Edge Diffraction of Complex Rays

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Abstract

The extension of the Geometrical Theory of Diffraction to complex rays is considered. The key difference with real rays arises in the determination of the regions of existence of the incident and reflected fields, which are governed by Stokes phenomenon. The procedure is demonstrated using the simple example of diffraction by a finite plate.

1 Introduction

The application of traditional ray theory to diffraction problems has been well understood since the introduction of the Geometrical Theory of Diffraction (GTD)[3]. However, the need to obtain accurate computation, especially in deep shadows or in the dark extremities of beams or complex plane waves, motivates the continuation of ray theory asymptotic expansions to numbers of terms where algebraically small contributions in terms of the inverse wave number can be comparable to exponentially small ones. This can happen, for example in the study of radar scattering in the atmosphere [4]. Additionally, the mathematical structure of high frequency solutions of linear wave equations in the frequency domain is revealed much more clearly if complex ray contributions are included than if they are not.

Unfortunately, the generalisation of real ray theory to cater for complex-valued phases is a difficult problem in asymptotic analysis and, relatively speaking, only a handful of papers have been devoted to the subject see, e.g. [2, 6, 7, 8]. In particular, since including complex rays necessarily involves analytic continuation of the boundary and boundary data, the application of complex ray theory to problems in which the boundary contains singularities requires especial care.

In this note, we aim to demonstrate the marriage of complex ray theory with GTD for the case of edge diffraction. We give as an example the diffraction of a complex plane wave, satisfying a scalar, two-dimensional Helmholtz equation, by a plate of finite length. For the scattering of a real plane wave, the methodology of GTD is first to solve the outer ray problem and then to consider a local problem in the vicinity of the singularities of the boundary, i.e. the ends of the plate. The far-field expansion of the solution to this inner problem will contain not only the incident and reflected rays, but also diffracted rays originating from the tip of the plate, which must be added to the outer ray expansion.

A similar methodology can be applied in principle when the incoming wave is a complex plane wave. However several practical difficulties arise: for example, whereas for a real plane wave the domain of existence of the incident and reflected fields is determined by the shadow boundaries formed by the rays through the ends of the plate, for a complex plane wave these domains of existence are not so obvious, and are determined by Stokes phenomenon.

To make progress we must solve an inner problem corresponding to the diffraction of a complex plane wave by a semi-infinite plate, which was first studied in [1]. In this case a closed form solution is available in the form of an integral, and the domains of existence of the rays in the far field can be discerned. This local information can then be matched with the outer ray solution to determine the domains of existence of the rays away from the tips of the plate.

2 A Procedure for Locating Complex Rays

The basic problem with any complex ray theory is that when solutions $\phi \sim Ae^{iku}(1 + O(\frac{1}{k}))$ are sought for Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0, \quad k \quad \text{real} \tag{1}$$

with appropriate boundary data, so that u satisfies the eikonal equation

$$|\nabla u|^2 = 1 \tag{2}$$

ideas of directionality of propagation are lost unless the phase u is real. Thus, although it may be easy to speculate about all the possible values of u at any point, there is no obvious procedure for deciding which values actually contribute to the physically relevant solution at that point. This problem can sometimes be circumvented when a contour integral representation is available for the solution of (1); in particular, if we know the configuration of the phase contours relative to the saddle points and any singularities of the integral, it is relatively easy to see how complex waves corresponding to the saddles are 'switched on and off' across certain boundaries in physical space. These important boundaries are the intersections with real space of the Stokes surfaces for the solution, which are the generalisation to partial differential equations of the Stokes lines that occur in WKB approximation to solutions of holomorphic ordinary differential equations. Now it is easy to guess candidate Stokes surfaces simply by looking at the *lowest* order WKB solutions to the differential equation, be it ordinary or partial. Indeed, the guess is clear from studying cases where an integral transform representation is available, because the topology of the steepest descent path can change only when two saddle points, corresponding to waves $u = u_1, u_2$ say, lie on the same connected component of this path, which implies that

$$\mathrm{Re}(u_1) = \mathrm{Re}(u_2). \tag{3}$$

However, those Stokes surfaces which are actually 'active', i.e. actually do switch on/off a sub-dominant solution, can only be discerned by a criterion that is more subtle than (3), since if switching is to occur the saddles must also be "adjacent". Here we will in fact be able to identify the relevant the Stokes surfaces through using GTD because, in the inner problem, an integral representation of the solution is available; therefore the Stokes surfaces are easy to determine locally, and we can then appeal to continuity to extend the active Stokes surfaces into the outer region.

3 The Inner Problem: Stokes Surfaces for the Sommerfeld Problem

The Sommerfeld problem of diffraction of a plane wave by a semi-infinite plate with Dirchlet boundary data is

$$\begin{aligned}(\nabla^2 + k^2)\phi &= 0,\\ \phi &= 0 \quad \text{on} \quad y = 0 \quad x < 0,\\ \phi &\sim e^{ik(-x\cos\alpha + y\sin\alpha)}.\end{aligned}$$

The exact solution can be calculated by the method of Sommerfeld [5], the Wiener-Hopf technique or the Kantorovich-Lebedev transform and is given by

$$\phi = \frac{e^{-i\pi/4+ikr}}{\sqrt{\pi}}\left[-\operatorname{Fr}\left(\sqrt{2kr}\cos\left(\frac{\theta-\alpha}{2}\right)\right) + \operatorname{Fr}\left(-\sqrt{2kr}\cos\left(\frac{\theta+\alpha}{2}\right)\right)\right], \qquad (4)$$

where

$$\operatorname{Fr}(z) = e^{-iz^2}\int_z^\infty e^{iv^2}\,dv, \qquad (5)$$

is the Fresnel integral. Note that the point of nonanalyticity of the boundary $r = 0$ is a branch point of the solution. The ray approximation, $k \to \infty$, corresponds to the far field and is given to two terms by

$$\begin{aligned}\phi \sim\ & e^{-ikr\cos(\theta+\alpha)} - e^{-ikr\cos(\theta-\alpha)}\\ & -\frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), -\pi < \theta < -\pi+\alpha, \qquad (6)\\ \phi \sim\ & e^{-ikr\cos(\theta+\alpha)}\\ & -\frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), -\pi+\alpha < \theta < \pi-\alpha, \qquad (7)\\ \phi \sim\ & -\frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), \pi-\alpha < \theta < \pi. \qquad (8)\end{aligned}$$

for real α and θ. We can observe that there are three ray fields; the incident (I),

$$\begin{aligned} u &= -r\cos(\theta+\alpha), && (9)\\ A &= 1, && (10)\end{aligned}$$

the reflected (R),

$$\begin{aligned} u &= -r\cos(\theta-\alpha), && (11)\\ A &= -1, && (12)\end{aligned}$$

and the diffracted (D), given to the lowest order by

$$\begin{aligned} u &= r, && (13)\\ A &= -\frac{e^{i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)}+\frac{1}{\cos((\theta-\alpha)/2)}\right). && (14)\end{aligned}$$

For complex θ the incident and reflected fields are switched off by the diffracted field across Stokes surfaces. The candidates for these Stokes surfaces are the equal phase surfaces of the incident/reflected rays and the diffracted ray. These are given by

$$\cos(\theta_1 \pm \alpha)\cosh\theta_2 = -1$$

where $\theta = \theta_1 + i\theta_2$, θ_1, θ_2 real. Which of these are Stokes surfaces can be determined by examining the behaviour of the contour of integration in the Fresnel integral. The Stokes surfaces and other 'inactive' equal-phase surfaces are shown in Fig (1). As expected, the Stokes surfaces intersect real space in the shadow boundaries $\theta = -\pi + \alpha$ and $\theta = \pi - \alpha$.

Having examined the case of real plane waves we extend this analysis to a complex plane wave. As has been demonstrated in [1] the solution (4) maybe analytically continued for complex values of α and, in the far field limit, three complex ray fields are obtained. Unlike the real case, it is not clear from the outset where the shadow boundaries (that is, the intersection of the Stokes surfaces with real space) are located. For α complex, the equal-phase surfaces are given by

$$\cos(\theta_1 + \alpha_1)\cosh(\theta_2 + \alpha_2) = -1 \qquad (15)$$

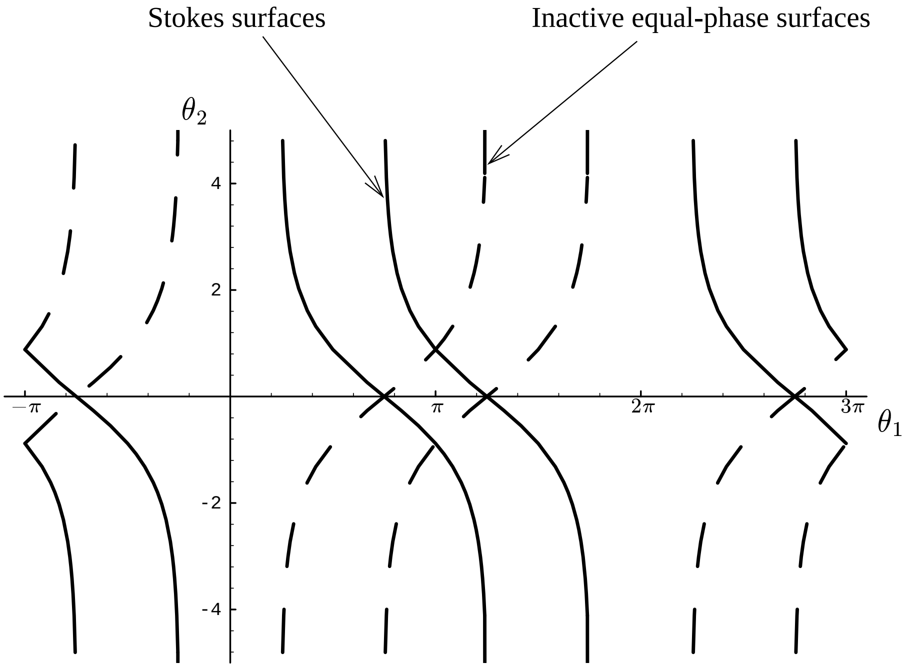

Figure 1: Stokes surfaces of the Sommerfeld problem with $\alpha = \frac{\pi}{4}$ where $\theta = \theta_1 + i\theta_2, \theta_1, \theta_2 \in \mathbb{R}$.

where $\alpha = \alpha_1 + \alpha_2$, α_1, α_2 real. Again, the behaviour of the integration contour in the Fresnel integral can be used to determine which of these are Stokes surfaces; since the diffracted field is responsible for switching off either the incident or reflected fields, the Stokes surfaces must be those surfaces described by (15) on which the diffracted field dominates. The intersection with real space then gives the ray approximation

$$\begin{aligned}
\phi &\sim e^{-ikr\cos(\theta+\alpha)} - e^{-ikr\cos(\theta-\alpha)} \\
&\quad - \frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), -\pi < \theta < \theta^*, &(16)\\
\phi &\sim e^{-ikr\cos(\theta+\alpha)} \\
&\quad - \frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), \theta^* < \theta < \theta^{**}, &(17)\\
\phi &\sim -\frac{e^{ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)} + \frac{1}{\cos((\theta-\alpha)/2)}\right), \theta^{**} < \theta < \pi, &(18)
\end{aligned}$$

where θ^* and θ^{**} are given by

$$\begin{aligned}
\theta^* &= -\alpha_1 + \cos^{-1}\left(\frac{-1}{\cosh\alpha_2}\right), &(19)\\
\theta^{**} &= \alpha_1 + \cos^{-1}\left(\frac{-1}{\cosh\alpha_2}\right), &(20)
\end{aligned}$$

the branch of arccos being chosen so that the imaginary part of the incident/reflected phase is larger than the imaginary part of the diffracted phase.

Figure 2 shows the Stokes surfaces for complex θ with $\alpha = \pi/4 + 2i$, and the behaviour of the ray solution in real space. Since the directions of the rays are complex we can no longer draw a conventional geometrical optics picture in real x and y, but the solution can be visualised by drawing the lines along which the amplitude of the wavefields decay most slowly (i.e. algebraically rather than exponentially as $k \to \infty$), which are the level curves of $\mathrm{Im}(u)$. These are also the normals to the level curves of $\mathrm{Re}(u)$, which would be the ray directions, were the rays to be real. The arrows indicate the direction of energy flow. We see that the domains of existence of incident and reflected waves may be extremely counter-intuitive. However, we should remember that the diffracted rays are exponentially large compared to the incident and reflected rays to the left of the line AB in Figure 2.

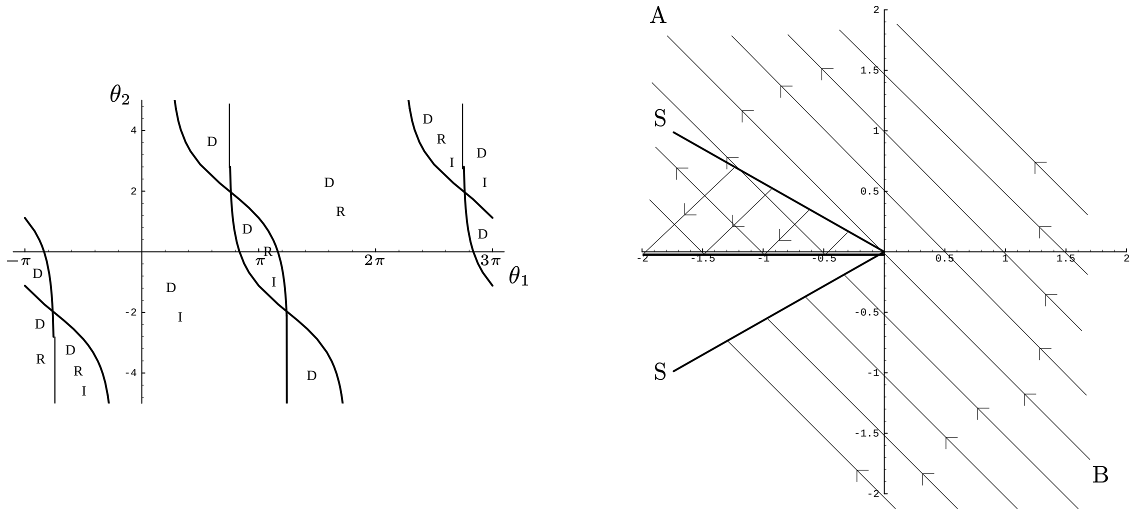

Figure 2: Stokes surfaces for the Sommerfeld problem with $\alpha = \frac{\pi}{4} + 2i$.

4 Edge diffraction of an arbitrary complex wave-field

Suppose a complex rayfield $\phi_i \sim A_0(x, y)e^{iku_0(x,y)}$ is incident upon a plate along the negative x-axis which has an edge at the origin. With homogeneous Dirichlet boundary data this will generate a reflected field $-A_0(x, -y)e^{iku_0(x,-y)}$.

Unlike the case of real rayfields, the region of existence of these incident and reflected rays is again not obvious *a priori*. The ray which hits the edge of the plate will also generate a diffracted rayfield, and it is this rayfield which is responsible for turning on and off the incident and reflected fields through Stokes phenomenon.

To determine the diffracted field we need to consider the inner problem in the vicinity of the end of the plate. This inner problem is exactly the diffraction of a complex plane wave by a semi-infinite plate considered in the previous section, since near to the origin

$$\phi_i \sim A_0(0, 0)e^{iku_0(0,0)}e^{ik(xu_{0x}(0,0)+yu_{0y}(0,0))}. \tag{21}$$

Hence the (complex) angle of incidence of the plane wave is given by

$$\alpha = -\tan^{-1}(u_{0y}(0,0)/u_{0x}(0,0)). \tag{22}$$

From the results of the previous section we can read off the diffracted field, and add its outer expansion to the outer ray solution:

$$\phi_d \sim -A_0(0,0)\frac{e^{iku_0(0,0)+ikr+i\pi/4}}{2\sqrt{2\pi kr}}\left(\frac{1}{\cos((\theta+\alpha)/2)}+\frac{1}{\cos((\theta-\alpha)/2)}\right). \tag{23}$$

From the Stokes surfaces in the inner problem we see that in the vicinity of the tip the reflected field is present for $-\pi < \theta < \theta^*$ while the incident field is present for $-\pi < \theta < \theta^{**}$ where θ^* and θ^{**} are given by (19) and (20).

To complete the problem we need to return to the outer ray problem and determine the regions of existence of the incident and reflected fields. The candidate Stokes surfaces are given by the equal-phase surfaces of the incident and reflected fields and the diffracted field, giving

$$\mathrm{Re}(u_0(0,0)+r) = \mathrm{Re}(u_0(x,y)), \tag{24}$$

and

$$\mathrm{Re}(u_0(0,0)+r) = \mathrm{Re}(u_0(x,-y)), \tag{25}$$

respectively. For general u_0 these will intersect real space in a set of curves. To determine which components are Stokes surfaces we simply match with the known solution in the inner region. Since a Stokes surface can only end where it meets another Stokes surface, we can appeal to continuity to identify the Stokes surfaces in the outer problem.

Thus, although there is no integral formulation of an exact solution for the full problem, a complex ray outer expansion can be used together with an exact inner solution to determine the incident, reflected and diffracted fields and their regions of existence.

5 Example: diffraction of a complex plane wave by a finite plate

As an example of the procedure, let us consider an incident field $e^{ik(-x\cos\alpha+y\sin\alpha)}$ impinging on a finite plate, as illustrated for α real in Figure 3. We expect there

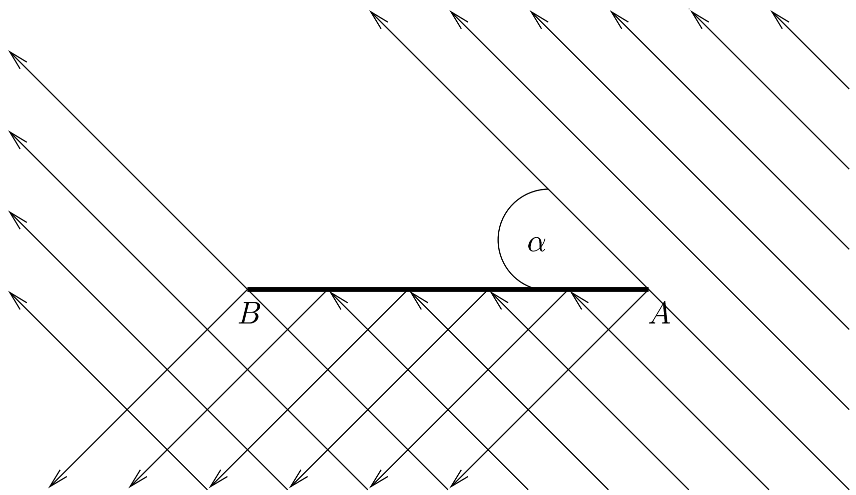

Figure 3: Diffraction of a real plane wave by a finite plate.

to be incident and reflected complex plane waves and two edge diffracted waves emanating from A and B respectively. When each diffracted wave hits the other end of the plate if will produce a second, double-diffracted wave. This process will continue, so that there will be an infinite family of diffracted waves from each end of the plate. With homogeneous Dirichlet data,say, an n-tuply diffracted ray will be of order $k^{1-3n/2}$. Even though the incident and reflected rays are complex, all the diffracted rays are real (albeit, perhaps, with an exponentially small amplitude prefactor), and the multiply diffracted rays can be handled by the usual GTD. Although the diffracted rays will dominate exponentially over the incident and reflected waves in much of the domain, when enough multiple diffractions are taken into account (depending on the value of k), at some point the behaviour of the incident and reflected rays becomes important. Since this is the part of the problem involving complex rays, we will focus our attention on determining the region of existence of the incident and reflected rays. As for the semi-infinite plate, these regions are determined by the position of the Stokes surfaces, across which the diffracted field turns off the incident and reflected fields.

The first step is therefore to determine the leading order (singly) diffracted field from the leading end of the plate. As usual, the inner problem in the vicinity of this end is the diffraction of a complex plane wave by a semi-infinite plate. Thus we can read off the diffracted field, and add its outer expansion to the outer ray solution. The candidate Stokes surfaces are then given by the equal phase surfaces of the incident and reflected fields with this diffracted field. As in §4, from the outer solution alone it is difficult to determine which of the equal phase surfaces will be Stokes surfaces, since we have no integral representation of the full solution. However, close to the tip we do know which are the Stokes surfaces, and we can determine the Stokes surfaces in the outer solution by matching with the inner solution as before.

We note that since the incoming field is a plane wave, the intersection of the Stokes surfaces with real space are straight lines in this case, and the Stokes surfaces in the outer solution are simply the continuation of the Stokes surfaces in the inner solution. However, an exactly similar analysis could easily be performed if the incoming wave-field were not so simple, say a Gaussian beam, in which case the Stokes surfaces in the outer solution would be curved.

The next step is to solve the local diffraction problem in the vicinity of the trailing edge of the plate. Some care needs to be taken here, since we have seen in Figure 2 that the incoming field here may actually be the reflected field if the decay of the complex plane wave is strong enough (that is, if $\mathrm{Im}(\alpha)$ is sufficiently large and negative). However, having determined whether it is the incident or reflected field which hits the tip, the inner problem is again the diffraction of a complex plane wave by a semi-infinite plate, and we can again read off the diffracted field. Note that this diffracted field will be exponentially smaller/larger that the one from the leading edge (for $\mathrm{Im}(\alpha)$ negative and positive respectively), because the ray which hits the trailing edge is exponentially smaller/larger than the ray which hits the leading edge due to the exponential decay of the complex plane wave.

The equal phase lines between the incident/reflected and the diffracted rays give the candidate Stokes surfaces, and matching with the inner solution will identify which of these are active.

Thus we are able to determine the region of existence of the incident and reflected fields. Examples for various values of α are shown in Fig. 4.

6 Conclusion

We have been concerned with edge diffraction of complex rays. The points of nonanalyticity of the boundary are branch points of the solution, and the key new feature introduced by the complex rays is in the determination the regions

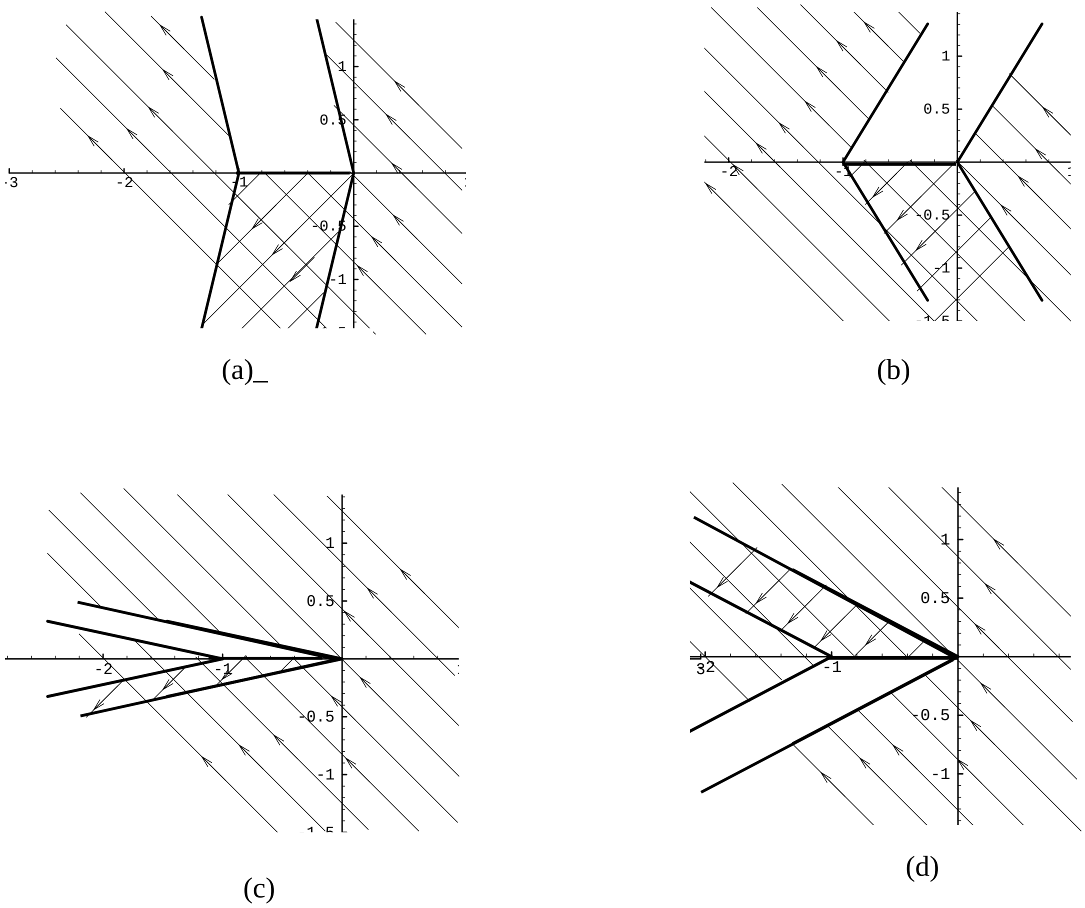

Figure 4: Stokes surfaces for the diffraction of a complex plane wave by a finite plate with (a) $\alpha = \frac{\pi}{4} + 0.6i$, (b) $\alpha = \frac{\pi}{4} + 2i$, (c) $\alpha = \frac{\pi}{4} - 2i$ and (d) $\alpha = \frac{\pi}{4} - 2i$. The arrows indicate the normals to the level curves of $\mathrm{Re}(u)$, which is the direction of energy flow. These are also the level curves of $\mathrm{Im}(u)$, and thus indicate lines of constant magnitude of the wavefield.

in which the incident and reflected waves exist. These regions are bounded by Stokes surfaces which emerge form the points of nonanalyticity of the boundary.

An inner problem in the vicinity of each point of nonanalyticity can be solved, to produce a diffracted wave exactly as in real GTD. Candidate Stokes surfaces in the outer ray problem can be determined by finding the equal phase surfaces between the diffracted and incident/reflected fields. In general there will be no simple closed form solution for the full problem, and determining which of the equal-phase surfaces are active Stokes surfaces is non-trivial. However, closed form solutions in the form of integrals do exist for the inner problems, so that the Stokes surfaces in the inner regions may be easily determined. Matching with the outer ray region then selects which of the outer equal-phase surfaces are Stokes surfaces.

We have demonstrated this procedure for diffraction of a complex plane wave by a finite plate. In this case all the Stokes surfaces are straight, but the results may still be quite surprising for a strongly decaying incoming wave (see Fig. 4). If the incident field is not a plane wave, but a Gaussian beam for example, then the Stokes surfaces would be curved in the outer solution.

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