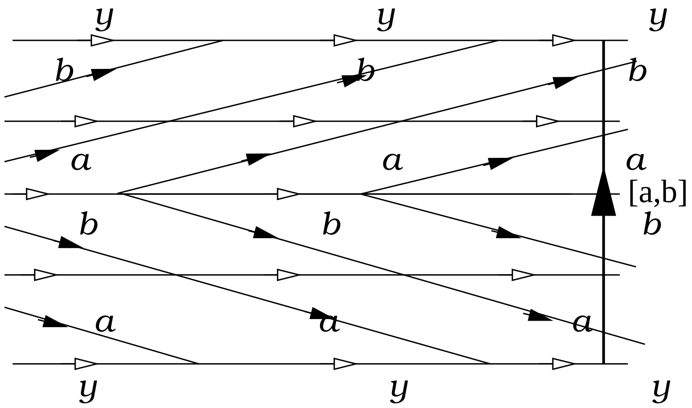

Figure 6: A lift of a geodesic for $[a,b]$

Finally, let $\bar{Y}$ be the piecewise-Euclidean 2-complex obtained by attaching to Y along geodesic circles four cylinders that realise the conjugacies $s_1^{-1}as_1 = [a,b]$, $t_1^{-1}bt_1 = [a,b]$, $s_2^{-1}bs_2 = [b,a]$ and $t_2^{-1}bt_2 = [b,a]$.

2.4 Proposition.

(1) $\bar{Y}$ *is non-positively curved.*
(2) $\pi_1\bar{Y} = \hat{\Gamma}$.
(3) $\bar{Y}$ *is homotopy equivalent to* $\hat{X}$.

Proof. (1) is an immediate consequence of the local gluing lemma for non-positively curved spaces ([BH], II.11.13), and (2) is immediate from van Kampen's theorem (the isomorphism is implicit in our labelling of loops). (3), of course, follows from general considerations, because $\hat{Y}$ and $\hat{X}$ are both models for $K(\hat{\Gamma},1)$, but it can also be seen directly (cf. the parenthetical remark at the end of paragraph (2.1)). □

3. Proof of the Theorem

The theorem stated in the introduction is an immediate consequence of Propositions 2.4, 3.1 and 3.3.

3.1 Proposition. *The group* $\Gamma = \pi_1 X$ *constructed in paragraph (2.1) does not act properly by semisimple isometries on any 2-dimensional* CAT(0) *space.*

Proof. Suppose that such an action were to exist, say $\Gamma \to \mathrm{Isom}(M)$, and consider

$$\mathrm{Min}(\gamma^2) = \left\{p \in M \;\middle|\; d(\gamma^2\cdot p, p) = |\gamma^2|\right\}.$$

According to ([BH], II.6.2), there is an isometric splitting $\mathrm{Min}(\gamma^2) = N \times \mathbb{R}$. Moreover $C(\gamma^2) \subseteq \Gamma$, the centralizer of γ^2, preserves $\mathrm{Min}(\gamma^2)$ and its splitting. The action of $C(\gamma^2)$ on the second factor $\mathbb{R}$ is by translations. $C(\gamma^2)$ obviously contains $gp\{\gamma,a,b\}$ (in fact this subgroup is the whole of $C(\gamma^2)$). Since $\gamma a\gamma^{-1} = a^{-1}$ and $\gamma b\gamma^{-1} = b^{-1}$, the image of both a and b in the torsion-free abelian group $\mathrm{Isom}_+\mathbb{R}$ is trivial. Thus the free subgroup $F := gp\{a,b\} \subseteq \Gamma$ leaves $N = N \times \{0\} \subseteq M$ invariant.

Because $N \subseteq M$ is convex, the restriction of every $w \in F$ to N has the same translation length as $w \in \mathrm{Isom}(M)$ (see [BH], II.6.2). Moreover, the action of F on N is proper ([BH], II.6.10).

In Γ the elements a, b and $[a, b]$ are conjugate, therefore the translation numbers of these elements (as isometries of M and N) are equal. But since $N \times \mathbb{R} \subseteq M$ is at most 2-dimensional, N is an $\mathbb{R}$-tree (see lemma below) and thus we have a contradiction to Proposition 1.1. □

In the following statement the product $X = Y \times \mathbb{R}$ is assumed to be a metric product, i.e. the metric on X is related to the metrics on Y and $\mathbb{R}$ by the Pythagorean formula.

3.2 Lemma. *If $X = Y \times \mathbb{R}$ is a 2-connected geodesic space with covering dimension $\dim X = 2$, then Y is an $\mathbb{R}$-tree.*

Proof. Y is convex in X and therefore it is a geodesic space. If it were not an $\mathbb{R}$-tree then it would contain a topologically embedded circle $f : \mathbb{S}^1 \to Y$. Identify $\mathbb{S}^2$ with the quotient of $\mathbb{S}^1 \times [-1, 1]$ by the relation that collapses $\mathbb{S}^1 \times \{-1\}$ to a point and $\mathbb{S}^1 \times \{1\}$ to a point. Extend f to $F : \mathbb{S}^2 \to Y \times \mathbb{R}$ so that $F(\theta, t) = (f(\theta), t)$ for $|t| \leq 1/2$, and $F|_{[-1,-1/2]}$ and $F|_{[1/2,1]}$ are null-homotopies of $(f(\mathbb{S}^1), \pm 1/2)$ in $Y \times \{\pm 1/2\}$.

Let $G : F(\mathbb{S}^2) \to \mathbb{S}^2$ be the continuous map that sends $F(\theta, t)$ to $(\theta, 2t)$ for $|t| \leq 1/2$ and is constant on each of the remaining components of $F(\mathbb{S}^2)$. Note that $G \circ F$ is homotopic to $\mathrm{id}_{\mathbb{S}^2}$. Since $F(\mathbb{S}^2)$ is null-homotopic in X (we assumed $\pi_2 X = 0$), the map G does not have a continuous extension to X. This contradicts the fact that $\dim X \leq 2$ (see [HW], Theorem VI.4). □

3.3 Proposition. *The group Γ described in (2.1) is the fundamental group of a compact, non-positively curved, 3-dimensional cubed complex.*

Proof. Consider the squared complex T described in paragraph 1.1. There is a unique orientation-preserving isometry of T that interchanges the two 1-cells in the boundary curve of T; this isometry is a cellular map, call it ϕ. The outer automorphism of $F(a, b)$ induced by ϕ is the outer automorphism class of the automorphism $a \mapsto a^{-1}$, $b \mapsto b^{-1}$.

Consider the mapping cylinder $M(\phi)$ of φ: this is the non-positively curved three dimensional cube complex obtained by taking the quotient of $T \times [0, 1]$ by the relation $[(x, 0) \sim \big(\varphi(x), 1\big)\ \forall x \in T]$. There are closed geodesics in the 1-skeleton of $T \times \{0\} \subset M(\varphi)$ representing a, b and $[a, b]$. Let $A = S(2) \times [0, 1]$, where $S(2)$ is a circle of length 2. Let Z be the cubed complex obtained from $M(\varphi)$ by attaching the ends of two copies of the cylinder A to the loops representing a and $[a, b]$ (respectively b and $[a, b]$) by maps that are cellular isometries with respect to the obvious squared-complex structure on A.

Proposition II.11.13 of [BH] assures us that Z is non-positively curved, and van Kampen's theorem tells us that $\pi_1 Z \cong \Gamma$ (and in fact it is easy to construct a homotopy equivalence from T to X). □

References

[Ba] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature,*, DMV Seminar Band 25, Birkhäuser, Basel, 1995.

[BC] N. Brady and J. Crisp, *Two-dimensional Artin groups with CAT(0) dimension three*, preprint, Université de Dijon, August 2000.

[Br] M.R. Bridson, *On the semisimplicity of polyhedral isometries*, Proc. Amer. Math. Soc. **127**, 2143–2146.

[BH] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grund. Math. Wiss. 319, Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[Ch] I.M. Chiswell, *Abstract length functions in groups*, Math. Proc. Camb. Phil. Soc. **80** (1976), 451–463.

[CM] M. Culler and J.W. Morgan, *Group actions on $\mathbb{R}$-trees*, Proc. London Math. Soc. (3) **55** (1987), 571–604.

[HW] W. Hurewicz and H. Wallmann, *Dimension Theory*, Princeton Univ. Press, Princeton NJ, 1941.

[Ki] I. Kim, *Marked length specturm of rank 1 symmetric spaces and their products*, Topology, to appear.

[LS] R.C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer Verlag, Berlin-Heidelberg-New York, 1977.

[Ot] J-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. **131** (1990), 151–162.

[Th] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), 417–431.

[Ti] J. Tits, *A 'Theorem of Lie-Kolchin' for trees*, Contributions to algebra: papers dedicated to Ellis Kolchin, Academic Press, New York, 1977, pp. 377–388.

Mathematical Institute, 24–29 St Giles', Oxford, OX1 3LB