THE CONJUGACY AND ISOMORPHISM PROBLEMS FOR COMBABLE GROUPS

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Abstract. There exist combable groups in which the conjugacy problem is unsolvable. The isomorphism problem is unsolvable for certain recursive sequences of finite presentations of combable groups.

Introduction

The class of combable groups is one of a number of classes that have been extensively studied in the last fifteen years in connection with manifestations of non-positive curvature in group theory. The other classes closely associated to it are word-hyperbolic groups, automatic groups, semihyperbolic groups, bicombable groups, and the fundamental groups of compact non-positively curved (orbi)spaces.

Each of these classes is defined in terms of a convexity condition, the common core of which is the "fellow-traveller" condition that forms the definition of a combable group: a group Γ with finite generating set $\mathcal{A}$ is said to be *combable* if there is a family of words $\{\sigma_\gamma \mid \gamma \in \Gamma\}$ in the letters $\mathcal{A}^{\pm 1}$ and a constant $k > 0$ such that for each $\gamma \in \Gamma$ and $a \in \mathcal{A}$, the paths in the Cayley graph $\mathcal{C}_{\mathcal{A}}\Gamma$ that begin at the identity vertex and are labelled σ_γ and $\sigma_{\gamma a}$ remain uniformly k-close.

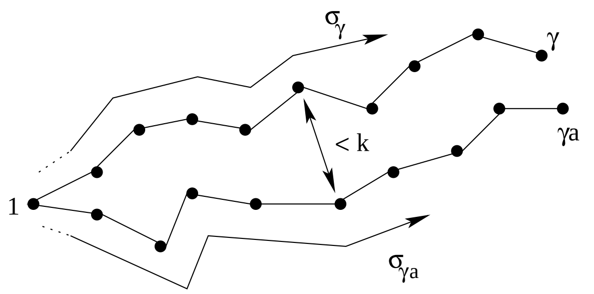

Figure 1: the fellow-traveller property

The results in [5] established that the class of combable groups is strictly larger than all of the other classes listed above. There is an effective solution to the word problem in any combable group [8], [4].

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The conjugacy problem is known to be solvable in all of the above classes except for the automatic groups and the combable groups, in which cases it has been the subject of considerable speculation.

Theorem A. *There exist combable groups in which the conjugacy problem is unsolvable.*

The smallest and most intensively studied of the above classes is that formed by the hyperbolic groups. A remarkable theorem of Zlil Sela shows that the isomorphism problem is solvable among hyperbolic groups[1] [17]. In the light of the work of Farrell and Jones on topological rigidity [9], Sela's result implies that the homeomorphism problem is solvable among closed (high-dimensional) manifolds that admit metrics of negative curvature. The results of Farrell and Jones remain valid for non-positively curved manifolds, and there is therefore considerable interest in the open question of whether or not the isomorphism problem is solvable among presentations of the fundamental groups of such manifolds — cf. [6] pp. 494. The solvability of the isomorphism problem in the related classes of groups listed above has also remained open.

Theorem B. *The isomorphism problem is unsolvable in the class of combable groups.*

In order to prove this theorem we shall exhibit recursive sequences of finite subsets $\mathcal{R}_n$ of a finitely generated free group $F = F(\mathcal{A})$ such that each of the groups $\Gamma_n = \langle \mathcal{A} \mid \mathcal{R}_n \rangle$ is combable, $n = 1, 2, \dots$, but there is no algorithm to decide which are isomorphic to Γ_1. In order to obtain such sequences we combine the construction of [5] with a suitable encoding of the fact that there is no algorithm to decide which m-element subsets generate a direct product of free groups, and likewise for certain hyperbolic groups.

The groups Γ_n that we construct are neither bicombable nor automatic, and the isomorphism problem remains open for these classes.

1. The Seeds of Undecidability

The existence among combable groups of the undecidability phenomenon asserted in Theorem B will be established by means of a suitable encoding of the fact that many finitely presented groups Γ associated with non-positive curvature have the following property:

> **Ψ:** If X is a finite generating set for Γ and m is a sufficiently large integer, then there exists a recursive sequence (S_n) of subsets of the free group $F(X)$ such that each S_n has cardinality m and there is no algorithm to determine whether or not $\langle S_n \rangle = \Gamma$; moreover if equality fails then $\langle S_n \rangle$ is not finitely presented.

[1] In the reference cited it is assumed that the groups are torsion-free and freely indecomposable but Sela has a proof in the general case.

Proposition 1.1. *If F is a non-abelian free group, $F \times F$ has property Ψ.*

This result is due to C.F. Miller III ([13], [12] page 194 and [14]).

Proof. It is enough to consider $X = \{(a,1),(1,a) \mid a \in A\}$ where A is a basis for F. Associated to each finite group presentation $\mathcal{P} = \langle A \mid R\rangle$ one has the fibre product $P = \mathrm{gp}\{(a,a),(r,1) \mid a \in A, r \in R\} \subset F \times F$. Let $S(\mathcal{P})$ be (the obvious words representing) the given generators of P. If $\mathcal{P}$ is trivial then $P = F \times F$. If $\langle A \mid R\rangle$ is an infinite group then P is not finitely presented (see [11]).

It is well known that there exist recursive sequences of finite presentations $\mathcal{P}_n = \langle A \mid R_n\rangle$, with R_n of fixed cardinality, such that each group presented is either trivial or infinite and there is no algorithm to recognise which are trivial (see [13]). Define $S_n = S(\mathcal{P}_n)$. $\square$

By applying a modification of the Rips construction [16] to the standard presentation of $F \times F$ one can translate the undecidability expressed in Proposition 1.1 into the realm of hyperbolic groups (cf. [2] and [7]).

Proposition 1.2. *There exists a compact negatively curved 2-complex whose fundamental group has property Ψ.*

Recall that a subgroup H of a finitely generated group Γ is said to have an *unsolvable membership problem* if there does not exist an algorithm that takes as input words w in the generators of Γ and decides whether or not $w \in H$.

Notation. We write $Z_G(g)$ to denote the centralizer in a group G of an element $g \in G$ and $Z_G(S)$ to denote the centralizer of a subset $S \subset G$.

Lemma 1.3. *If F is a non-abelian free group then there exist finitely generated subgroups $H \subset F \times F$ and elements $h \in H$ such that H has an unsolvable membership problem and $Z_{F\times F}(h) \subset H$.*

Proof. Let A be a basis for F and let $\langle A \mid R\rangle$ be a finitely presented group whose word problem is unsolvable. Consider the associated fibre product $P \subset F \times F$, as in the proof of (1.1). Now, given a word w in the letters $A^{\pm 1}$, ask if $(w,1) \in P$. The answer is YES if and only if $w = 1$ in $\langle A \mid R\rangle$, and the validity of this equality cannot be determined algorithmically. Define $H = P$ and $h = (a,a)$, where $a \in A$. $\square$

Once more, by applying the Rips construction one can translate the above phenomenon into the realm of hyperbolic groups — see [6] p.488.

Lemma 1.4. *There exists a compact negatively curved 2-complex whose fundamental group Γ has the following property: there is a finitely generated subgroup $H \subset \Gamma$ and an element $h \in H$ such that H has an unsolvable membership problem and $Z_\Gamma(h) \subset H$.*

2. Further Preliminaries

This section contains four lemmas that we shall need in the proofs of Theorems A and B. The first two lemmas concern conjugacy and centralizers in amalgamated free products. These are special cases of standard results that are proved in a combinatorial manner in [12], for example; they also admit simple geometric proofs in the language of Bass-Serre theory.

Consider an amalgamated free product $\Gamma = A *_C B$.

Lemma 2.1. *For all $a, a' \in A \smallsetminus C$ and all $b, b' \in B \smallsetminus C$, if ab is conjugate to $a'b'$ in Γ then there exists $c \in C$ such that $c(ab)c^{-1} = a'b'$.*

Lemma 2.2. *If $S \subset A \smallsetminus C$ then $Z_\Gamma(S) = Z_A(S)$.*

Our third lemma also involves Bass-Serre theory. Recall that a group is said to have *property FA* if every action of the group on a tree has a fixed point (equivalently, the group cannot be written as a non-trivial amalgamated free product or HNN extension).

Proposition 2.3. *Let A be a group that has property FA, let Q be a group such that there are no non-trivial homomorphisms $A \to Q$, and let B be a group that does not contain an isomorphic copy of A. Let $G = (A * B) \times Q$, let $\phi : Q \to B$ be a homomorphism, and let $\hat{Q} = \{(\phi(q), q) \mid q \in Q\} \subset G$. Let D be the amalgamation of two copies of G along $\hat{Q}$.*

Then $K_\phi := \ker \phi \subset \hat{Q}$ is a characteristic subgroup of D. More precisely, K_ϕ is the intersection of the centralizers of those subgroups $H \subset D$ that are isomorphic to A and are not strictly contained in any subgroup isomorphic to A.

Proof. First we claim that H must lie in a conjugate of one of the obvious copies of G in D. To see that this is the case, consider the action of $D = G *_{\hat{Q}} \overline{G}$ on the Bass-Serre tree associated to the given decomposition: $H \cong A$ has property FA and hence is contained in a vertex stabilizer.

Now, since there are no non-trivial homomorphisms from A to Q, if $H \subset G = (A * B) \times Q$ is isomorphic to A then $H \subset A * B$. Since A has property FA, it is *a fortiori* freely indecomposable, so the Kurosh subgroup theorem tells us that $H \cong A$ is contained in a conjugate of one of the vertex groups $A, B \subset A * B$. And since B does not contain a copy of A, only the former possibility can occur. It follows that if H is maximal among subgroups isomorphic to A then it must equal a conjugate of A. Thus the maximal $H \subset D$ isomorphic to A are exactly the conjugates of the obvious subgroups $A \subset G \subset D$ and $\overline{A} \subset \overline{G} \subset D$.

It remains to calculate the intersection of the centralizers of the conjugates of A and $\overline{A}$. It is easy to check that this intersection contains $\ker \phi \subset \hat{Q}$. For the reverse inclusion, note that since $A \cap \hat{Q}$ is trivial, the centralizer of A in $D = G *_{\hat{Q}} \overline{G}$ is simply its centralizer in G (Lemma 2.2), which is $Z(A) \times Q$.

Likewise, if $b \in B$ is non-trivial, then the centralizer of $A^b = bAb^{-1}$ in D is $Z_G(bAb^{-1}) = bZ_G(A)b^{-1} \times Q$. Thus the centralizer in D of $A \cup A^b$ is Q. Similarly, the centralizer in D of $\overline{A} \cup \overline{A}^b$ is $\overline{Q}$. And $Q \cap \overline{Q} = Q \cap \hat{Q} = \ker \phi$. $\square$

In our proof of Theorem B we shall need the following variation on Rapaport's Theorem [15].

Lemma 2.4. *Let Q be a group generated by $\{g_1, \dots, g_m\}$ and let F be a free group with basis $\{\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m\}$. Let $\phi : F \to Q$ be an epimorphism such that $\phi(\tau_i) = 1$ for $i = 1, \dots, m$. Then there exists an automorphism $\Phi : F \to F$ such that $\phi \circ \Phi(\sigma_i) = 1$ and $\phi \circ \Phi(\tau_i) = g_i$ for $i = 1, \dots, m$.*

Proof. Since ϕ is onto, for each $i = 1, \dots, m$ there exists a word u_i in the letters σ_j and their inverses such that $g_i = \phi(u_i)$ in Q. And since the g_j generate Q, for each $i = 1, \dots, m$ there exists a word v_i in the letters g_j and their inverses such that $\phi(\sigma_i) = v_i$ in Q.

Let $\Phi_1 : F_{2m} \to F_{2m}$ be the automorphism that sends each σ_i to itself and sends τ_i to $\tau_i' := \tau_i u_i$ for $i = 1, \dots, m$.

Let V_i be the word obtained from v_i by substituting the word τ_j' for each occurrence of the symbol g_j, for $j = 1, \dots, m$.

Let $\Phi_2 : F_{2m} \to F_{2m}$ be the automorphism that sends each τ_i' to itself and sends σ_i to $\sigma_i' := \sigma_i V_i^{-1}$ for $i = 1, \dots, m$.

(To see that Φ_1 and Φ_2 are automorphisms, note that each is the composition of right Nielsen moves.)

Define $\Phi = \Phi_2 \circ \Phi_1$. $\square$

3. Unsolvable Conjugacy Problems

In this section we shall prove Theorem A.

Definition 3.1. Given a group G and a subgroup $H \subset G$, we write $\Delta(G; H)$ to denote the *double of G along H*, i.e. the free product of two copies of G, written G and $\overline{G}$, with the subgroups H and $\overline{H}$ amalgamated by the identification $h = \overline{h}\ \forall h \in H$.

Theorem 3.2. *Let N and F be finitely generated groups and let $\phi : F \to N$ be a homomorphism. Let $\hat{F} = \{(\phi(x), x) \mid x \in F\} \subset N \times F$.*

If the membership problem for im ϕ *is unsolvable and there exists $a_0 \in N$ such that $Z_N(a_0) \subset \text{im } \phi$, then $D = \Delta(N \times F; \hat{F})$ has an unsolvable conjugacy problem.*

Proof. Let $I = \text{im } \phi$. We claim that our hypotheses imply that the following problem is algorithmically unsolvable:

$$\boxed{\text{Given}^2\ b \in N, \text{ determine whether } \exists h \in I \text{ with } h a_0 h^{-1} = b a_0 b^{-1}.}$$

To see this, first note that $ha_0h^{-1} = ba_0b^{-1}$ if and only $h^{-1}b \in Z_D(a_0)$. Since h and $Z_D(a_0)$ are assumed to lie in I, this means that such an h exists if and only if $b \in I$. And we are assuming that there is no algorithm to decide whether $b \in I$.

We adopt the notation $\hat{x} = (\phi(x), x)$ for elements of $\hat{F}$. Identifying N and F with their images in $N \times F$ we have

$$\hat{x} n \hat{x}^{-1} = \phi(x)\, n\, \phi(x)^{-1}\ \forall n \in N\ \forall x \in F.$$

To see that the conjugacy problem in D is unsolvable we consider the following sub-problem:

$$\boxed{\text{given } b \in N, \text{ decide if } (ba_0b^{-1})(\overline{ba_0b^{-1}}) \text{ is conjugate to } a_0\overline{a}_0 \text{ in } D.}$$

Lemma 2.1 tells us that $(ba_0b^{-1})(\overline{ba_0b^{-1}})$ is conjugate to $a_0\overline{a}_0$ in $D = (N \times F) *_{\hat{F}} (N \times F)$ if and only if there exists $\hat{x} \in \hat{F}$ conjugating one element to the other, i.e.

$$\hat{x} a_0 \overline{a}_0 \hat{x}^{-1} = (ba_0b^{-1})(\overline{ba_0b^{-1}}).$$

And noting that $\hat{x} = \overline{\hat{x}}$, we have

$$\hat{x} a_0 \overline{a}_0 \hat{x}^{-1} = (\hat{x} a_0 \hat{x}^{-1})\overline{(\hat{x} a_0 \hat{x}^{-1})} = (\phi(x) a_0 \phi(x)^{-1})\overline{(\phi(x) a_0 \phi(x)^{-1})}.$$

Thus $(ba_0b^{-1})(\overline{ba_0b^{-1}})$ is conjugate to $a_0\overline{a}_0$ in D if and only if there exists an element $h \in I$ such that $(ba_0b^{-1})(\overline{ba_0b^{-1}}) = (ha_0h^{-1})(\overline{ha_0h^{-1}})$. Moreover, since $N, \overline{N} \subset D$ do not intersect the amalgamated subgroup $\hat{F}$, the subgroup they generate is the free product $N * \overline{N}$. Therefore $(ba_0b^{-1})(\overline{ba_0b^{-1}}) = (ha_0h^{-1})(\overline{ha_0h^{-1}})$ if and only if $ba_0b^{-1} = ha_0h^{-1}$.

Thus we have shown that, given $b \in N$, the elements $(ba_0b^{-1})(\overline{ba_0b^{-1}})$ and $a_0\overline{a}_0$ are conjugate in D if and only if there exists $h \in I$ such that $ba_0b^{-1} = ha_0h^{-1}$. In the first paragraph of this proof we explained that there is no algorithm to decide whether such an h exists. □

We use the term *semihyperbolic* in the sense of [1]. In particular, biautomatic groups are semihyperbolic, as are the fundamental groups of compact non-positively curved (orbi)spaces.

As a special case of the main construction in [5] we have the following:

Theorem 3.3. *Let N and F be groups, let $\phi : F \to N$ be a homomorphism, and let $\hat{F} = \{(\phi(x), x) \mid x \in F\}$.*

If N is semihyperbolic and F is combable, then $\Delta(N \times F; \hat{F})$ is combable.

3.4. The Proof of Theorem A

Lemmas 1.3 and 1.4 provide examples of semihyperbolic groups N and finitely generated subgroups $H \subseteq N$ such that the membership problem for H is unsolvable, and there exists $h \in H$ such that $Z_N(h) \subset H$. Fix a surjection $\phi : F \to H$, where F is a finitely generated free group. Theorem 3.3 tells us that $D = \Delta(N \times F; \hat{F})$ is combable and Theorem 3.2 tells us that D has an unsolvable conjugacy problem. □

Remark 3.5. The results in [5] show that each of the examples D constructed in the above proof has a cubic Dehn function and is Ind-combable, where Ind is the class of indexed languages. At the time of writing, I do not know of similar examples that are Reg-combable, i.e. automatic.

4. Unsolvable Isomorphism Problems

Our purpose in this section is to prove Theorem B. We shall achieve this by applying the following general criterion to the groups from Section 1 that satisfy Property Ψ.

Theorem 4.1. *Let A be a group with property FA and let B be a finitely presented group that does not contain a subgroup isomorphic to A. If B has property Ψ and F is a finitely generated subgroup of sufficiently large rank $2m$, then there is a recursive sequence of subsets $S_n = \{s_{n,1}, \dots, s_{n,m}\} \subset B$ such there is no algorithm to determine which of the groups*

$$\Gamma_n = \Delta((A * B) \times F; \Sigma_n)$$

are isomorphic, where $\Sigma_n \subset B \times F$ is the subgroup $\langle (s_{n,i}, \sigma_i), (1, \tau_i); i = 1, \dots, m\rangle$, with $\{\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m\}$ a fixed basis for F.

Before turning to the proof of this result we note that the groups Γ_n are obtained in a natural way from a fixed free group by adding a fixed number of relations. Specifically, if we fix a presentation $\langle \mathcal{X} \mid \mathcal{R}\rangle$ for $(A * B) \times F$, with $\{\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m\} \subset \mathcal{X}$ a basis for F, then

$$\Gamma_n = \langle \mathcal{X}, \overline{\mathcal{X}} \mid \mathcal{R}, \overline{\mathcal{R}}; \tau_i = \overline{\tau}_i,\ \sigma_i s_{n,i} = \overline{\sigma_i s_{n,i}} \text{ for } i = 1, \dots, m\rangle.$$

Proof. F is the free group with basis $\{\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m\}$. The subgroup $\Sigma_n \subset B \times F$ is $\{(\phi_n(x), x) \mid x \in F\}$, where $\phi_n : F \to B$ is the homomorphism that sends each σ_i to $s_{n,i} \in S_n$ and sends each τ_i to the identity. The hypothesis that B has property Ψ means that we may assume that there is no algorithm to determine which of the subgroups $\langle S_n\rangle$ are equal to B; we may also assume that those which are not equal are not finitely presentable. We proceed with these assumptions in place. *Scheme of Proof:* In Step 1 we prove that if $\langle S_n\rangle \subset B$ is finitely presentable and $\langle S_{n'}\rangle \subset B$ is not, then $\Gamma_n \ncong \Gamma_{n'}$. In Step 2 we argue that if $\langle S_n\rangle \subset B$ and $\langle S_{n'}\rangle \subset B$ are both finitely presentable (hence equal to B) then $\Gamma_n \cong \Gamma_{n'}$. Since there is no algorithm to determine which of the subgroups $\langle S_n\rangle \subset B$ are finitely presentable, this will complete the proof.

Step 1: Proposition 2.3 tells us that any isomorphism from Γ_n to $\Gamma_{n'}$ must map $\ker \phi_n \subset \Sigma_n$ to $\ker \phi_{n'}$ isomorphically. (We may apply Proposition 2.3 because no group with property FA admits a non-trivial homomorphism to a free group.) Thus we shall be done if we can prove that $\ker \phi_n$ is the normal closure of a finite subset of Γ_n if and only if $\langle S_n\rangle$ is finitely presentable.

ϕ_n is an epimorphism from the finitely generated free group F to the group $\langle S_n\rangle$. Thus $\ker\phi_n$ is the normal closure of a finite subset of F if and only if $\langle S_n\rangle$ is finitely presentable. (The finite presentability of a group is independent of the choice of finite generating set.)

There is a natural retraction of Γ_n onto F: first, writing $G = (A * B) \times F$, retract $\Gamma_n = G *_{\Sigma_n} \overline{G}$ onto G by sending both $g \in G$ and $\overline{g} \in \overline{G}$ to g for every $g \in G$; now project onto the second direct summand in G.

If a subgroup H is a retract of a group Γ, then the normal closure in H of any subset of $U \subset H$ is the same as the image in H (under retraction) of the normal closure of U in Γ. Thus, combining the conclusions of the previous two paragraphs, we see that $\ker\phi_n$ is the normal closure of a finite subset of Γ_n if and only if $\langle S_n\rangle$ is finitely presentable.

Step 2: We now assume that $\langle S_n\rangle \subset B$ and $\langle S_{n'}\rangle \subset B$ are both finitely presentable. In this case ϕ_n and $\phi_{n'}$ have the same image, namely B. Thus we may apply Lemma 2.4. This lemma provides us with an automorphism $\Phi : F \to F$ such that $\phi_n \circ \Phi(\sigma_i) = 1$ and $\phi_n \circ \Phi(\tau_i) = s_{n',i}$ for $i = 1, \dots, m$.

Consider the automorphism α of $G = (A * B) \times F$ that is the identity on $A * B$ and is defined on F by $\alpha(\sigma_i) = \Phi(\tau_i)$ and $\alpha(\tau_i) = \Phi(\sigma_i)$. We claim that α sends $\Sigma_{n'}$ isomorphically to Σ_n. Since these are free groups of equal rank, it suffices to check that $\alpha(\Sigma_{n'}) \subset \Sigma_n$.

By definition, for $i = 1, \dots, n$ we have $\alpha(s_{n',i}, \sigma_i) = (s_{n',i}, \Phi(\tau_i))$, which is in $\Sigma_n = \{(\phi(x), x) \mid x \in F\}$ because $\phi_n \circ \Phi(\tau_i) = s_{n',i}$.

Because α maps $\Sigma_{n'}$ isomorphically onto Σ_n, we obtain an isomorphism from $\Gamma_{n'} = G *_{\Sigma_{n'}} \overline{G}$ to Γ_n by defining $g \mapsto \alpha(g)$ and $\overline{g} \mapsto \overline{\alpha(g)}$.

This completes the proof of Theorem 4.1. $\square$

4.1. **The Proof of Theorem B.** There are many semihyperbolic groups that have property FA, for example finite groups, mapping class groups, higher-rank lattices, and hyperbolic groups that have property (T). To be definite, we take A to be a finite group and we take B to be a torsion-free semihyperbolic group that has property Ψ (examples are discussed below). By applying the construction of Theorem 4.1 to $G := (A*B)\times F$ we obtain a recursive sequence of groups $\Gamma_n = \Delta(G;\Sigma_n)$ for which the isomorphism problem is unsolvable. According to Theorem 3.3, the groups Γ_n are all combable. $\square$

Example 4.2. Let F be a finitely generated free group of rank at least 2. The arguments in Section 1 yield explicit sequences of subsets $S_n \subset F \times F$ of cardinality m enjoying Property Ψ. Let F_m be a free group of rank $2m$. Let $G = [\mathbb{Z}_2 * (F \times F)] \times F_m$. The construction of Theorem 4.1 shows that there is a recursive sequence of free subgroups $\Sigma_n \subset G$ (each a retract of G in fact) such that there is no algorithm to decide isomorphism among the combable groups $\Gamma_n = \Delta(G;\Sigma_n)$.

Example 4.3. The arguments in Section 1 also yield torsion-free hyperbolic groups Γ with explicit sequences of subsets $S_n \subset \Gamma$ of cardinality m enjoying Property Ψ. As above, in $G = (\mathbb{Z}_2 * \Gamma) \times F$ one finds a recursive sequence of free subgroups $\Sigma_n \subset G$ (each a retract of G) such that there is no algorithm to decide isomorphism among the combable groups $\Gamma_n = \Delta(G; \Sigma_n)$. i

Remark 4.4. If one wishes to obtain torsion-free examples, one can replace $\mathbb{Z}_2$ in the above constructions by a torsion-free hyperbolic group with Property (T).

Remark 4.5. The results in [5] show that all of the groups in the sequences (Γ_n) considered above satisfy a cubic Dehn isoperimetric inequality and are Ind-combable, where Ind is the class of indexed languages. At the time of writing, I do not know of such sequences in which all of the groups are Reg-combable, i.e. automatic.

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