Automorphism groups of free groups, surface groups and free abelian groups

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The group of 2×2 matrices with integer entries and determinant ± 1 can be identified either with the group of outer automorphisms of a rank two free group or with the group of isotopy classes of homeomorphisms of a 2-dimensional torus. Thus this group is the beginning of three natural sequences of groups, namely the general linear groups $\mathrm{GL}(n, \mathbb{Z})$, the groups $\mathrm{Out}(F_n)$ of outer automorphisms of free groups of rank $n \geq 2$, and the mapping class groups $\mathrm{Mod}^{\pm}(S_g)$ of orientable surfaces of genus $g \geq 1$. Much of the work on mapping class groups and automorphisms of free groups is motivated by the idea that these sequences of groups are strongly analogous, and should have many properties in common. This program is occasionally derailed by uncooperative facts but has in general proved to be a successful strategy, leading to fundamental discoveries about the structure of these groups. In this article we will highlight a few of the most striking similarities and differences between these series of groups and present some open problems motivated by this philosophy.

Similarities among the groups $\mathrm{Out}(F_n)$, $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Mod}^{\pm}(S_g)$ begin with the fact that these are the outer automorphism groups of the most primitive types of torsion-free discrete groups, namely free groups, free abelian groups and the fundamental groups of closed orientable surfaces $\pi_1 S_g$. In the case of $\mathrm{Out}(F_n)$ and $\mathrm{GL}(n, \mathbb{Z})$ this is obvious, in the case of $\mathrm{Mod}^{\pm}(S_g)$ it is a classical theorem of Nielsen. In all cases there is a *determinant* homomorphism to $\mathbb{Z}/2$; the kernel of this map is the group of "orientation-preserving" or "special" automorphisms, and is denoted $\mathrm{SOut}(F_n)$, $\mathrm{SL}(n, Z)$ or $\mathrm{Mod}(S_g)$ respectively.

1 Geometric and topological models

A natural geometric context for studying the global structure of $\mathrm{GL}(n, \mathbb{Z})$ is provided by the symmetric space X of positive-definite, real symmetric matrices of determinant 1 (see [78] for a nice introduction to this subject). This is a non-positively curved manifold diffeomorphic to $\mathbb{R}^d$, where $d = \frac{1}{2}n(n+1) - 1$. $\mathrm{GL}(n, \mathbb{Z})$ acts properly by isometries on X with a quotient of finite volume.

Each $A \in X$ defines an inner product on $\mathbb{R}^n$ and hence a Riemannian metric ν of constant curvature and volume 1 on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. One can recover A from the metric ν and an ordered basis for $\pi_1 T^n$. Thus X is homeomorphic to the space of equivalence classes of *marked* Euclidean tori (T^n, ν) of volume 1, where a *marking* is a homotopy class of homeomorphisms $\rho : T^n \to (T^n, \nu)$ and two marked tori are considered equivalent if there is an isometry $i : (T_1^n, \nu_1) \to (T_2^n, \nu_2)$ such that $\rho_2^{-1} \circ i \circ \rho_1$ is homotopic to the identity. The natural action of $\mathrm{GL}(n, \mathbb{Z}) = \mathrm{Out}(\mathbb{Z}^n)$ on $T^n = K(\mathbb{Z}^n, 1)$ twists the markings on tori, and when one traces through the identifications this is the standard action on X.

If one replaces T^n by S_g and follows exactly this formalism with marked metrics of constant curvature[1] and fixed volume, then one arrives at the definition of *Teichmüller space* and the natural action of $\mathrm{Mod}^{\pm}(S_g)= \mathrm{Out}(\pi_1 S_g)$ on it. Teichmüller space is again homeomorphic to a Euclidean space, this time $\mathbb{R}^{6g-6}$.

In the case of $\mathrm{Out}(F_n)$ there is no canonical choice of classifying space $K(F_n, 1)$ but rather a finite collection of natural models, namely the finite graphs of genus n with no vertices of valence less than 3. Nevertheless, one can proceed in essentially the same way: one considers metrics of fixed volume (sum of the lengths of edges =1) on the various models for $K(F_n, 1)$, each equipped with a marking, and one makes the obvious identifications as the homeomorphism type of a graph changes with a sequence of metrics that shrink an edge to length zero. The space of marked metric structures obtained in this case is Culler and Vogtmann's Outer space [27], which is stratified by manifold subspaces corresponding to the different homeomorphism types of graphs that arise. This space is not a manifold, but it is contractible and its local homotopical structure is a natural generalization of that for a manifold (cf. [80]).

One can also learn a great deal about the group $\mathrm{GL}(n, \mathbb{Z})$ by examining its actions on the Borel-Serre bordification of the symmetric space X and on the spherical Tits building, which encodes the asymptotic geometry of X. Teichmüller space and Outer space both admit useful bordifications that are closely analogous to the Borel-Serre bordification [44, 53, 2]. And in place of the spherical Tits building for $\mathrm{GL}(n, \mathbb{Z})$ one has the complex of curves

[1] if $g \geq 2$ then the curvature will be negative

[46] for $\mathrm{Mod}^{\pm}(S_g)$, which has played an important role in recent advances concerning the large scale geometry of $\mathrm{Mod}^{\pm}(S_g)$. For the moment this complex has no well-established counterpart in the context of $\mathrm{Out}(F_n)$.

These closely parallel descriptions of geometries for the three families of groups have led mathematicians to try to push the analogies further, both for the geometry and topology of the "symmetric spaces" and for purely group-theoretic properties that are most naturally proved using the geometry of the symmetric space. For example, the symmetric space for $\mathrm{GL}(n, \mathbb{Z})$ admits a natural equivariant deformation retraction onto an $n(n-1)/2$-dimensional cocompact subspace, the *well-rounded retract* [1]. Similarly, both Outer space and the Teichmüller space of a punctured or bounded orientable surface retract equivariantly onto cocompact simplicial spines [27, 44]. In all these cases, the retracts have dimension equal to the virtual cohomological dimension of the relevant group. For closed surfaces, however, the question remains open:

Question 1 *Does the Teichmüller space for S_g admit an equivariant deformation retraction onto a cocompact spine whose dimension is equal to $4g-5$, the virtual cohomological dimension of $\mathrm{Mod}^{\pm}(S_g)$?*

Further questions of a similar nature are discussed in (2.1).

The issues involved in using these symmetric space analogs to prove purely group theoretic properties are illustrated in the proof of the Tits alternative, which holds for all three classes of groups. A group Γ is said to satisfy the Tits alternative if each of its subgroups either contains a non-abelian free group or else is virtually solvable. The strategy for proving this is similar in each of the three families that we are considering: inspired by Tits's original proof for linear groups (such as $\mathrm{GL}(n, \mathbb{Z})$), one attempts to use a ping-pong argument on a suitable boundary at infinity of the symmetric space. This strategy ultimately succeeds but the details vary enormously between the three contexts, and in the case of $\mathrm{Out}(F_n)$ they are particularly intricate ([4, 3] versus [9]). One finds that this is often the case: analogies between the three classes of groups can be carried through to theorems, and the architecture of the expected proof is often a good guide, but at a more detailed level the techniques required vary in essential ways from one class to the next and can be of completely different orders of difficulty.

Let us return to problems more directly phrased in terms of the geometry of the symmetric spaces. The symmetric space for $\mathrm{GL}(n, \mathbb{Z})$ has a left-invariant metric of non-positive curvature, the geometry of which is relevant to many areas of mathematics beyond geometric group theory. Teichmüller space has two natural metrics, the Teichmüller metric and the Weyl-Petersen metric, and again the study of each is a rich subject. In contrast, the metric theory of Outer space has not been developed, and in fact there is no obvious candidate for a natural metric. Thus, the following question has been left deliberately vague:

Question 2 *Develop a metric theory of Outer space.*

The elements of infinite order in $\mathrm{GL}(n, \mathbb{Z})$ that are diagonalizable over $\mathbb{C}$ act as loxodromic isometries of X. When $n = 2$, these elements are the hyperbolic matrices; each fixes two points at infinity in $X = \mathbb{H}^2$, one a source and one a sink. The analogous type of element in $\mathrm{Mod}^{\pm}(S_g)$ is a pseudo-Anosov, and in $\mathrm{Out}(F_n)$ it is an *iwip* (irreducible with irreducible powers). In both cases, such elements have two fixed points at infinity (i.e. in the natural boundary of the symmetric space analog), and the action of the cyclic subgroup generated by the element exhibits the north-south dynamics familiar from the action of hyperbolic matrices on the closure of the Poincaré disc [62], [54]. In the case of $\mathrm{Mod}^{\pm}(S_g)$ this cyclic subgroup leaves invariant a unique geodesic line in Teichmüller space, i.e. pseudo-Anosov's are axial like the semi-simple elements of infinite order in $\mathrm{GL}(n, \mathbb{Z})$. Initial work of Handel and Mosher [43] shows that in the case of iwips one cannot hope to have a unique axis in the same metric sense, but leaves open the possibility that there may be a reasonable notion of axis in a weaker sense. (We highlighted this problem in an earlier version of the current article.) In a more recent preprint [42] they have addressed this last point directly, defining an *axis bundle* associated to any iwip, cf. [63]. Nevertheless, many interesting questions remain (some of which are highlighted by Handel and Mosher). Thus we retain a modified version of our original question:

Question 3 *Describe the geometry of the axis bundle (and associated objects) for an iwip acting on Outer Space.*

2 Actions of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ on other spaces

Some of the questions that we shall present are more naturally stated in terms of $\mathrm{Aut}(F_n)$ rather than $\mathrm{Out}(F_n)$, while some are natural for both. To avoid redundancy, we shall state only one form of each question.

2.1 Baum-Connes and Novikov conjectures

Two famous conjectures relating topology, geometry and functional analysis are the Novikov and Baum-Connes conjectures. The Novikov conjecture for closed oriented manifolds with fundamental group Γ says that certain *higher signatures* coming from $H^*(\Gamma;\mathbb{Q})$ are homotopy invariants. It is implied by the Baum-Connes conjecture, which says that a certain *assembly map* between two K-theoretic objects associated to Γ is an isomorphism. Kasparov [57] proved the Novikov conjecture for $\mathrm{GL}(n,\mathbb{Z})$, and Guenther, Higson and Weinberger proved it for all linear groups [40]. The Baum-Connes conjecture for $\mathrm{GL}(n,\mathbb{Z})$ is open when $n \geq 4$ (cf. [61]).

Recently Storm [79] pointed out that the Novikov conjecture for mapping class groups follows from results that have been announced by Hamenstädt [41] and Kato [59], leaving open the following:

Question 4 *Do mapping class groups or* $\mathrm{Out}(F_n)$ *satisfy the Baum-Connes conjecture? Does* $\mathrm{Out}(F_n)$ *satisfy the Novikov conjecture?*

An approach to proving these conjectures is given by work of Rosenthal [75], generalizing results of Carlsson and Pedersen [23]. A contractible space on which a group Γ acts properly and for which the fixed point sets of finite subgroups are contractible is called an $\underline{\mathrm{E}}\Gamma$. Rosenthal's theorem says that the Baum-Connes map for Γ is split injective if there is a cocompact $\underline{\mathrm{E}}\Gamma = E$ that admits a compactification X, such that

1. the Γ-action extends to X;
2. X is metrizable;
3. X^G is contractible for every finite subgroup G of Γ
4. E^G is dense in X^G for every finite subgroup G of Γ
5. compact subsets of E become small near $Y = X \setminus E$ under the Γ-action: for every compact $K \subset E$ and every neighborhood $U \subset X$ of $y \in Y$, there exists a neighborhood $V \subset X$ of y such that $\gamma K \cap V \neq \emptyset$ implies $\gamma K \subset U$.

The existence of such a space E also implies the Novikov conjecture for Γ.

For $\mathrm{Out}(F_n)$ the spine of Outer space mentioned in the previous section is a reasonable candidate for the required $\underline{\mathrm{E}}\Gamma$, and there is a similarly defined candidate for $\mathrm{Aut}(F_n)$. For mapping class groups of punctured surfaces the complex of arc systems which fill up the surface is a good candidate (note that this can be identified with a subcomplex of Outer space, as in [47], section 5).

Question 5 *Does there exist a compactification of the spine of Outer space satisfying Rosenthal's conditions? Same question for the complex of arc systems filling a punctured surface.*

In all of the cases mentioned above, the candidate space E has dimension equal to the virtual cohomological dimension of the group. G. Mislin [68] has constructed a cocompact $\underline{\mathrm{E}}G$ for the mapping class group of a closed surface, but it has much higher dimension, equal to the dimension of the Teichmüller space. This leads us to a slight variation on Question 1.

Question 6 *Can one construct a cocompact $\underline{E}G$ with dimension equal to the virtual cohomological dimension of the mapping class group of a closed surface?*

2.2 Properties (T) and FA

A group has Kazdhan's Property (T) if any action of the group by isometries on a Hilbert space has fixed vectors. Kazdhan proved that $\mathrm{GL}(n, \mathbb{Z})$ has property (T) for $n \geq 3$.

Question 7 *For $n > 3$, does $\mathrm{Aut}(F_n)$ have property (T)?*

The corresponding question for mapping class groups is also open. If $\mathrm{Aut}(F_n)$ were to have Property (T), then an argument of Lubotzky and Pak [64] would provide a conceptual explanation of the apparently-unreasonable effectiveness of certain algorithms in computer science, specifically the Product Replacement Algorithm of Leedham-Green *et al.*

If a group has Property (T) then it has Serre's property FA: every action of the group on an $\mathbb{R}$-tree has a fixed point. When $n \geq 3$, $\mathrm{GL}(n, \mathbb{Z})$ has property FA, as do $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$, and mapping class groups in genus ≥ 3 (see [28]). In contrast, McCool [67] has shown that $\mathrm{Aut}(F_3)$ has a subgroup of finite-index with positive first betti number, i.e. a subgroup which maps onto $\mathbb{Z}$. In particular this subgroup acts by translations on the line and therefore does not have property FA or (T). Since property (T) passes to finite-index subgroups, it follows that $\mathrm{Aut}(F_3)$ does not have property (T).

Question 8 *For $n > 3$, does $\mathrm{Aut}(F_n)$ have a subgroup of finite index with positive first betti number?*

Another finite-index subgroup of $\mathrm{Aut}(F_3)$ mapping onto $\mathbb{Z}$ was constructed by Alex Lubotzky, and was explained to us by Andrew Casson. Regard F_3 as the fundamental group of a graph R with one vertex. The single-edge loops provide a basis $\{a, b, c\}$ for F_3. Consider the 2-sheeted covering $\hat{R} \to R$ with fundamental group $\langle a, b, c^2, cac^{-1}, cbc^{-1}\rangle$ and let $G \subset \mathrm{Aut}(F_3)$ be the stabilizer of this subgroup. G acts on $H_1(\hat{R}, \mathbb{Q})$ leaving invariant the eigenspaces of the involution that generates the Galois group of the covering. The eigenspace corresponding to the eigenvalue -1 is two dimensional with basis $\{a - cac^{-1},\ b - cbc^{-1}\}$. The action of G with respect to this basis gives an epimorphism $G \to \mathrm{GL}(2, \mathbb{Z})$. Since $\mathrm{GL}(2, \mathbb{Z})$ has a free subgroup of finite-index, we obtain a subgroup of finite index in $\mathrm{Aut}(F_3)$ that maps onto a non-abelian free group.

One can imitate the essential features of this construction with various other finite-index subgroups of F_n, thus producing subgroups of finite index in $\mathrm{Aut}(F_n)$ that map onto $\mathrm{GL}(m, \mathbb{Z})$. In each case one finds that $m \geq n - 1$.

Question 9 *If there is a homomorphism from a subgroup of finite index in* $\mathrm{Aut}(F_n)$ *onto a subgroup of finite index in* $\mathrm{GL}(m, \mathbb{Z})$, *then must* $m \geq n - 1$*?*

Indeed one might ask:

Question 10 *If* $m < n - 1$ *and* $H \subset \mathrm{Aut}(F_n)$ *is a subgroup of finite index, then does every homomorphism* $H \to \mathrm{GL}(m, \mathbb{Z})$ *have finite image?*

Similar questions are interesting for the other groups in our families (cf. section 3). For example, if $m < n - 1$ and $H \subset \mathrm{Aut}(F_n)$ is a subgroup of finite index, then does every homomorphism $H \to \mathrm{Aut}(F_m)$ have finite image?

A positive answer to the following question would answer Question 8; a negative answer would show that $\mathrm{Aut}(F_n)$ does not have property (T).

Question 11 *For* $n \geq 4$, *do subgroups of finite index in* $\mathrm{Aut}(F_n)$ *have Property FA?*

A promising approach to this last question breaks down because we do not know the answer to the following question.

Question 12 *Fix a basis for* F_n *and let* $A_{n-1} \subset Aut(F_n)$ *be the copy of* $Aut(F_{n-1})$ *corresponding to the first* $n-1$ *basis elements. Let* $\phi : Aut(F_n) \to G$ *be a homomorphism of groups. If* $\phi(A_{n-1})$ *is finite, must the image of* ϕ *be finite?*

Note that the obvious analog of this question for $\mathrm{GL}(n,\mathbb{Z})$ has a positive answer and plays a role in the foundations of algebraic K-theory.

A different approach to establishing Property (T) was developed by Zuk [85]. He established a combinatorial criterion on the links of vertices in a simply connected G-complex which, if satisfied, implies that G has property (T): one must show that the smallest positive eigenvalue of the discrete Laplacian on links is sufficiently large. One might hope to apply this criterion to one of the natural complexes on which $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ act, such as the spine of Outer space. But David Fisher has pointed out to us that the results of Izeki and Natayani [55] (alternatively, Schoen and Wang – unpublished) imply that such a strategy cannot succeed.

2.3 Actions on CAT(0) spaces

An $\mathbb{R}$-tree may be defined as a complete CAT(0) space of dimension[2] 1. Thus one might generalize property FA by asking, for each $d \in \mathbb{N}$, which groups must fix a point whenever they act by isometries on a complete CAT(0) space of dimension $\leq d$.

Question 13 *What is the least integer δ such that $\mathrm{Out}(F_n)$ acts without a global fixed point on a complete CAT(0) space of dimension δ? And what is the least dimension for the mapping class group $\mathrm{Mod}^{\pm}(S_g)$?*

The action of $\mathrm{Out}(F_n)$ on the first homology of F_n defines a map from $\mathrm{Out}(F_n)$ to $\mathrm{GL}(n,\mathbb{Z})$ and hence an action of $\mathrm{Out}(F_n)$ on the symmetric space of dimension $\frac{1}{2}n(n+1)-1$. This action does not have a global fixed point and hence we obtain an upper bound on δ. On the other hand, since $\mathrm{Out}(F_n)$ has property FA, $\delta \geq 2$. In fact, motivated by work of Farb on $\mathrm{GL}(n,\mathbb{Z})$, Bridson [14] has shown that using a Helly-type theorem and the structure of finite subgroups in $\mathrm{Out}(F_n)$, one can obtain a lower bound on δ that grows as a linear function of n. Note that a lower bound of $3n-3$ on δ would imply that Outer Space did not support a complete $\mathrm{Out}(F_n)$-equivariant metric of non-positive curvature.

If X is a CAT(0) polyhedral complex with only finitely many isometry types of cells (e.g. a finite dimensional cube complex), then each isometry of X is either elliptic (fixes a point) or hyperbolic (has an axis of translation) [15]. If $n \geq 4$ then a variation on an argument of Gersten [36] shows that in any action of $\mathrm{Out}(F_n)$ on X, no Nielsen generator can act as a hyperbolic isometry.

[2] topological covering dimension

Question 14 *If $n \geq 4$, then can* $\mathrm{Out}(F_n)$ *act without a global fixed point on a finite-dimensional* $\mathrm{CAT}(0)$ *cube complex?*

2.4 Linearity

Formanek and Procesi [33] proved that $\mathrm{Aut}(F_n)$ is not linear for $n \geq 3$ by showing that $\mathrm{Aut}(F_3)$ contains a "poison subgroup", i.e. a subgroup which has no faithful linear representation.

Since $\mathrm{Aut}(F_n)$ embeds in $\mathrm{Out}(F_{n+1})$, this settles the question of linearity for $\mathrm{Out}(F_n)$ as well, except when $n = 3$.

Question 15 *Does* $\mathrm{Out}(\mathrm{F}_3)$ *have a faithful representation into* $\mathrm{GL}(\mathrm{m}, \mathbb{C})$ *for some $m \in \mathbb{N}$?*

Note that braid groups are linear [8] but it is unknown if mapping class groups of closed surfaces are. Brendle and Hamidi-Tehrani [13] showed that the approach of Formanek and Procesi cannot be adapted directly to the mapping class groups. More precisely, they prove that the type of "poison subgroup" described above does not arise in mapping class groups.

The fact that the above question remains open is an indication that $\mathrm{Out}(F_3)$ can behave differently from $\mathrm{Out}(F_n)$ for n large; the existence of finite index subgroups mapping onto $\mathbb{Z}$ was another instance of this, and we shall see another in our discussion of automatic structures and isoperimetric inequalities.

3 Maps to and from $\mathrm{Out}(F_n)$

A particularly intriguing aspect of the analogy between $\mathrm{GL}(n, \mathbb{Z})$ and the two other classes of groups is the extent to which the celebrated rigidity phenomena for lattices in higher rank semisimple groups transfer to mapping class groups and $\mathrm{Out}(F_n)$. Many of the questions in this section concern aspects of this rigidity; questions 9 to 11 should also be viewed in this light.

Bridson and Vogtmann [21] showed that any homomorphism from $\mathrm{Aut}(F_n)$ to a group G has finite image if G does not contain the symmetric group Σ_{n+1}; in particular, any homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n-1})$ has image of order at most 2.

Question 16 *If $n \geq 4$ and $g \geq 1$, does every homomorphism from* $\mathrm{Aut}(F_n)$ *to* $\mathrm{Mod}^{\pm}(S_g)$ *have finite image?*

By [21], one cannot obtain homomorphisms with infinite image unless $\mathrm{Mod}^{\pm}(S_g)$ contains the symmetric group Σ_{n+1}. For large enough genus, you can realize any symmetric group; but the order of a finite group of symmetries is at most 84g-6, so here one needs $84g-6 \geq (n+1)!$.

There are no *injective* maps from $\mathrm{Aut}(F_n)$ to mapping class groups. This follows from the result of Brendle and Hamidi-Tehrani that we quoted earlier. For certain g one can construct homomorphisms $\mathrm{Aut}(F_3) \to \mathrm{Mod}^{\pm}(S_g)$ with infinite image, but we do not know the minimal such g.

Question 17 *Let Γ be an irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank at least 2. Does every homomorphism from Γ to $\mathrm{Out}(F_n)$ have finite image?*

This is known for non-uniform lattices (see [16]; it follows easily from the Kazdhan-Margulis finiteness theorem and the fact that solvable subgroups of $\mathrm{Out}(F_n)$ are virtually abelian [5]). Farb and Masur provided a positive answer to the analogous question for maps to mapping class groups [32]. The proof of their theorem was based on results of Kaimanovich and Masur [56] concerning random walks on Teichmüller space. (See [54] and, for an alternative approach, [6].)

Question 18 *Is there a theory of random walks on Outer space similar to that of Kaimanovich and Masur for Teichmüller space?*

Perhaps the most promising approach to Question 17 is via bounded cohomology, following the template of Bestvina and Fujiwara's work on subgroups of the mapping class group [6].

Question 19 *If a subgroup $G \subset \mathrm{Out}(F_n)$ is not virtually abelian, then is $H^2_b(G;\mathbb{R})$ infinite dimensional?*

If $m \geq n$ then there are obvious embeddings $\mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(m,\mathbb{Z})$ and $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_m)$, but there are no obvious embeddings $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$. Bogopolski and Puga [10] have shown that, for $m = 1+(n-1)kn$, where k is an arbitrary natural number coprime to $n-1$, there is in fact an embedding, by restricting automorphisms to a suitable characteristic subgroup of F_m.

Question 20 *For which values of m does $\mathrm{Out}(F_n)$ embed in $\mathrm{Out}(F_m)$? What is the minimal such m, and is it true for all sufficiently large m?*

It has been shown that when n is sufficiently large with respect to i, the homology group $H_i(\mathrm{Out}(F_n), \mathbb{Z})$ is independent of n [50, 51].

Question 21 *Is there a map* $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ *that induces an isomorphism on homology in the stable range?*

A number of the questions in this section and (2.2) ask whether certain quotients of $\mathrm{Out}(F_n)$ or $\mathrm{Aut}(F_n)$ are necessarily finite. The following quotients arise naturally in this setting: define $Q(n,m)$ to be the quotient of $\mathrm{Aut}(F_n)$ by the normal closure of λ^m, where λ is the Nielsen move defined on a basis $\{a_1, \ldots, a_n\}$ by $a_1 \mapsto a_2a_1$. (All such Nielsen moves are conjugate in $\mathrm{Aut}(F_n)$, so the choice of basis does not alter the quotient.)

The image of a Nielsen move in $\mathrm{GL}(n,\mathbb{Z})$ is an elementary matrix and the quotient of $\mathrm{GL}(n,\mathbb{Z})$ by the normal subgroup generated by the m-th powers of the elementary matrices is the finite group $\mathrm{GL}(n,\mathbb{Z}/m)$. But Bridson and Vogtmann [21] showed that if m is sufficiently large then $Q(n,m)$ is infinite because it has a quotient that contains a copy of the free Burnside group $B(n-1,m)$. Some further information can be gained by replacing $B(n-1,m)$ with the quotients of F_n considered in subsection 39.3 of A.Yu. Ol'shanskii's book [73]. But we know very little about the groups $Q(n,m)$. For example:

Question 22 *For which values of* n *and* m *is* $Q(n,m)$ *infinite? Is* $Q(3,5)$ *infinite?*

Question 23 *Can* $Q(n,m)$ *have infinitely many finite quotients? Is it residually finite?*

4 Individual elements and mapping tori

Individual elements $\alpha \in \mathrm{GL}(n,\mathbb{Z})$ can be realized as diffeomorphisms $\hat{\alpha}$ of the n-torus, while individual elements $\psi \in \mathrm{Mod}^{\pm}(S_g)$ can be realized as diffeomorphisms $\hat{\psi}$ of the surface S_g. Thus one can study α via the geometry of the torus bundle over $\mathbb{S}^1$ with holonomy $\hat{\alpha}$ and one can study ψ via the geometry of the 3-manifold that fibres over $\mathbb{S}^1$ with holonomy $\hat{\psi}$. (In each case the manifold depends only on the conjugacy class of the element.)

The situation for $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ is more complicated: the natural choices of classifying space $Y = K(F_n, 1)$ are finite graphs of genus n, and no element of infinite order $\phi \in \mathrm{Out}(F_n)$ is induced by the action on $\pi_1(Y)$ of a homeomorphism of Y. Thus the best that one can hope for in this situation is to identify a graph Y_ϕ that admits a homotopy equivalence inducing ϕ

and that has additional structure well-adapted to ϕ. One would then form the mapping torus of this homotopy equivalence to get a good classifying space for the algebraic mapping torus $F_n \rtimes_\phi \mathbb{Z}$.

The *train track technology* of Bestvina, Feighn and Handel [7, 4, 3] is a major piece of work that derives suitable graphs Y_ϕ with additional structure encoding key properties of ϕ. This results in a decomposition theory for elements of $\mathrm{Out}(F_n)$ that is closely analogous to (but more complicated than) the Nielsen-Thurston theory for surface automorphisms. Many of the results mentioned in this section are premised on a detailed knowledge of this technology and one expects that a resolution of the questions will be too.

There are several natural ways to define the *growth* of an automorphism ϕ of a group G with finite generating set A; in the case of free, free-abelian, and surface groups these are all asymptotically equivalent. The most easily defined growth function is $\gamma_\phi(k)$ where $\gamma_\phi(k) := \max\{d(1, \phi^k(a) \mid a \in A\}$. If $G = \mathbb{Z}^n$ then $\gamma_\phi(k) \simeq k^d$ for some integer $d \leq n-1$, or else $\gamma_\phi(k)$ grows exponentially. If G is a surface group, the Nielsen-Thurston theory shows that only bounded, linear and exponential growth can occur. If $G = F_n$ and $\phi \in \mathrm{Aut}(F_n)$ then, as in the abelian case, $\gamma_\phi(k) \simeq k^d$ for some integer $d \leq n-1$ or else $\gamma_\phi(k)$ grows exponentially.

Question 24 *Can one detect the growth of a surface or free-group homomorphism by its action on the homology of a characteristic subgroup of finite index?*

Notice that one has to pass to a subgroup of finite index in order to have any hope because automorphisms of exponential growth can act trivially on homology. A. Piggott [74] has answered the above question for free-group automorphisms of polynomial growth, and linear-growth automorphisms of surfaces are easily dealt with, but the exponential case remains open in both settings.

Finer questions concerning growth are addressed in the on-going work of Handel and Mosher [43]. They explore, for example, the implications of the following contrast in behaviour between surface automorphisms and free-group automorphisms: in the surface case the exponential growth rate of a pseudo-Anosov automorphism is the same as that of its inverse, but this is not the case for iwip free-group automorphisms.

For mapping tori of automorphisms of free abelian groups $G = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$, the following conditions are equivalent (see [17]): G is automatic; G is a

CAT(0) group[3]; G satisfies a quadratic isoperimetric inequality. In the case of mapping tori of surface automorphisms, all mapping tori satisfy the first and last of these conditions and one understands exactly which $S_g \rtimes \mathbb{Z}$ are CAT(0) groups.

Brady, Bridson and Reeves [12] show that there exist mapping tori of free-group automorphisms $F \rtimes \mathbb{Z}$ that are not automatic, and Gersten showed that some are not CAT(0) groups [36]. On the other hand, many such groups do have these properties, and they all satisfy a quadratic isoperimetric inequality [18].

Question 25 *Classify those $\phi \in \mathrm{Aut}(F_n)$ for which $F_n \rtimes_\phi \mathbb{Z}$ is automatic and those for which it is* CAT(0)*.*

Of central importance in trying to understand mapping tori is:

Question 26 *Is there an alogrithm to decide isomorphism among groups of the form $F \rtimes \mathbb{Z}$.*

In the purest form of this question one is given the groups as finite presentations, so one has to address issues of how to find the decomposition $F \rtimes \mathbb{Z}$ and one has to combat the fact that this decomposition may not be unique. But the heart of any solution should be an answer to:

Question 27 *Is the conjugacy problem solvable in $\mathrm{Out}(F_n)$?*

Martin Lustig posted a detailed outline of a solution to this problem on his web page some years ago [65], but neither this proof nor any other has been accepted for publication. This problem is of central importance to the field and a clear, compelling solution would be of great interest. The conjugacy problem for mapping class groups was shown to be solvable by Hemion [52], and an effective algorithm for determining conjugacy, at least for pseudo-Anosov mapping classes, was given by Mosher [70]. The isomorphism problem for groups of the form $S_g \rtimes \mathbb{Z}$ can be viewed as a particular case of the solution to the isomorphism problem for fundamental groups of geometrizable 3-manifolds [76]. The solvability of the conjugacy problem for $\mathrm{GL}(n, \mathbb{Z})$ is due to Grunewald [39]

[3] this means that G acts properly and cocompactly by isometries on a CAT(0) space

5 Cohomology

In each of the series of groups $\{\Gamma_n\}$ we are considering, the ith homology of Γ_n has been shown to be independent of n for n sufficiently large. For $\mathrm{GL}(n,\mathbb{Z})$ this is due to Charney [24], for mapping class groups to Harer [45], for $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ to Hatcher and Vogtmann [48, 50], though for $\mathrm{Out}(F_n)$ this requires an erratum by Hatcher, Vogtmann and Wahl [51]. With trivial rational coefficients, the stable cohomology of $\mathrm{GL}(n,\mathbb{Z})$ was computed in the 1970's by Borel [11], and the stable rational cohomology of the mapping class group computed by Madsen and Weiss in 2002 [66]. The stable rational cohomology of $\mathrm{Aut}(F_n)$ (and $\mathrm{Out}(F_n)$) was very recently determined by S. Galatius [34] to be trivial.

The exact stable range for trivial rational coefficients is known for $\mathrm{GL}(n,\mathbb{Z})$ and for mapping class groups of punctured surfaces. For $\mathrm{Aut}(F_n)$ the best known result is that the ith homology is independent of n for $n > 5i/4$ [49], but the exact range is unknown:

Question 28 *Where precisely does the rational homology of* $\mathrm{Aut}(F_n)$ *stabilize? And for* $\mathrm{Out}(F_n)$*?*

There are only two known non-trivial classes in the (unstable) rational homology of $\mathrm{Out}(F_n)$ [49, 26]. However, Morita [69] has defined an infinite series of cycles, using work of Kontsevich which identifies the homology of $\mathrm{Out}(F_n)$ with the cohomology of a certain infinite-dimensional Lie algebra. The first of these cycles is the generator of $H_4(\mathrm{Out}(F_4);\mathbb{Q}) \cong \mathbb{Q}$, and Conant and Vogtmann showed that the second also gives a non-trivial class, in $H_8(\mathrm{Out}(F_6);\mathbb{Q})$ [26]. Both Morita and Conant-Vogtmann also defined more general cycles, parametrized by odd-valent graphs.

Question 29 *Are Morita's original cycles non-trivial in homology? Are the generalizations due to Morita and to Conant and Vogtmann non-trivial in homology?*

No other classes have been found to date in the homology of $\mathrm{Out}(F_n)$, leading naturally to the question of whether these give all of the rational homology.

Question 30 *Do the Morita classes generate all of the rational homology of* $\mathrm{Out}(F_n)$*?*

The maximum dimension of a Morita class is about $4n/3$. Morita's cycles lift naturally to $\mathrm{Aut}(F_n)$, and again the first two are non-trivial in homology.

By Galatius' result, all of these cycles must eventually disappear under the stabilization map $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1})$. Conant and Vogtmann show that in fact they disappear immediately after they appear, i.e. one application of the stabilization map kills them [25]. If it is true that the Morita classes generate all of the rational homology of $\mathrm{Out}(F_n)$ then this implies that the stable range is significantly lower than the current bound.

We note that Morita has identified several conjectural relationships between his cycles and various other interesting objects, including the image of the Johnson homomorphism, the group of homology cobordism classes of homology cylinders, and the motivic Lie algebra associated to the algebraic mapping class group (see Morita's article in this volume).

Since the stable rational homology of $\mathrm{Out}(F_n)$ is trivial, the natural maps from mapping class groups to $\mathrm{Out}(F_n)$ and from $\mathrm{Out}(F_n)$ to $\mathrm{GL}(n, \mathbb{Z})$ are of course zero. However, the unstable homology of all three classes of groups remains largely unkown and in the unstable range these maps might well be nontrivial. In particular, we note that $H_8(\mathrm{GL}(6, \mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}$ [30]; this leads naturally to the question

Question 31 *Is the image of the second Morita class in $H_8(\mathrm{GL}(6, \mathbb{Z}); \mathbb{Q}))$ non-trivial?*

For further discussion of the cohomology of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ we refer to [81].

6 Generators and Relations

The groups we are considering are all finitely generated. In each case, the most natural set of generators consists of a single orientation-reversing generator of order two, together with a collection of simple infinite-order special automorphisms. For $\mathrm{Out}(F_n)$, these special automorphisms are the Nielsen automorphisms, which multiply one generator of F_n by another and leave the rest of the generators fixed; for $\mathrm{GL}(n, \mathbb{Z})$ these are the elementary matrices; and for mapping class groups they are Dehn twists around a small set of non-separating simple closed curves.

These generating sets have a number of important features in common. First, implicit in the description of each is a choice of generating set for the group B on which Γ is acting. In the case of $\mathrm{Mod}^{\pm}(S_g)$ this "basis" can be taken to consist of $2g+1$ simple closed curves representing the standard generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, of $\pi_1(S_g)$ together with $z = a_2^{-1} b_3 a_3 b_3^{-1}$.

In the case of $\mathrm{Out}(F_n)$ and $\mathrm{GL}(n,\mathbb{Z})$, the generating set is a basis for F_n and $\mathbb{Z}^n$ respectively.

Note that in the cases $\Gamma =$Out(F_n) or $\mathrm{GL}(n,\mathbb{Z})$, the universal property of the underlying free objects $B = F_n$ or $\mathbb{Z}^n$ ensures that Γ acts transitively on the set of preferred generating sets (bases). In the case $B = \pi_1 S_g$, the corresponding result is that any two collections of simple closed curves with the same pattern of intersection numbers and complementary regions are related by a homeomorphism of the surface, hence (at the level of π_1) by the action of Γ.

If we identify $\mathbb{Z}^n$ with the abelianization of F_n and choose bases accordingly, then the action of $\mathrm{Out}(F_n)$ on the abelianization induces a homomorphism $\mathrm{Out}(F_n) \to \mathrm{GL}(n,\mathbb{Z})$ that sends each Nielsen move to the corresponding elementary matrix (and hence is surjective). Correspondingly, the action $\mathrm{Mod}^{\pm}(S_g)$ on the abelianization of $\pi_1 S_g$ yields a homomorphism onto the symplectic group $Sp(2g,\mathbb{Z})$ sending the generators of $\mathrm{Mod}^{\pm}(S_g)$ given by Dehn twists around the a_i and b_i to transvections. Another common feature of these generating sets is that they all have linear growth (see section 4).

Smaller (but less transparent) generating sets exist in each case. Indeed B.H. Neumann [72] proved that $\mathrm{Aut}(F_n)$ (hence its quotients $\mathrm{Out}(F_n)$ and $\mathrm{GL}(n,\mathbb{Z})$) is generated by just 2 elements when $n \geq 4$. Wajnryb [83] proved that this is also true of mapping class groups.

In each case one can also find generating sets consisting of finite order elements, involutions in fact. Zucca showed that $\mathrm{Aut}(F_n)$ can be generated by 3 involutions two of which commute [84], and Kassabov, building on work of Farb and Brendle, showed that mapping class groups of large enough genus can be generated by 4 involutions [58].

Our groups are also all finitely presented. For $\mathrm{GL}(n,\mathbb{Z})$, or more precisely for $\mathrm{SL}(n,\mathbb{Z})$, there are the classical Steinberg relations, which involve commutators of the elementary matrices. For the special automorphisms $\mathrm{SAut}(F_n)$, Gersten gave a presentation in terms of corresponding commutator relations of the Nielsen generators [35]. Finite presentations of the mapping class groups are more complicated. The first was given by Hatcher and Thurston, and worked out explicitly by Wajnryb [82].

Question 32 *Is there a set of simple Steinberg-type relations for the mapping class group?*

There is also a presentation of $\mathrm{Aut}(F_n)$ coming from the action of $\mathrm{Aut}(F_n)$ on the subcomplex of Auter space spanned by graphs of degree at most 2. This is simply-connected by [48], so Brown's method [22] can be used to write down a presentation. The vertex groups are stabilizers of marked graphs, and the edge groups are the stabilizers of pairs consisting of a marked graph and a forest in the graph. The quotient of the subcomplex modulo $\mathrm{Aut}(F_n)$ can be computed explicitly, and one finds that $\mathrm{Aut}(F_n)$ is generated by the (finite) stabilizers of seven specific marked graphs. In addition, all of the relations except two come from the natural inclusions of edge stabilizers into vertex stabilizers, i.e. either including the stabilizer of a pair (graph, forest) into the stabilizer of the graph, or into the stabilizer of the quotient of the graph modulo the forest. Thus the whole group is almost (but not quite) a pushout of these finite subgroups. In the terminology of Haefliger (see [19], II.12), the complex of groups is not simple.

Question 33 *Can* $\mathrm{Out}(F_n)$ *and* $\mathrm{Mod}^{\pm}(S_g)$ *be obtained as a pushout of a finite subsystem of their finite subgroups, i.e. is either the fundamental group of a developable simple complex of finite groups on a 1-connected base?*

6.1 IA automorphisms

We conclude with a well-known problem about the kernel $\mathrm{IA}(n)$ of the map from $\mathrm{Out}(F_n)$ to $\mathrm{GL}(n, Z)$. The notation "IA" stands for *identity on the abelianization*; these are (outer) automorphisms of F_n which are the identity on the abelianization Z^n of F_n. Magnus showed that this kernel is finitely generated, and for $n = 3$ Krstic and McCool showed that it is not finitely presentable [60]. It is also known that in some dimension the homology is not finitely generated [77]. But that is the extent of our knowledge of basic finiteness properties.

Question 34 *Establish finiteness properties of the kernel* $\mathrm{IA}(n)$ *of the map from* $\mathrm{Out}(F_n)$ *to* $\mathrm{GL}(n, \mathbb{Z})$*. In particular, determine whether* $\mathrm{IA}(n)$ *is finitely presentable for* $n > 3$.

The subgroup $\mathrm{IA}(n)$ is analogous to the Torelli subgroup of the mapping class group of a surface, which also remains quite mysterious in spite of having been extensively studied.

7 Automaticity and Isoperimetric Inequalities

In the foundational text on automatic groups [31], Epstein gives a detailed account of Thurston's proof that if $n \geq 3$ then $\mathrm{GL}(n, \mathbb{Z})$ is not automatic. The argument uses the geometry of the symmetric space to obtain an exponential lower bound on the $(n-1)$-dimensional isoperimetric function of

$\mathrm{GL}(n,\mathbb{Z})$; in particular the Dehn function of $\mathrm{GL}(3,\mathbb{Z})$ is shown to be exponential.

Bridson and Vogtmann [20], building on this last result, proved that the Dehn functions of $\mathrm{Aut}(F_3)$ and $\mathrm{Out}(F_3)$ are exponential. They also proved that for all $n \geq 3$, neither $\mathrm{Aut}(F_n)$ nor $\mathrm{Out}(F_n)$ is biautomatic. In contrast, Mosher proved that mapping class groups are automatic [71] and Hamenstädt [41] proved that they are biautomatic; in particular these groups have quadratic Dehn functions and satisfy a polynomial isoperimetric inequality in every dimension. Hatcher and Vogtmann [47] obtain an exponential upper bound on the isoperimetric function of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ in every dimension.

An argument sketched by Thurston and expanded upon by Gromov [37], [38] (cf. [29]) indicates that the Dehn function of $\mathrm{GL}(n,\mathbb{Z})$ is quadratic when $n \geq 4$. More generally, the isoperimetric functions of $\mathrm{GL}(n,\mathbb{Z})$ should parallel those of Euclidean space in dimensions $m \leq n/2$.

Question 35 *What are the Dehn functions of* $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$ *for* $n > 3$*?*

Question 36 *What are the higher-dimensional isoperimetric functions of* $\mathrm{GL}(n,\mathbb{Z})$*,* $\mathrm{Aut}(F_n)$*and* $\mathrm{Out}(F_n)$*?*

Question 37 *Is* $\mathrm{Aut}(F_n)$ *automatic for* $n > 3$*?*

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