Asymptotic analysis of the bifurcation diagram for symmetric one-dimensional solutions of the Ginzburg-Landau equations

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July 5, 2000

Abstract

The bifurcation of symmetric superconducting solutions from the normal solution is considered for the one-dimensional Ginzburg-Landau equations by the methods of formal asymptotics. The behaviour of the bifurcating branch depends of the parameters d, the size of the superconducting slab, and κ, the Ginzburg-Landau parameter. It was found numerically in [2] that there are three distinct regions of the (κ, d) plane, labelled S_1, S_2 and S_3, in which there are at most one, two and three symmetric solutions of the Ginzburg-Landau system respectively.

The curve in the (κ, d) plane across which the bifurcation switches from being subcritical to supercritical is identified, which is the boundary between S_2 and $S_1 \cup S_3$, and the bifurcation diagram is analysed in its vicinity.

The triple point, corresponding to the point at which S_1, S_2 and S_3 meet, is determined, and the bifurcation diagram and the boundaries of S_1, S_2 and S_3 are analysed in its vicinity.

The results provide formal evidence for the resolution of some of the conjectures of [2].

1 Introduction

When a superconducting slab of constant thickness, between the planes $x = -d$ and $x = d$ is submitted to an exterior magnetic field $(0, 0, h_0)$, the state of the slab can be described by the functions $(\Psi(x), A(x))$ satisfying the Ginzburg-Landau system:

$$\begin{cases} \Psi'' = \kappa^2 \Psi(\Psi^2 + A^2 - 1) \quad \text{in} \quad (-d, d), \\ \Psi'(\pm d) = 0, \\ A'' = \Psi^2 A \quad \text{in} \quad (-d, d), \\ A'(\pm d) = H. \end{cases} \tag{GL_d}$$

Here, Ψ is the superconducting order parameter, which can be thought of as an averaged wave function of the superconducting electrons, and $(0, A, 0)$ is the magnetic vector potential, so that $(0, 0, A')$ is the magnetic field. This model was first introduced by Ginzburg and Landau [12]. For a more detailed description of the model, one may refer to [2], [10] or [11] and the references therein.

*Supported by a Royal Society University Research Fellowship

Notice that $\Psi \equiv 0$ and $A = H(x+e)$ is always a solution for any real e. From now on, we will call this a *normal solution*. It is well known that when H is too large, superconductivity is destroyed and the only solution of (GL_d) is the normal solution.

There are two types of physically important solutions of (GL_d): symmetric solutions and asymmetric solutions. We define a *symmetric solution* to be a solution of (GL_d) such that $\Psi > 0$, Ψ is even and A is odd on $[-d, d]$. Thus, a symmetric solution satisfies the following problem:

$$\begin{cases} \Psi'' = \kappa^2 \Psi(\Psi^2 + A^2 - 1) \quad \text{in} \quad (0, d), \\ \Psi(0) = \beta, \quad \Psi'(0) = 0, \\ A'' = \Psi^2 A \quad \text{in} \quad (0, d), \\ A(0) = 0, \quad A'(0) = \alpha, \end{cases} \tag{GL_{sym}}$$

for $\beta \in (0, 1)$ and $\alpha \geq 0$. We need to choose α and β such that $\Psi'(d) = 0$. Then (Ψ, A) will be a solution of (GL_d) with $H = A'(d)$.

For the existence of symmetric solutions, Kwong [14] has proved the following important result.

Theorem 1.1 *(Kwong [14]) For each β in $(0, 1)$, there exists a unique $\alpha > 0$ such that the solution (Ψ, A) of (GL_{sym}) satisfies $\Psi'(d) = 0$. Moreover, α is a continuous, decreasing function of β,*

$$\lim_{\beta \to 0} \alpha(\beta) > 0 \quad \textit{and} \quad \lim_{\beta \to 1} \alpha(\beta) = 0. \tag{1.1}$$

Let $H(\beta) = A'(d)$ for this choice of $\alpha(\beta)$. Then H is continuous and

$$\lim_{\beta \to 0} H(\beta) = H_s > 0 \quad \textit{and} \quad \lim_{\beta \to 1} H(\beta) = 0. \tag{1.2}$$

A complete numerical study of the curve $H(\beta)$ has been done in [2] in terms of the parameters d and κ. Recall that for symmetric solutions, $\beta = \|\Psi\|_\infty$. It is found that there are three possible behaviours of the curve $H(\beta)$, which are recalled in Figures 1, 2 and 3: they correspond respectively to regimes when there are at most one, two or three symmetric solutions of (GL_d). Notice that instead of graphing $H(\beta)$ against β, we have put β on the vertical axis and H on the horizontal axis in keeping with the more usual convention for response diagrams, since H is our bifurcation parameter.

In [2], the regimes of parameters d and κ where each behaviour of the curve $H(\beta)$ holds have been classified. The results of these numerical investigations are shown graphically in Figure 4. They indicate that the (κ, d) plane is the union of three connected sets S_1, S_2 and S_3. In S_1, the behaviour of $H(\beta)$ of Figure 1 holds. Likewise, S_2 and S_3 reflect the behaviour of $H(\beta)$ of Figures 2 and 3 respectively.

In [2] the authors make the following conjecture.

Conjecture 1.2 *([2]) There exist three continuous functions $\kappa_1(d)$, $\kappa_2(d)$ and $\kappa_3(d)$ which separate the (κ, d) plane into three connected regions S_1, S_2 and S_3, as shown in Figure 4. There exists exactly one point d_* called the triple point (with approximate value 1.6) such that $\kappa_1(d_*) = \kappa_2(d_*) = \kappa_3(d_*) = \kappa_* \simeq 0.93$. Moreover, $\kappa_1(d)$ is defined and monotone increasing on $[\sqrt{5}/2, d_*]$, $\kappa_3(d)$ is defined and monotone increasing on $[d_*, \infty]$ and $\kappa_2(d)$ is defined and monotone decreasing on $[d_*, \infty]$, and*

$$\lim_{d \to \sqrt{5}/2} \kappa_1(d) = 0, \quad \lim_{d \to \infty} \kappa_2(d) = 1/\sqrt{2}, \quad \lim_{d \to \infty} \kappa_3(d) = +\infty. \tag{1.3}$$

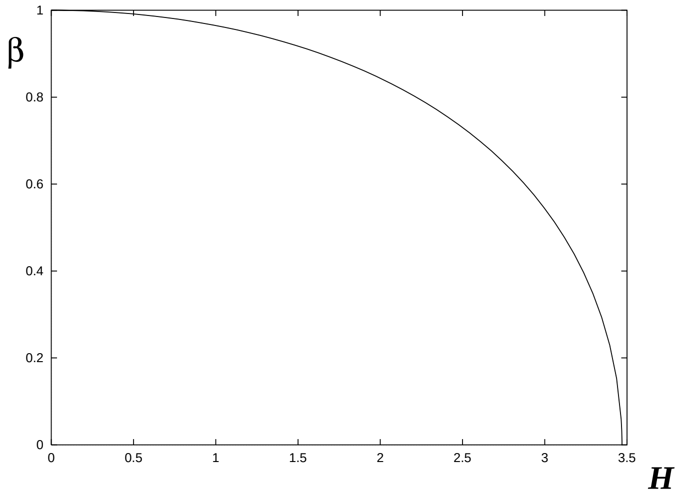

Figure 1: Curve $H(\beta)$ for $d = 0.5$ and $\kappa = 0.4$

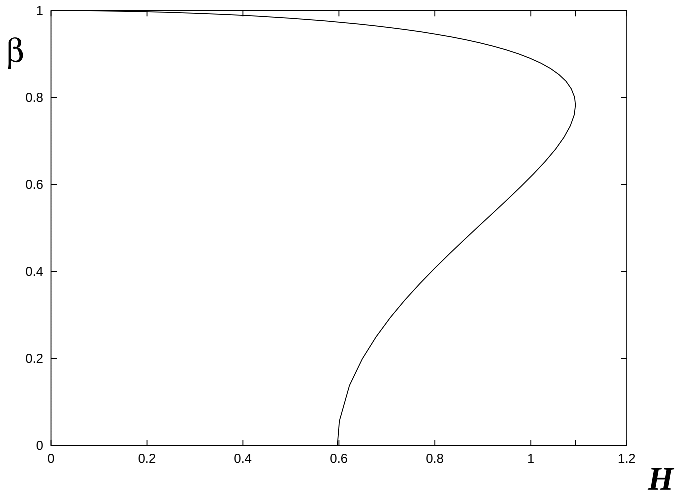

Figure 2: Curve $H(\beta)$ for $d = 3$ and $\kappa = 0.3$

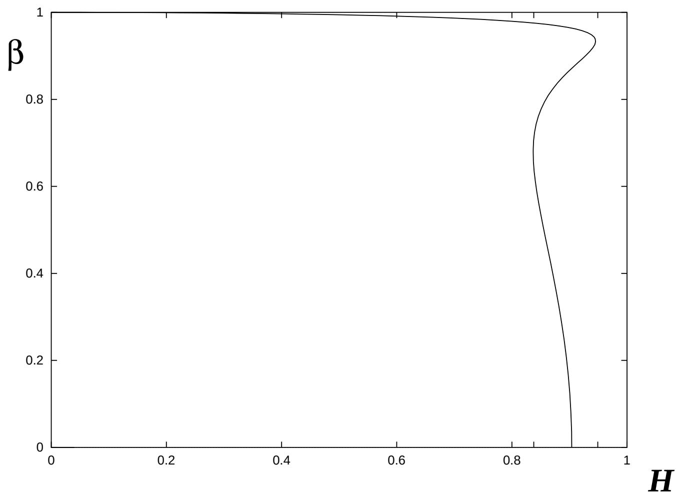

Figure 3: Curve $H(\beta)$ for $d = 3$ and $\kappa = 0.9$

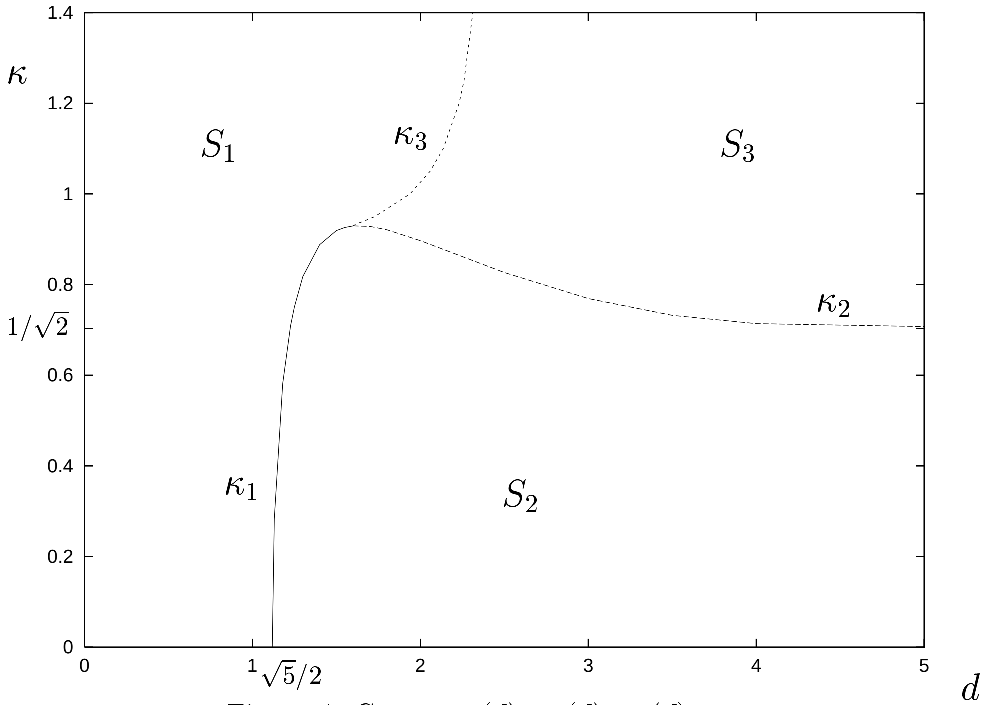

Figure 4: Curves $\kappa_1(d)$, $\kappa_2(d)$, $\kappa_3(d)$.

We will be concerned here only with the bifurcation of symmetric solutions of (GL_d) from the normal solution, which is a study of the curve $H(\beta)$ near $\beta = 0$. As explained in [2], the curves $\kappa_1(d)$ and $\kappa_2(d)$ correspond to the points (κ, d) where this bifurcation switches from being supercritical in $S_1 \cup S_3$ to subcritical in S_2. This bifurcation was first studied by Ginzburg & Landau on an infinite domain in [12], and by Odeh on a finite slab in [16]. Millman & Keller developed a systematic perturbation theory to study the bifurcation in [15], which was used to study the stability of the bifurcating solutions in [8, 9]. We note also that there has been a great deal of recent rigorous work on the symmetric solutions of the one-dimensional Ginzburg-Landau model including [1], the work of Bolley & Helffer [4, 5, 6, 7] and that of Hastings et al. [13]. For an overview of all these results we refer the reader to [2]. In particular we note that in [7] the curve $\kappa_1(d)$ was identified near $d = \sqrt{5}/2$ and the curve $\kappa_2(d)$ near $\kappa = 1/\sqrt{2}$, by studying the change of bifurcation near the normal solution.

Here we are concerned with the special values of κ and d for which the standard bifurcation analysis introduced by Millman & Keller needs to be modified. In Section 2 we study asymptotically the bifurcation from the normal solution to a superconducting solution, and identify the curve $\kappa_1 \cup \kappa_2$ across which this bifurcation switches from being sub to supercritical. We then study the response diagram in the vicinity of $\kappa_1 \cup \kappa_2$, and identify the triple point.

In Section 3 we study the response diagram in the vicinity of the triple point. We find that κ_3 bifurcates from $\kappa_1 \cup \kappa_2$ tangentially at the triple point, and that it does so at a value of d smaller than that corresponding to the maximum of $\kappa_1 \cup \kappa_2$. Thus we find that κ_2 is not monotonic decreasing but has a maximum.

Finally, in Section 4, we present our conclusions.

2 Asymptotic analysis of the bifurcation from the normal state

Since the size of the domain d is a parameter which we will want to vary, we begin by rescaling distance with d so that the domain will remain fixed as $-1 < x < 1$. The Ginzburg-Landau system is then

$$\begin{aligned} \frac{1}{\kappa^2 d^2}\Psi'' &= \Psi^3 - \Psi + \Psi A^2, && (2.4)\\ A'' &= d^2\Psi^2 A, && (2.5)\\ \Psi'(\pm 1) &= 0, && (2.6)\\ A'(\pm 1) &= Hd. && (2.7) \end{aligned}$$

We wish to examine the bifurcation from the normal state to a superconducting state, and in particular to determine the behaviour of the curve $H(\beta)$ near $\beta = 0$. Close to the bifurcation point Ψ will be small. We quantify this smallness by introducing a small parameter ε and setting

$$\begin{aligned} \Psi &= \varepsilon^{1/2} f, && (2.8)\\ A &= Hdx + \varepsilon q, && (2.9) \end{aligned}$$

as in [3, 8, 9, 15]. The relative scaling of f and q here is motivated by the fact that we want the nonlinear term Ψ^3 to balance with the correction to ΨA^2 in (2.4). For ease of notation we set

$$p = \frac{1}{\kappa^2 d^2}, \qquad h = Hd. \qquad (2.10)$$

Then f and q satisfy

$$\begin{aligned} pf'' &= \varepsilon f^3 - f + f(h^2x^2 + 2\varepsilon hxq + \varepsilon^2 q^2), && (2.11)\\ q'' &= d^2 f^2 (hx + \varepsilon q), && (2.12)\\ f'(\pm 1) &= 0, && (2.13)\\ q'(\pm 1) &= 0. && (2.14) \end{aligned}$$

We formally expand all quantities in powers of ε:

$$\begin{aligned} f &= f_0 + \varepsilon f_1 + \cdots, && (2.15)\\ q &= q_0 + \varepsilon q_1 + \cdots, && (2.16)\\ h &= h_0 + \varepsilon h_1 + \cdots, && (2.17)\\ p &= p_0 + \varepsilon p_1 + \cdots, && (2.18)\\ d &= d_0 + \varepsilon d_1 + \cdots. && (2.19) \end{aligned}$$

Note that d, p and h are expanded as well as f and q. This is to allow us to consider what happens for (d, p) close to a point (d_0, p_0) on the curve $\kappa_1 \cup \kappa_2$, and for h close to the bifurcation point h_0. Note also that the expansions for p and h can be related to the corresponding expansions for the original parameters κ and H by substituting both expansions into (2.10) and equating coefficients of powers of ε.

Leading order Substituting the expansions (2.15)-(2.19) into (2.11)-(2.12) and equating coefficients of powers of ε we find that at leading order

$$Lf_0 \equiv p_0 f_0'' + f_0 - f_0 h_0^2 x^2 = 0, \tag{2.20}$$

$$q_0'' = d_0^2 f_0^2 h_0 x, \tag{2.21}$$

with homogeneous Neumann boundary data. Let h_0 be the leading eigenvalue of (2.20), and θ_0 the corresponding normalised eigenfunction (with the L^∞ norm, say). Then $f_0 = C\theta_0$, where C is undetermined at present. We see that $q_0 = C^2 a_0$, where

$$a_0'' = d_0^2 \theta_0^2 h_0 x, \tag{2.22}$$

with homogeneous Neumann boundary data. Thus q_0 is determined once C is known. Note that there is a solvability condition on (2.22) since the homogeneous version of the equation and boundary conditions is satisfied by a constant, so that by the Fredholm alternative the integral of the right-hand side must be zero. However, this solvability condition is automatically satisfied since θ_0 is even.

To determine C we need to proceed to first order in the expansions.

First order At first order in (2.11), (2.12) we find

$$Lf_1 = -p_1 f_0'' + f_0^3 + 2h_0 h_1 x^2 f_0 + 2h_0 x f_0 q_0, \tag{2.23}$$

$$q_1'' = 2d_0 d_1 f_0^2 h_0 x + 2d_0^2 f_0 f_1 h_0 x + d_0^2 f_0^2 (h_1 x + q_0), \tag{2.24}$$

with homogeneous Neumann boundary data. Since f_0 satisfies the homogeneous version of (2.23) there is a solvability condition for f_1 by the Fredholm alternative. This condition is derived by multiplying (2.23) by f_0, integrating and using (2.20), to give

$$h_1 = \alpha_0 C^2 + \Gamma_1 p_1, \tag{2.25}$$

where

$$\alpha_0 = -\frac{\int_{-1}^{1}\left(\theta_0^4 - \frac{2(a_0')^2}{d_0^2}\right) dx}{2h_0 \int_{-1}^{1} x^2 \theta_0^2 \, dx}, \tag{2.26}$$

$$\Gamma_1 = -\frac{\int_{-1}^{1} (\theta_0')^2 \, dx}{2h_0 \int_{-1}^{1} x^2 \theta_0^2 \, dx}. \tag{2.27}$$

If $\alpha_0 \neq 0$ then this equation gives C in terms of h_1, and so completes the leading order solution.

Since $||\Psi||_\infty = \varepsilon^{1/2} C$ to leading order, the graph of $h(C)$ is an approximation to the bifurcation diagram $H(\beta)$ for small β. If $\alpha_0 > 0$ then, in the vicinity of the bifurcation point, the solution exists for values of the magnetic field slightly greater than the bifurcation value; the bifurcation is subcritical and (κ, d) is in S_2 (Figure 2). For $\alpha_0 < 0$ the solution exists for values of the magnetic field slightly smaller than the bifurcation value; the bifurcation is supercritical and (κ, d) is in S_1 (Figure 1) or S_3 (Figure 3).

The curve $\alpha_0 = 0$ is the curve across which the bifurcation changes from being subcritical to supercritical, and therefore it corresponds to the curve $\kappa_1 \cup \kappa_2$ in [2] (see Conjecture 1.2). A numerical evaluation of the zero set of α_0 is shown in Figure 5; henceforth we assume its existence. Note that Figure 5 indicates that $\kappa_1 \cup \kappa_2$ is uniquely determined as a function of d, which we

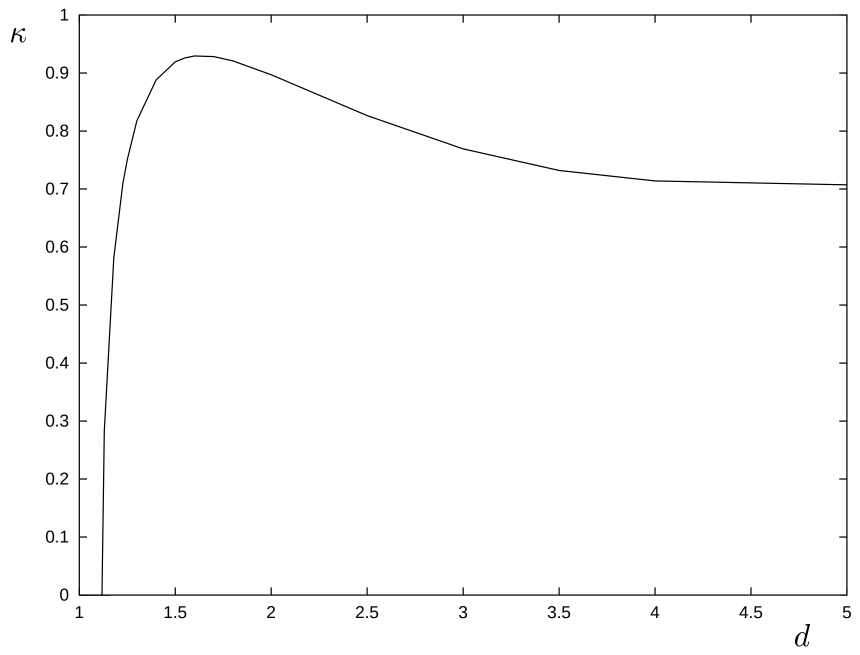

Figure 5: The curve in the (κ_0, d_0) plane corresponding to $\alpha_0 = 0$.

henceforth assume. Note, though, that a similar analysis could be performed if this were not the case.

In the case $\alpha_0 = 0$, C is not determined by (2.25), and we must proceed further down the asymptotic expansion to determine it. To examine the behaviour in the vicinity of $\kappa_1 \cup \kappa_2$ let us suppose that p_0 and d_0 are such that $\alpha_0 = 0$, so that (κ, d) lies on the curve $\kappa_1 \cup \kappa_2$ to leading order. Then we find that

$$f_1 = C^3\theta_1 + Cp_1\phi_1 + D\theta_0, \tag{2.28}$$

where D is arbitrary at present and

$$\begin{aligned} L\theta_1 &= \theta_0^3 + 2\theta_0 h_0 x a_0, && (2.29)\\ L\phi_1 &= 2\theta_0\Gamma_1 h_0 x^2 - \theta_0'', && (2.30) \end{aligned}$$

with homogeneous Neumann boundary data. Note that the solvability conditions which guarantee the existence of a solution to (2.29) and (2.30) are $\alpha_0 = 0$ and (2.27) respectively. Since we may add any multiple of θ_0 to θ_1 and ϕ_1, for definiteness we fix them by requiring that they are equal to zero at the origin.

Substituting (2.28) into (2.24) we find that q_1 is given by

$$q_1 = C^4 a_1 + \frac{2C^2 d_1 a_0}{d_0} + C^2 p_1 b_1 + 2CDa_0, \tag{2.31}$$

where

$$\begin{aligned} a_1'' &= 2d_0^2\theta_0\theta_1 h_0 x + d_0^2\theta_0^2 a_0, && (2.32)\\ b_1'' &= 2d_0^2\theta_0\phi_1 h_0 x + d_0^2\theta_0^2\Gamma_1 x. && (2.33) \end{aligned}$$

Note that we have chosen q_1 to be odd, that θ_1, ϕ_1, a_1 and b_1 depend only on p_0 and d_0, and not on p_1 and d_1. Note also that θ_1 and ϕ_1 are even, while a_1 and b_1 are odd, so that the solvability conditions on (2.32) and (2.33) are automatically satisfied.

Second order Proceeding to the next order in the expansion in (2.11) we find

$$\begin{aligned} Lf_2 &= -p_2 f_0'' - p_1 f_1'' + 3f_0^2 f_1 + f_1(2h_0h_1x^2 + 2h_0xq_0) \\ &\qquad + f_0(2h_0h_2x^2 + 2h_0xq_1 + (h_1x+q_0)^2). \end{aligned} \tag{2.34}$$

As before, since f_0 satisfies the homogeneous version of this equation there is a solvability condition which is derived by multiplying by f_0 and integrating. We find, after some manipulation that when $\alpha_0 = 0$, the terms involving D cancel (as they should), leaving

$$h_2 = \alpha_1 C^4 + \beta_1 C^2 + \gamma_1, \tag{2.35}$$

where

$$\begin{aligned} \alpha_1 &= -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\left(\frac{2}{d_0^2}\int_{-1}^1 a_0''a_1\,dx + \int_{-1}^1 \theta_0^2 a_0^2\,dx \right. \\ &\qquad\qquad \left. + 2h_0\int_{-1}^1 \theta_0 x a_0\theta_1\,dx + 3\int_{-1}^1 \theta_0^3\theta_1\,dx\right), \qquad (2.36) \\ \beta_1 &= -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\left(\lambda d_1 + \mu p_1\right), \qquad (2.37) \\ \gamma_1 &= p_1^2\Gamma_2 + p_2\Gamma_1, \qquad (2.38) \\ \lambda &= -\frac{4}{d_0^3}\int_{-1}^1 (a_0')^2\,dx, \qquad (2.39) \\ \mu &= \frac{2}{d_0^2}\int_{-1}^1 a_0''b_1\,dx + 2\Gamma_1\int_{-1}^1 \theta_0^2 x a_0\,dx + 2h_0\Gamma_1\int_{-1}^1 x^2\theta_0\theta_1\,dx + 2h_0\int_{-1}^1 \theta_0 x a_0\phi_1\,dx \\ &\qquad + 3\int_{-1}^1 \theta_0^3\phi_1\,dx + \int_{-1}^1 \theta_0'\theta_1'\,dx, \qquad (2.40) \\ \Gamma_2 &= -\frac{1}{2h_0\int_{-1}^1 x^2\theta_0^2\,dx}\left(\Gamma_1^2\int_{-1}^1 x^2\theta_0^2\,dx + 2h_0\Gamma_1\int_{-1}^1 x^2\theta_0\phi_1\,dx + \int_{-1}^1 \theta_0'\phi_1'\,dx\right). \qquad (2.41) \end{aligned}$$

Equation (2.35) gives C as a function of h_2, thus completing the leading order solution and the bifurcation diagram.

Now α_1 depends only on p_0 and d_0, that is, it is a function only of the position of (p_0, d_0) on the curve $\kappa_1 \cup \kappa_2$. This position is determined uniquely by d_0; a numerical evaluation of the graph of α_1 against d_0 is shown in Figure 6. We see that there is a point $d_0 = d_* \approx 1.55$ at which $\alpha_1 = 0$; for $d_0 < d_*$, $\alpha_1 < 0$, while for $d_0 > d_*$, $\alpha_1 > 0$.

On the other hand, β_1 is linear in p_1 and d_1 and so is dependent on the deviation of (p, d) from the curve $\kappa_1 \cup \kappa_2$; β_1 is zero if (p, d) is exactly on the curve, and changes sign as (p, d) crosses it. The easiest way to determine the sign of β_1 on each side is to consider the case in which $p_1 = 0$. Then

$$\beta_1 = \frac{2d_1\int_{-1}^1 (a_0')^2\,dx}{d_0^3 h_0\int_{-1}^1 x^2\theta_0^2\,dx},$$

so that β_1 is the same sign as d_1 when $p_1 = 0$. This means that β_1 is positive in S_2 and negative in $S_1 \cup S_3$.

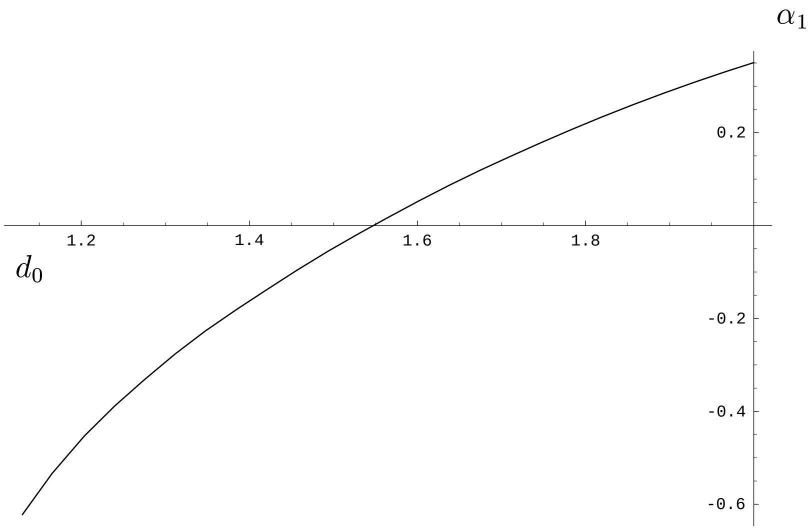

Figure 6: The graph of α_1 as a function of d_0.

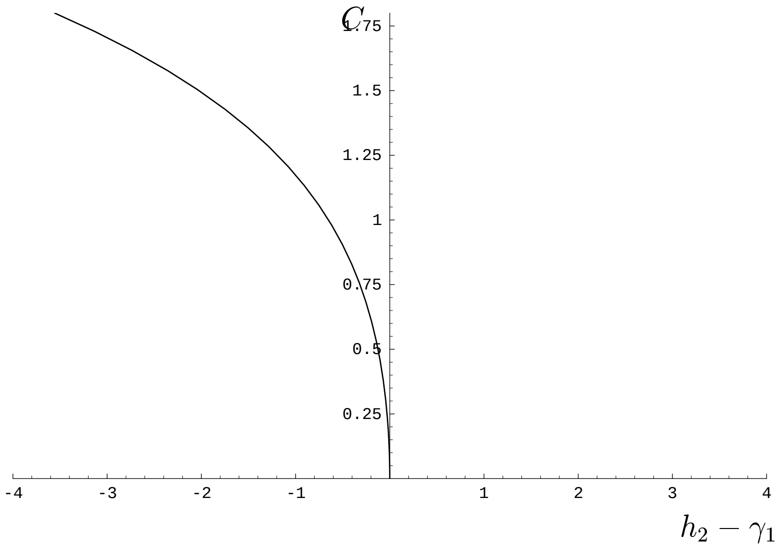

Figure 7: The bifurcation curve for $\alpha_1 = -0.2$, $\beta_1 = -0.45$, corresponding to $d_0 = 1.37$, $p_1 = 0$, $d_1 = -0.38$, which is in region S_1.

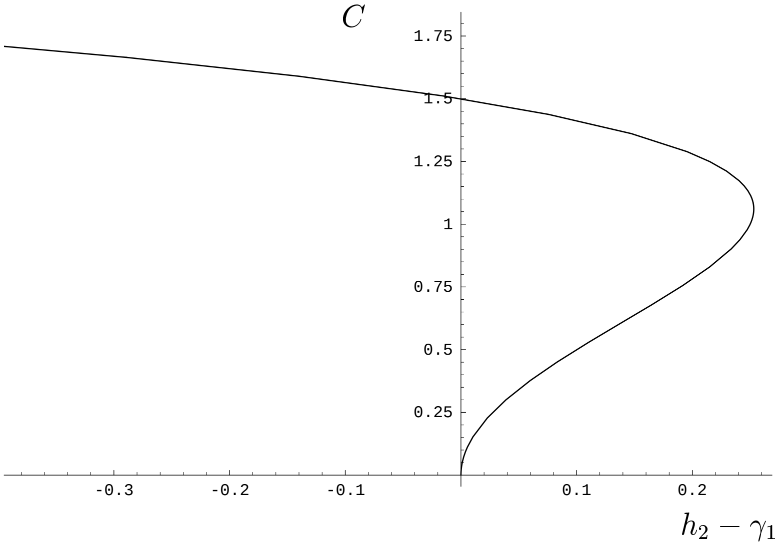

Figure 8: The bifurcation curve for $\alpha_1 = -0.2$, $\beta_1 = 0.45$, corresponding to $d_0 = 1.37$, $p_1 = 0$, $d_1 = 0.38$, which is in region S_2.

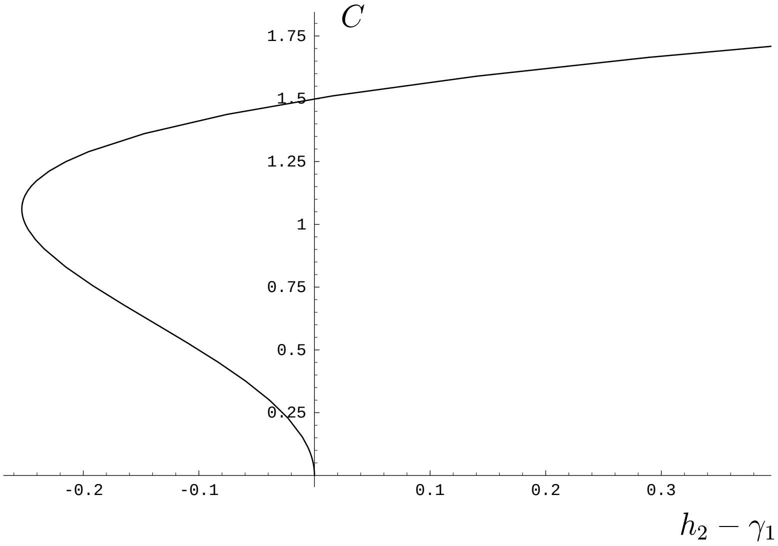

Figure 9: The bifurcation curve for $\alpha_1 = 0.2$, $\beta_1 = -0.45$, corresponding to $d_0 = 1.77$, $p_1 = 0$, $d_1 = -0.43$, which is in region S_3.

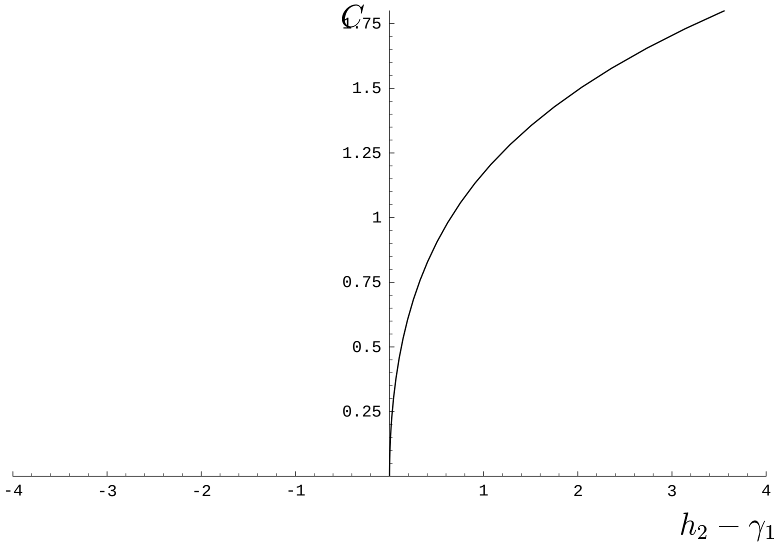

Figure 10: The bifurcation curve for $\alpha_1 = 0.2$, $\beta_1 = 0.45$, corresponding to $d_0 = 1.77$, $p_1 = 0$, $d_1 = 0.43$, which is in region S_2.

Figures 7-8 show typical bifurcation curves for $d_0 < d_*$. These two figures indicate what happens as (d,p) cross κ_1: as (d,p) moves from S_1 into S_2, a fold in the bifurcation curve (see Figure 2) appears near $C = 0$. In the case of Figure 7, $d_1 < 0$ so the bifurcation is supercritical, and (d,p) is in S_1. In the case of Figure 8, $d_1 > 0$ so the bifurcation is subcritical, and (d,p) is in S_2.

We recall that $||\Psi|| = \varepsilon^{1/2}C$ to leading order so that the folds in Figures 7 and 8 lie very close to the normal solution.

Figures 9-10 show typical bifurcation curves for $d_0 > d_*$. These two figures indicate what happens as (d,p) cross κ_2: as (d,p) moves from S_2 into S_3, the lower fold in the bifurcation curve (see Figure 3) appears near $C = 0$. In the case of Figure 9, $d_1 < 0$ so the bifurcation is supercritical, and it has a fold so (d,p) is in S_3. In the case of Figure 10, $d_1 > 0$ so the bifurcation is subcritical, and (d,p) is in S_2.

If $d_0 = d_*$ then $\alpha_0 = \alpha_1 = 0$, and C is not determined by (2.35) if d_1 and p_1 are such that $\beta_1 = 0$. This corresponds to the triple point, since it corresponds to the switch in behaviour from that shown in Figures 7-8 to that shown in Figures 9-10. In this case we must proceed even further down the expansions to determine C.

Note that $d_* \approx 1.55$ is to the left of the maximum of $\kappa_1 \cup \kappa_2$ shown in Figure 5, which occurs for $d \approx 1.64$. Since d_* divides this curve into κ_1 and κ_2 we conclude that κ_2 is not monotone decreasing but in fact has a maximum.

3 The triple point

To examine the behaviour in the vicinity of the triple point, let us suppose that p_0 and d_0 are such that $\alpha_1 = 0$ as well as $\alpha_0 = 0$, that is (p_0, d_0) is exactly the triple point. Then we must also take p_1 and d_1 to be such that $\beta_1 = 0$. This immediately tells us that the curve κ_3 bifurcates from the curve $\kappa_1 \cup \kappa_2$ tangentially, which will become explicitly clear later.

Since we wish to see how the behaviour of the solution changes as we move around the (d, p) plane, it makes sense to rotate coordinates in this plane so that one of the coordinate directions corresponds to $\beta_1 = 0$. Therefore, in the light of (2.37), we set

$$z = \frac{\lambda}{\lambda^2+\mu^2}(\lambda d + \mu p), \tag{3.42}$$

$$y = \frac{\lambda}{\lambda^2+\mu^2}(-\mu d + \lambda p), \tag{3.43}$$

so that

$$d = z - \frac{\mu y}{\lambda}, \tag{3.44}$$

$$p = \frac{\mu z}{\lambda} + y. \tag{3.45}$$

We expand z and y in powers of ε, exactly as for p and d. Then $\beta_1 = 0$ implies that $z_1 = 0$. The solution of (2.34) is given by

$$f_2 = C^5\theta_2 + C^3 y_1\phi_2 + C\left(\frac{\mu z_2}{\lambda} + y_2\right)\phi_1 + C y_1^2\psi_2 + 3C^2 D\theta_1 + D y_1\phi_1 + E\theta_0, \tag{3.46}$$

where

$$L\theta_2 = 3\theta_0^2\theta_1 + 2h_0 x a_0\theta_1 + 2h_0 x\theta_0 a_1 + \theta_0 a_0^2, \tag{3.47}$$

$$L\phi_2 = -\theta_1'' + 2h_0\Gamma_1 x^2\theta_1 + 3\theta_0^2\phi_1 + 2h_0 x a_0\phi_1 - \frac{4\mu a_0 h_0 x\theta_0}{d_0\lambda} + 2h_0 x\theta_0 b_1 + 2\Gamma_1 x a_0\theta_0, \tag{3.48}$$

$$L\psi_2 = -\phi_1'' + 2h_0\Gamma_1 x^2\phi_1 + 2h_0\Gamma_2 x^2\theta_0 + \Gamma_1^2 x^2\theta_0, \tag{3.49}$$

and E is arbitrary at present. Note that the solvability conditions guaranteeing the existence of a solution to (3.47), (3.48) and (3.49) are $\alpha_1 = 0$, (2.39) and (2.40), and (2.41) respectively. As before we may add any multiple of θ_0 to θ_2, ϕ_2 and ψ_2. Again, for definiteness, we fix these functions by requiring that they are equal to zero at the origin. At second order in (2.12) we find

$$\begin{aligned} q_2'' &= (2d_0 d_2 + d_1^2) f_0^2 h_0 x + 2d_0 d_1\left[2f_0 f_1 h_0 x + f_0^2(h_1 x + q_0)\right] \\ &\quad + d_0^2\left[(2f_0 f_2 + f_1^2)h_0 x + 2f_0 f_1(h_1 x + q_0) + f_0^2(h_2 x + q_1)\right]. \end{aligned} \tag{3.50}$$

Substituting in the expressions for f_1 and f_2 etc. we find

$$\begin{aligned} q_2 &= C^6 a_2 + C^4 y_1 b_2 + C^2\left(y_1^2 c_2 + \left(y_2 + \frac{\mu z_2}{\lambda}\right) b_1 + \left(z_2 - \frac{\mu y_2}{\lambda}\right)\frac{2a_0}{d_0}\right) \\ &\quad + 4C^3 D a_1 + 2CD\left(b_1 - \frac{2\mu a_0}{d_0\lambda}\right) + (D^2 + 2CE)a_0, \end{aligned} \tag{3.51}$$

where

$$a_2'' = 2d_0^2\theta_0\theta_2 h_0 x + h_0 x\theta_1^2 d_0^2 + 2\theta_0\theta_1 a_0 d_0^2 + d_0^2\theta_0^2 a_1, \tag{3.52}$$

$$\begin{aligned} b_2'' &= -\frac{4\mu}{\lambda}d_0\theta_0 h_0 x\theta_1 - \frac{2\mu}{\lambda}d_0\theta_0^2 a_0 + 2d_0^2\theta_0\phi_2 h_0 x + 2d_0^2 h_0 x\theta_1\phi_1 \\ &\quad + 2d_0^2\theta_0\theta_1\Gamma_1 x + 2d_0^2\theta_0\phi_1 a_0 + d_0^2\theta_0^2 b_1, \end{aligned} \tag{3.53}$$

$$\begin{aligned} c_2'' &= \frac{\mu^2}{\lambda^2}\theta_0^2 h_0 x - \frac{4\mu}{\lambda}d_0\theta_0 h_0 x\phi_1 - \frac{2\mu}{\lambda}d_0\theta_0^2\Gamma_1 x + 2d_0^2\theta_0\psi_2 h_0 x \\ &\quad + d_0^2 h_0 x\phi_1^2 + 2d_0^2\theta_0\phi_1\Gamma_1 x + d_0^2\theta_0^2\Gamma_2 x, \end{aligned} \tag{3.54}$$

and q_2 is chosen to be odd. As usual, θ_2, ϕ_2 and ψ_2 are even, while a_2, b_2 and c_2 are odd, so that the solvability conditions on (3.52)-(3.54) are automatically satisfied.

Third order Proceeding to third order in (2.11) we find

$$\begin{aligned} Lf_3 \;=\; & -p_3 f_0'' - p_2 f_1'' - p_1 f_2'' + 3f_0^2 f_2 + 3f_0 f_1^2 + 2h_0 h_1 x^2 f_2 + 2h_0 x f_2 q_0 + f_1 h_1^2 x^2 \\ & + 2f_1 x q_0 h_1 + f_1 q_0^2 + 2h_0 x^2 h_2 f_1 + 2f_1 h_0 x q_1 + 2h_1 h_2 x^2 f_0 + 2h_1 x q_1 f_0 \\ & + 2h_2 x q_0 f_0 + 2f_0 q_0 q_1 + 2f_0 h_0 h_3 x^2 + 2h_0 x q_2 f_0. \end{aligned} \tag{3.55}$$

As usual there is a solvability condition. Multiplying by f_0 and integrating, using the expressions for f_1 etc., we find after some manipulation that the terms involving D and E cancel (again, as they should) to leave

$$h_3 = \alpha_2 C^6 + \beta_2 y_1 C^4 + (\gamma_{21} y_1^2 + \gamma_{22} z_2) C^2 + \delta_2, \tag{3.56}$$

where

$$\begin{aligned} \alpha_2 \;=\; & -\frac{1}{2h_0 \int_{-1}^1 x^2\theta_0^2\,dx}\left(3\int_{-1}^1 \theta_0^3\theta_2\,dx + 3\int_{-1}^1 \theta_0^2\theta_1^2\,dx + 2h_0\int_{-1}^1 x\theta_0\theta_2 a_0\,dx + \int_{-1}^1 \theta_0\theta_1 a_0^2\,dx \right.\\ & \left. + 2h_0\int_{-1}^1 x\theta_0\theta_1 a_1\,dx + 2\int_{-1}^1 \theta_2^2 a_0 a_1\,dx + 2h_0\int_{-1}^1 x\theta_0^2 a_2\,dx\right), \end{aligned} \tag{3.57}$$

$$\begin{aligned} \beta_2 \;=\; & -\frac{1}{2h_0 \int_{-1}^1 x^2\theta_0^2\,dx}\left(\int_{-1}^1 \theta_0'\theta_2'\,dx + 3\int_{-1}^1 \theta_0^3\phi_2\,dx + 6\int_{-1}^1 \theta_0^2\theta_1\phi_1\,dx + 2h_0\Gamma_1\int_{-1}^1 x^2\theta_0\theta_2\,dx\right.\\ & + 2h_0\int_{-1}^1 x\theta_0\phi_2 a_0\,dx + 2\Gamma_1\int_{-1}^1 x\theta_0\theta_1 a_0\,dx + \int_{-1}^1 \theta_0\phi_1 a_0^2\,dx \\ & - 2h_0\int_{-1}^1 x\theta_0\theta_1\frac{2\mu a_0}{\lambda d_0}\,dx + 2h_0\int_{-1}^1 x\theta_0\theta_1 b_1\,dx + 2h_0\int_{-1}^1 x\theta_0\phi_1 a_1\,dx + 2\Gamma_1\int_{-1}^1 x\theta_0^2 a_1\,dx \\ & \left. - 2\int_{-1}^1 \theta_0^2 a_0\frac{2\mu a_0}{\lambda d_0}\,dx + 2\int_{-1}^1 \theta_0^2 a_0 b_1\,dx + 2h_0\int_{-1}^1 x\theta_0^2 b_2\,dx\right), \end{aligned} \tag{3.58}$$

$$\begin{aligned} \gamma_{21} \;=\; & -\frac{1}{2h_0 \int_{-1}^1 x^2\theta_0^2\,dx}\left(\int_{-1}^1 \theta_0'\phi_2'\,dx + 3\int_{-1}^1 \theta_0^3\psi_2\,dx + 3\int_{-1}^1 \theta_0^2\phi_1^2\,dx + 2h_0\Gamma_1\int_{-1}^1 x^2\theta_0\phi_2\,dx\right.\\ & + 2h_0\int_{-1}^1 x\theta_0\psi_2 a_0\,dx + \Gamma_1^2\int_{-1}^1 x^2\theta_0\theta_1\,dx + 2\Gamma_1\int_{-1}^1 x\theta_0\phi_1 a_0\,dx + 2h_0\Gamma_2\int_{-1}^1 x^2\theta_0\theta_1\,dx \\ & - 2h_0\int_{-1}^1 x\theta_0\phi_1\frac{2\mu a_0}{\lambda d_0}\,dx + 2h_0\int_{-1}^1 x\theta_0\phi_1 b_1\,dx - 2\Gamma_1\int_{-1}^1 x\theta_0^2\frac{2\mu a_0}{\lambda d_0}\,dx + 2\Gamma_1\int_{-1}^1 x\theta_0^2 b_1\,dx \\ & \left. + 2\Gamma_2\int_{-1}^1 x\theta_0^2 a_0\,dx + 2h_0\int_{-1}^1 x\theta_0^2 c_2\,dx\right), \end{aligned} \tag{3.59}$$

$$\gamma_{22} = -\frac{1}{2h_0 \int_{-1}^1 x^2\theta_0^2\,dx}\left(-\frac{4}{d_0^3}\left(1+\frac{\mu^2}{\lambda^2}\right)\int_{-1}^1 (a_0')^2\,dx\right), \tag{3.60}$$

$$\delta_2 = \left(\frac{\mu z_3}{\lambda} + y_3\right)\Gamma_1 + 2\left(\frac{\mu z_2}{\lambda} + y_2\right) y_1\Gamma_2 + y_1^3\Gamma_3, \tag{3.61}$$

$$\begin{aligned} \Gamma_3 \;=\; & -\frac{1}{2h_0 \int_{-1}^1 x^2\theta_0^2\,dx}\left(\int_{-1}^1 \theta_0'\psi_2'\,dx + 2h_0\Gamma_1\int_{-1}^1 x^2\theta_0\psi_2\,dx + \Gamma_1^2\int_{-1}^1 x^2\theta_0\phi_1\,dx\right.\\ & \left. + 2h_0\Gamma_2\int_{-1}^1 x^2\theta_0\phi_1\,dx + 2\Gamma_1\Gamma_2\int_{-1}^1 x^2\theta_0^2\,dx\right). \end{aligned} \tag{3.62}$$

Equation (3.56) now determines C as a function of h_3, so long as $\alpha_2 \neq 0$. Now, however, the coefficients α_2, β_2, γ_{21}, and γ_{22} are functions only of p_0 and d_0, which are equal to their values at the triple point. Hence these coefficients are fixed numbers. Numerically we find that

$$\alpha_2 = -0.145, \qquad \beta_2 = -0.681, \qquad \gamma_{21} = -2.155, \qquad \gamma_{22} = 1.589,$$

$$\lambda = -0.602, \qquad \mu = -0.826,$$

so that

$$h_3 = -0.145C^6 - 0.681y_1C^4 - (2.155y_1^2 - 1.589z_2)C^2 + \delta_2. \tag{3.63}$$

To determine the nature of the response diagram implied by (3.63), let us consider for a moment the equation

$$h_3 = \bar{\alpha}C^6 + \bar{\beta}C^4 + \bar{\gamma}C^2 + \bar{\delta}, \tag{3.64}$$

with $\bar{\alpha} < 0$. Since h_3 is then negative for large C, there are three possible behaviours for h_3, corresponding to S_1, S_2 and S_3 respectively.

If $\bar{\gamma} > 0$ then $h_3 > \bar{\delta}$ for small C, corresponding to S_2, while if $\bar{\gamma} < 0$ then $h_3 < \bar{\delta}$ for small C, corresponding to S_1 and S_3. Thus the delineation between S_2 and $S_1 \cup S_3$, namely $\kappa_1 \cup \kappa_2$, is given by $\bar{\gamma} = 0$ in the vicinity of the triple point.

Let us now consider

$$\frac{dh_3}{dC} = 6\bar{\alpha}C^5 + 4\bar{\beta}C^3 + 2\bar{\gamma}C. \tag{3.65}$$

The roots of this equation are at $C = 0$ and

$$C^2 = -\frac{1}{3\bar{\alpha}}\left(\bar{\beta} \pm (\bar{\beta}^2 - 3\bar{\alpha}\bar{\gamma})^{1/2}\right). \tag{3.66}$$

Now, since $\bar{\alpha} < 0$, if $\bar{\gamma} > 0$ there is exactly one positive root to (3.66), corresponding to S_2, while if $\bar{\gamma} < 0$ there are either two positive roots or no positive roots, corresponding to S_3 and S_1 respectively, in agreement with the argument above.

With $\bar{\gamma} < 0$ there are two positive roots if and only if

$$\bar{\beta} > 0 \text{ and } \bar{\beta}^2 > 3\bar{\alpha}\bar{\gamma}. \tag{3.67}$$

Hence the delineation between S_1 and S_3, namely κ_3, is given by

$$\bar{\beta} = (3\bar{\alpha}\bar{\gamma})^{1/2}.$$

Thus we find that κ_1 is given by $\bar{\gamma} = 0$, $\bar{\beta} < 0$, κ_2 is given by $\bar{\gamma} = 0$, $\bar{\beta} > 0$, and κ_3 is given by $\bar{\beta} = (3\bar{\alpha}\bar{\gamma})^{1/2}$.

In terms of y and z we find κ_1 is given by $z_2 = -(\gamma_{21}/\gamma_{22})y_1^2$, $y_1 > 0$, κ_2 is given by $z_2 = -(\gamma_{21}/\gamma_{22})y_1^2$, $y_1 < 0$, and κ_3 is given by $z_2 = (\beta_2^2 - 3\alpha_2\gamma_{21})/(3\alpha_2\gamma_{22})y_1^2$, $y_1 < 0$. We see explicitly now that the bifurcating curves meet tangentially in the (y, z) plane, then using (3.44)-(3.45) in the κd-plane. Evaluating the coefficients numerically we find κ_1 is given by $z_2 = 1.356y_1^2$, $y_1 > 0$, κ_2 is given by $z_2 = 1.356y_1^2$, $y_1 < 0$, and κ_3 is given by $z_2 = 0.685y_1^2$, $y_1 < 0$.

These curves are shown in the (y, z) plane in Figure 11, and in the (κ, d) plane in Figure 12.

Figures 13-15 show typical bifurcation curves in in vicinity of the triple point.

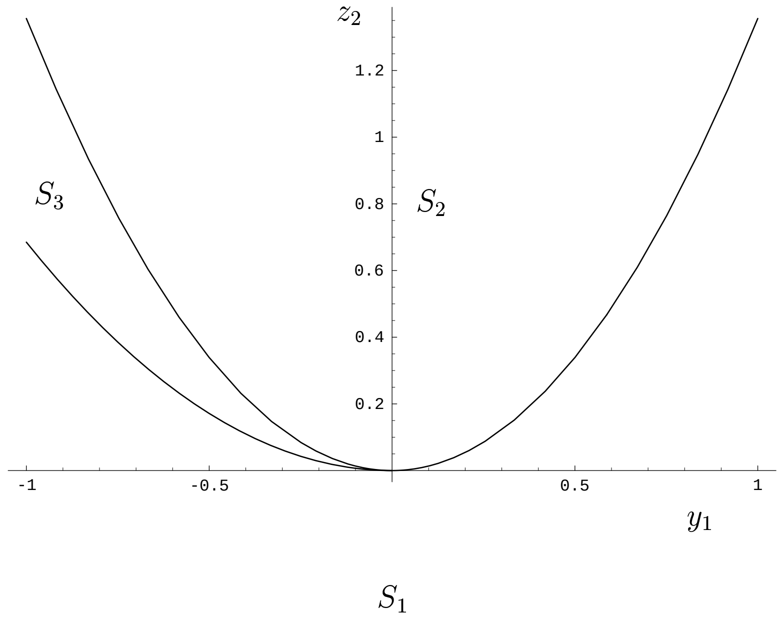

Figure 11: The boundaries of the regions S_1, S_2 and S_3 in the (y, z) plane.

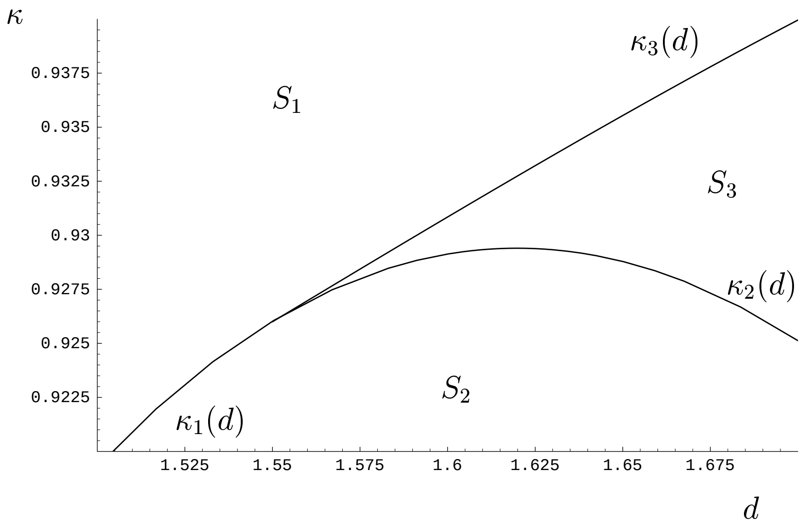

Figure 12: The curves $\kappa_1(d)$, $\kappa_2(d)$ and $\kappa_3(d)$, which are the boundaries of the regions S_1, S_2 and S_3 in the (κ, d) plane, near the triple point.

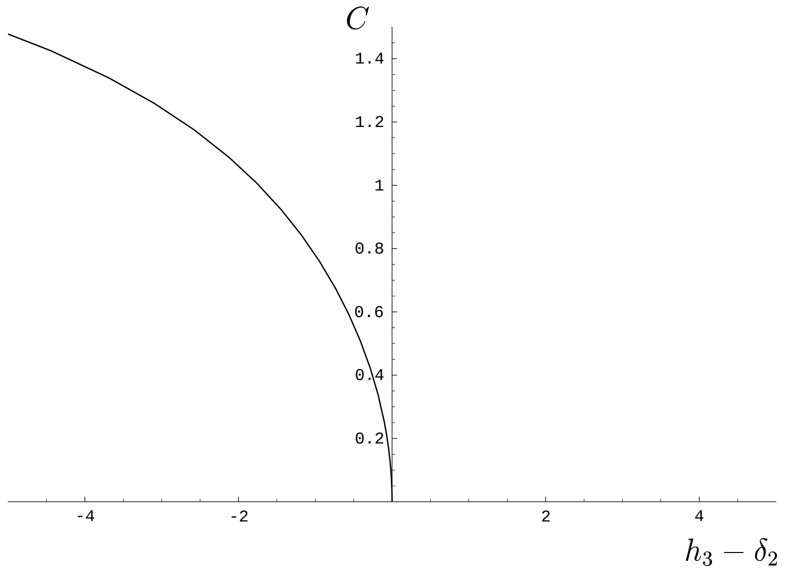

Figure 13: The bifurcation curve for the values $y_1 = 0$, $z_2 = -1$, which lies in the region S_1.

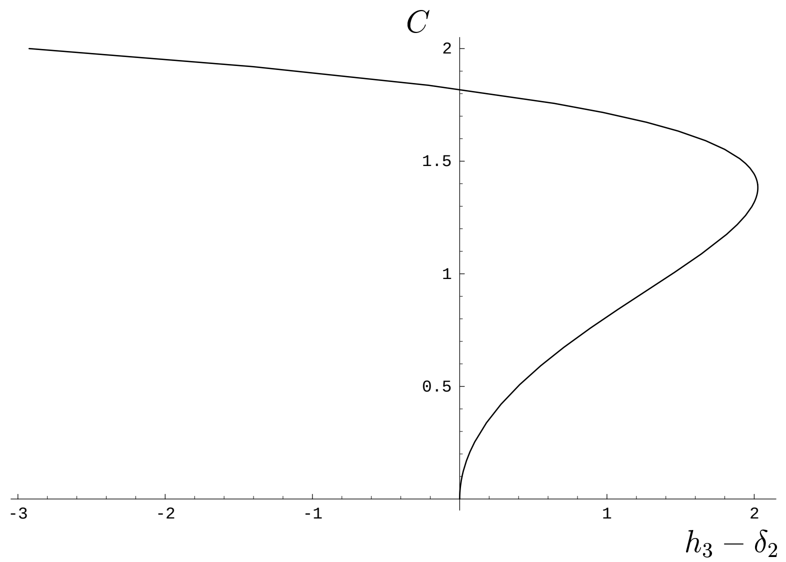

Figure 14: The bifurcation curve for the values $y_1 = 0$, $z_2 = 1$, which lies in the region S_2.

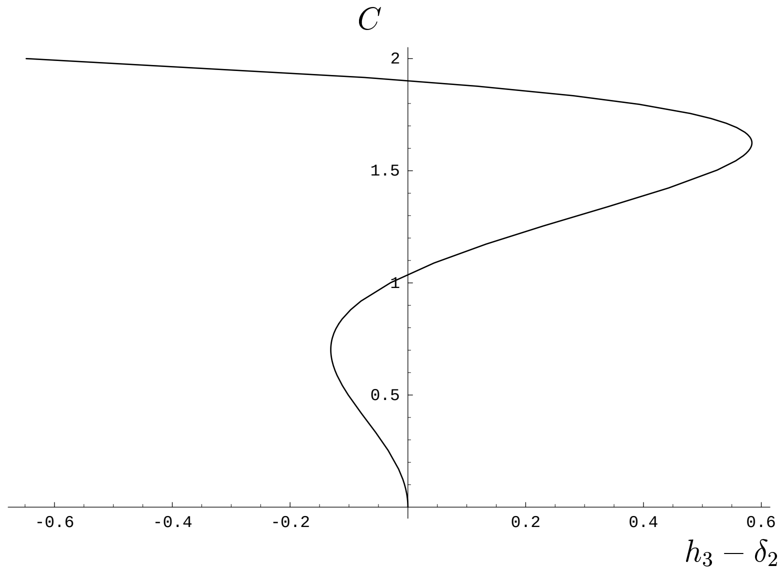

Figure 15: The bifurcation curve for the values $y_1 = -1$, $z_2 = 1$, which lies in the region S_3.

4 Conclusion

We have examined the bifurcation from the normal solution to a symmetric superconducting solution of the Ginzburg-Landau equations on a slab (GL_d), using the applied magnetic field as the bifurcation parameter. The behaviour of the bifurcating branch depends on the additional parameters d, the slab thickness, and κ, the Ginzburg-Landau parameter.

It was found numerically in [2] that there are three distinct regions of the κd-plane, which they labelled S_1, S_2 and S_3, in which there are at most one, two and three symmetric solutions of the Ginzburg-Landau system respectively, as shown in Figures 1, 2 and 3.

Since the leading-order problem corresponds to a linearization about the normal state, the amplitude of the leading-order solution is not determined at leading order but by a solvability condition at higher order. For most values of κ and d this solvability condition occurs at first order, and leads to the perturbation of the magnetic field from the bifurcation value being quadratic in the amplitude of the leading-order solution as given by (2.25).

The boundary between regions S_2 and $S_1 \cup S_3$ corresponds to the bifurcation from the normal state switching from sub to supercritical, which occurs when α_0 given by (2.26) is zero. In this case the amplitude of the leading-order solution is not determined even at first order. The solvability condition at second order leads to the perturbation of the magnetic field from the bifurcation value being quartic in the amplitude of the leading-order solution, as given by (2.35).

The triple point, where the regions S_1, S_2 and S_3 meet, is such that α_1 given by (2.36) as well as α_0 is zero. Then the amplitude of the leading-order solution is not determined even at second order. The solvability condition at third order leads to the perturbation of the magnetic field from the bifurcation value being sextic in the amplitude of the leading-order solution, as given by (3.56).

Thus we have been able to determine the boundary between S_2 and $S_1 \cup S_3$ and the behaviour of the bifurcating curve in its vicinity. We also have determined the curves κ_1, κ_2 and κ_3 in the vicinity of the triple point, and the behaviour of the bifurcating curve in the vicinity of the triple point.

Our results resolve formally some of the conjectures of [2]. In particular we have shown that the curve κ_3 bifurcates from the curve $\kappa_1 \cup \kappa_2$ tangentially, and that the triple point does not correspond to the maximum of $\kappa_1 \cup \kappa_2$, so that κ_2 is not monotone decreasing. Of course, our results still rely on the numerical evaluation of certain constants and the rigorous proof of these statements remains an interesting open problem.

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