SIMPLIFIED GINZBURG-LANDAU MODELS FOR SUPERCONDUCTIVITY VALID FOR HIGH KAPPA AND HIGH FIELDS

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Abstract. A formal asymptotic expansion is used to simplify the Ginzburg-Landau model of superconductivity in the limit of large values of the Ginzburg-Landau parameter and high applied magnetic field strengths. The convergence of solutions of the full Ginzburg-Landau equations to solutions of the leading order equations in the hierarchy is demonstrated for both boundary value and periodic problems. The results of computational experiments using the full Ginzburg-Landau model and the leading order model are given. These indicate that the simplified model yields solutions that are accurate approximations to solutions of the full model equations even for moderate values of the Ginzburg-Landau parameter.

1. Introduction

Superconductors are largely characterized by the value of the Ginzburg-Landau parameter κ. The recently discovered high critical temperature superconductors are known to have values of κ in excess of 50. Thus, for technological reasons, there is interest in exploring the behavior of superconductors in the limit of large κ. For the same reasons, high magnetic fields are also of interest. Thus, the subject of our study is models of the Ginzburg-Landau type and solutions of these models in the limit of high κ and high applied fields.

The discussion below is in the context of the homogeneous, isotropic Ginzburg-Landau model. Analogous simplifications may be effected in other models for superconductivity such as the anisotropic Ginzburg-Landau and Lawrence-Doniach models for layered high critical temperature superconductors. The latter two models are considered in [8], where part of the discussion is devoted to simplified models that are valid in the high κ, high field regime.

Of course, high values of κ implies that $\kappa > 1/\sqrt{2}$ so that we are considering type-II superconductors, one of whose salient characteristics is the appearance quantised tubes of magnetic flux each circled by a vortex of superconducting current. At the core of each vortex the density of superconducting charge carriers vanishes.

The high κ limit has received considerable attention in the literature; see, e.g., [2], [5], [18], [23], [27], and [28]. In most of these studies, the applied field strength was kept fixed, independent of κ. The most striking feature of the superconductor in this regime, i.e., $\kappa \to \infty$ with fixed field strength, is that the size of the vortex cores becomes increasingly small. Thus, the vortices may often be treated in isolation from each other. For example, in many studies, the order parameter is assumed to have constant modulus equal to the superconducting value and one obains a London equation for the magnetic field with Dirac delta function source terms providing the

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effect of the vortices on the field. Using stretched variables, one can also study the structure of the vortices near the core, but always in isolation from other vortices.

High field strengths have also received substantial attention. For example, almost any book on superconductivity will discuss Abrikosov's study (see [1] and [21]) of the structure of the vortex state for fields in the vicinity of the upper critical field $H_{c2} = \sqrt{2}\kappa H_c$, where H_c denotes the thermodynamic critical field that is an intrinsic property of superconductors and whose value is independent of κ. In this regime the vortex cores are closely packed. However, simplifications may be effected in this case as a result of the fact that the density of superconducting charge carriers is small, and thus the governing partial differential equations may be linearized about the normal state.

Here, we study simplifications to the Ginzburg-Landau model of superconductivity that may be effected for large κ and large applied field strengths, i.e., fields that are of $O(\kappa)$. (Note that having the applied field be of $O(\kappa)$ does not necessarily imply that it is close to the upper critical field H_{c2}; for example, we could have an applied field of strength $H_{c2}/2$.) In this high κ, high field regime the vortex cores overlap with each other so that they cannot be treated in isolation, while the density of superconducting charge carriers is not necessarily small. Thus, the simplifications that have been introduced in previous studies cannot be effected in the large κ, large field strength regime.

In [14] and [15], computational results were given for the periodic Ginzburg-Landau model of superconductivity. In this case, the physical variables, e.g., the magnetic field, the current, the density of superconducting charge carriers, etc., are all periodic with respect to a lattice in the plane. From these results one could infer that if solutions of the model where appropriately scaled with respect to the Ginzburg-Landau parameter κ, then, for large values of κ, these solutions are very nearly independent of κ. These results leads one to believe that these solutions can be approximated by solutions of a simplified model whose governing equations themselves are invariant with respect to κ.

Here, we perform a formal asymptotic expansion of the Ginzburg-Landau equations as $\kappa \to \infty$, leading to a hierarchy of simpler systems of partial differential equations from which increasingly accurate approximations to solutions of the full Ginzburg-Landau equations may be obtained. For each system in the hierarchy, the magnetic potential is uncoupled from the corresponding order parameter. (These are the primary dependent variables in the Ginzburg-Landau theory of superconductivity.) Thus, to leading order, the magnetic potential may be determined as the solution of a linear magnetostatics problem, and subsequently, the order parameter may be determined as the solution of a nonlinear elliptic partial differential equation. The higher order corrections to the magnetic potential may also be determined from linear magnetostatics problems while the correponding corrections to the order parameter are solutions of linear elliptic partial differential equations.

We also rigorously show that if solutions of the Ginzburg-Landau model are appropriately scaled, then indeed, as $\kappa \to \infty$ and if the applied field is of $O(\kappa)$, these solutions converge to solutions of the leading order equations of the simplified model, thus justifying the formal asymptotics. This program is carried out for both boundary value and periodic situations. We also provide a variety of other analytical results and the results of some numerical experiments which illustrate the accuracy of the leading order terms in the asymptotic expansion in approximating the full Ginzburg-Landau equations.

From a practical point of view, the simplified model is important because it enables a substantial reduction in the cost of performing computational simulations of superconducting phenomena in the high-κ, high field regime. We note that solutions of the leading order equations of the simpified model differ from those of the full Ginzburg-Landau model by terms of $O(1/\kappa^2)$ so that, certainly, for $\kappa \geq 50$, the former are indeed very accurate. In addition, the computational experiments we have performed indicate that solutions of the simplified model are "accurate" approximations to solutions of the full Ginzburg-Landau model even at moderate values of κ, e.g., $\kappa \approx 5$. This observation is discussed in more detail in §7.

We note that the simplification that can be effected in the high κ, high field limit is sometimes referred to as the "mean field decoupling scheme;" see, e.g., [6].

2. The Ginzburg-Landau equations

We begin with a short discussion concerning the well-known Ginzburg-Landau model for superconductivity. For details concerning the material of this section, one may consult [2], [4], [9], [10], [13], [18], [20], [23], [27], or [28].

In [19], Ginzburg and Landau introduced a phenomenological theory of superconductivity based on the minimization of a free energy. Our setting is that of a superconductor occupying the bounded region $\tilde{\Omega} \subset \mathbb{R}^3$. The region exterior to the superconductor is denoted by $\tilde{\Omega}_e = \mathbb{R}^3/\tilde{\Omega}$. The applied field is denoted by $\tilde{\mathbf{H}}_e$ and throughout this paper it is assumed to be a constant vector. The contribution to the free energy in $\tilde{\Omega}$ may be expressed in the form

$$\begin{aligned}\tilde{\mathcal{G}}_s(\tilde{\psi}, \tilde{\mathbf{A}}) = \int_{\tilde{\Omega}} \left(\alpha|\tilde{\psi}|^2 + \frac{\beta}{2}|\tilde{\psi}|^4 + \frac{1}{2m_s}\left| \left(i\hbar\tilde{\nabla} + \frac{e_s\tilde{\mathbf{A}}}{c} \right) \tilde{\psi} \right|^2 \right) d\tilde{\mathbf{x}} \\ + \frac{1}{8\pi}\int_{\tilde{\Omega}} \left| \tilde{\nabla}\times\tilde{\mathbf{A}} - \tilde{\mathbf{H}}_e \right|^2 d\tilde{\mathbf{x}}\,. \end{aligned} \tag{2.1}$$

(In fact, $\tilde{\mathcal{G}}_s$ is the difference between the free energy in the superconductor $\tilde{\Omega}$ in the presence of the applied field $\tilde{\mathbf{H}}_e$ and the free energy that would accrue due to that field should the superconductor be absent. Since $\tilde{\Omega}$ is finite, it is unimportant whether one uses the free energy itself, or the difference between the free energy and some reference value. For us, it will be more convenient to use the form given above.) Here, $\tilde{\psi}$ is the (complex-valued) order parameter, $\tilde{\mathbf{A}}$ is the (vector-valued) magnetic potential, α and β are constants (with respect to the space variable $\mathbf{x}$) whose values depend on the temperature, c is the speed of light, e_s and m_s are the charge and mass, respectively, of the superconducting charge-carriers, and $2\pi\hbar$ is Planck's constant.

The region exterior to the superconductor $\tilde{\Omega}_e$ is of infinite extent. Thus, in order to consider only finite quantities, it is now crucial to consider the difference between the energy in that region in the presence of the superconductor with that should the superconductor be absent. This energy difference is given by

$$\tilde{\mathcal{G}}_e(\tilde{\mathbf{A}}) = \frac{1}{8\pi}\int_{\tilde{\Omega}_e} |\tilde{\nabla}\times\tilde{\mathbf{A}} - \tilde{\mathbf{H}}_e|^2\, d\tilde{\mathbf{x}}\,. \tag{2.2}$$

The minimization of the total Gibbs free energy $\tilde{\mathcal{G}}_s + \tilde{\mathcal{G}}_e$ with respect to $\tilde{\psi}^*$, the conjugate of $\tilde{\psi}$, and $\tilde{\mathbf{A}}$ yields the celebrated Ginzburg-Landau equations

$$\frac{1}{2m_s}\left(i\hbar\tilde{\nabla} + \frac{e_s}{c}\tilde{\mathbf{A}} \right)^2 \tilde{\psi} + \alpha\tilde{\psi} + \beta|\tilde{\psi}|^2\tilde{\psi} = 0 \quad \text{in } \tilde{\Omega}\,, \tag{2.3}$$

$$\tilde\nabla\times\tilde\nabla\times\tilde{\mathbf{A}}+\frac{2\pi i e_s\hbar}{m_s c}(\tilde\psi^*\tilde\nabla\tilde\psi-\tilde\psi\tilde\nabla\tilde\psi^*)+\frac{4\pi e_s^2}{m_s c^2}|\tilde\psi|^2\tilde{\mathbf{A}}=\mathbf{0}\quad\text{in }\tilde\Omega\,, \tag{2.4}$$

and

$$\tilde\nabla\times\tilde\nabla\times\tilde{\mathbf{A}}=\mathbf{0}\quad\text{in }\tilde\Omega_e \tag{2.5}$$

along with the boundary conditions

$$\mathbf{n}\cdot(i\hbar\tilde\nabla+\frac{e_s}{c}\tilde{\mathbf{A}})\tilde\psi=0\quad\text{on }\tilde\Gamma\,, \tag{2.6}$$

$$[\tilde{\mathbf{A}}\times\mathbf{n}]=\mathbf{0}\quad\text{on }\tilde\Gamma\,, \tag{2.7}$$

$$\left[(\tilde\nabla\times\tilde{\mathbf{A}})\times\mathbf{n}\right]=\mathbf{0}\quad\text{on }\tilde\Gamma\,, \tag{2.8}$$

and

$$\tilde\nabla\times\tilde{\mathbf{A}}\to\tilde{\mathbf{H}}_e\quad\text{as }|\tilde{\mathbf{x}}|\to\infty\,, \tag{2.9}$$

where $[\cdot]$ denotes the jump of the enclosed quantity across $\tilde\Gamma$ and $|\tilde{\mathbf{x}}|$ is the distance from the origin. The boundary condition (2.6) is the natural boundary condition associated with the minimization, and corresponds to the superconductor being adjacent to a vacuum. We could easily fit more general boundary conditions, e.g., see [10], into our framework. The boundary conditions (2.8) and (2.9) are also natural to the minimization process. The boundary condition (2.7) is not a natural boundary condition and must be imposed on candidate minimizers so that the interface condition $[\tilde{\mathbf{H}}\cdot\mathbf{n}]=0$ is satisfied across $\tilde\Gamma$.

The coefficient α in (2.1) or (2.3) changes sign at the critical temperature T_c, with $\alpha<0$ for $T<T_c$. If the temperature of the sample Ω is lower than T_c, the sample is in the superconducting state; if $T>T_c$, then the sample is in the normal, i.e., nonsuperconducting, state.

Variables of physical interest may be determined from $\tilde\psi$ and $\tilde{\mathbf{A}}$. For example, $\tilde{\mathbf{H}}=\tilde\nabla\times\tilde{\mathbf{A}}$ is the magnetic field, $|\tilde\psi|^2$ determines the density of superconducting charge carriers, and

$$\tilde{\mathbf{J}}=\frac{c}{4\pi}\tilde\nabla\times\tilde{\mathbf{H}}=\frac{c}{4\pi}\tilde\nabla\times\tilde\nabla\times\tilde{\mathbf{A}}=-i\frac{e_s\hbar}{2m_s}(\tilde\psi^*\tilde\nabla\tilde\psi-\tilde\psi\tilde\nabla\tilde\psi^*)-\frac{e_s^2}{m_s c}|\tilde\psi|^2\tilde{\mathbf{A}}$$

is the current in the superconductor.

There are two important length scales associated with changes in the order parameter and the magnetic field. These are the *coherence length*

$$\xi=\left(-\frac{\hbar^2}{2m_s\alpha}\right)^{1/2}$$

and the *penetration depth*

$$\lambda=\left(-\frac{m_s\beta c^2}{4\pi\alpha e_s^2}\right)^{1/2}$$

which measure distances over which the order parameter and the magnetic field, respectively, undergo appreciable change. The nondimensional ratio $\kappa = \lambda/\xi$ is known as the *Ginzburg-Landau parameter.*

2.1. Non-dimensionalized form

We nondimensionalize lengths with respect to a length scale ℓ. (It is customary to pick $\ell = \lambda$ or $\ell = \xi$; we will discuss these two choices below.) Magnetic fields are nondimensionalized by $\sqrt{2}H_c$ where the critical field $H_c = \sqrt{4\pi\alpha^2/\beta}$, the magnetic potential by $\sqrt{2}\ell H_c$, the order parameter by $\sqrt{-\alpha/\beta}$, free energy densities by α^2/β, free energies by $\alpha^2\ell^3/\beta$, and currents by $cH_c/(\pi\ell\sqrt{8})$. Specifically, we introduce the nondimensional variables

$$\mathbf{x} = \frac{\tilde{\mathbf{x}}}{\ell}, \qquad \mathbf{A} = \frac{\tilde{\mathbf{A}}}{\sqrt{2}H_c\ell}, \qquad \psi = \frac{\tilde{\psi}}{\sqrt{-\alpha/\beta}},$$

$$\mathbf{H} = \frac{\tilde{\mathbf{H}}}{\sqrt{2}H_c}, \qquad \text{and} \qquad \mathbf{J} = \frac{\tilde{\mathbf{J}}}{cH_c/(\pi\ell\sqrt{8})}.$$

The region occupied by the superconductor is denoted, in terms of nondimensional variables, by Ω. The region exterior to it is denoted by Ω_e. The interface between Ω and Ω_e is denoted by Γ.

The nondimensional form of the total Gibbs free energy, i.e., the nondimensional form of the sum of (2.1) and (2.2), is then given by

$$\begin{aligned}\mathcal{G}(\psi, \mathbf{A}) = &\int_\Omega \left(\frac{1}{2}|\psi|^4 - |\psi|^2 + \left| \left(i\frac{\xi}{\ell}\nabla + \frac{\ell}{\lambda}\mathbf{A} \right) \psi \right|^2 \right) d\mathbf{x} \\ &+ \int_\Omega \left|\nabla \times \mathbf{A} - \mathbf{H}_e\right|^2 d\mathbf{x} + \int_{\Omega_e} |\nabla \times \mathbf{A} - \mathbf{H}_e|^2 \, d\mathbf{x}\,.\end{aligned} \tag{2.10}$$

The nondimensionalized counterparts to the Ginzburg-Landau equations and boundary conditions (2.3)-(2.9) are given by

$$\left(i\frac{\xi}{\ell}\nabla + \frac{\ell}{\lambda}\mathbf{A} \right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{2.12}$$

$$\nabla \times \nabla \times \mathbf{A} + \frac{i}{2}\frac{\xi}{\lambda}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \left(\frac{\ell}{\lambda}\right)^2 |\psi|^2\mathbf{A} = 0 \quad \text{in } \Omega\,, \tag{2.13}$$

$$\nabla \times \nabla \times \mathbf{A} = \mathbf{0} \quad \text{in } \Omega_e\,, \tag{2.14}$$

$$\mathbf{n} \cdot \left(i\frac{\xi}{\ell}\nabla + \frac{\ell}{\lambda}\mathbf{A} \right) \psi = 0 \quad \text{on } \Gamma\,, \tag{2.15}$$

$$[\mathbf{A} \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{2.16}$$

$$[(\nabla \times \mathbf{A}) \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{2.17}$$

and

$$\nabla \times \mathbf{A} \to \mathbf{H}_e \quad \text{as } |\mathbf{x}| \to \infty\,. \tag{2.18}$$

In nondimensional form, we have that the density of superconducting charge carriers is given by $|\psi|^2$, the magnetic field by $\mathbf{H} = \nabla \times \mathbf{A}$, and the current in the superconductor by

$$\mathbf{J} = \nabla \times \mathbf{H} = (\nabla \times \nabla \times \mathbf{A}) = -\frac{i}{2}\frac{\xi}{\lambda}(\psi^* \nabla\psi - \psi\nabla\psi^*) - \left(\frac{\ell}{\lambda}\right)^2 |\psi|^2 \mathbf{A}\,. \tag{2.19}$$

3. Simplified equations in the high-κ limit

We now wish to study the Ginzburg-Landau model in the limit of large κ. This limit is of interest since it is known that the value of κ for high-T_c superconductors is large. (Of course, the Ginzburg-Landau model discussed here is not necessarily relevant to high-T_c layered superconductors. However, the results of this paper may be easily extended to models for layered superconductors such as the anisotropic Ginzburg-Landau and Lawrence-Doniach models; see [8].)

The limiting behavior of the Ginzburg-Landau equations as $\kappa \to \infty$ depends very much on the density of vortices. If the vortices are separated by distances $O(\lambda)$, then the choice $\ell = \lambda$ is natural, so that the sample size remains fixed on the scale of the penetration depth, and the coherence length becomes small. This limit, which has been studied by various authors, is singular in the sense that the order of the governing partial differential equations is reduced in the limit of $\kappa \to \infty$. In fact in the outer region, i.e., away from vortex cores, the order parameter has constant modulus equal to the superconducting value, i.e., $|\psi| = 1$ in nondimensional variables, and the magnetic field satisfies the London equation with Dirac delta function source terms providing the effect of the vortices. Appropriate equations in the inner region, i.e., near the core of the vortices, may be obtained. The strength of the magnetic field in this regime is $o(1)$.

Here, we are interested in much larger magnetic field strengths, and therefore much greater vortex densities. In particular we take the vortices to be separated by distances $O(\xi)$, and magnetic field strangths to be $O(\kappa)$. Then we must choose the lengthscale $\ell = \xi$. Thus, now the sample size remains fixed on the scale of coherence length, but the penetration depth becomes large with respect to the sample size. This limit is regular in the sense that there is no loss in the order of the governing partial differential equations as $\kappa \to \infty$.

With $\ell = \xi$, the nondimensional Ginzburg-Landau system (2.12)-(2.19) becomes

$$\left(i\nabla + \frac{1}{\kappa}\mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{3.1}$$

$$\nabla \times \nabla \times \mathbf{A} + \frac{i}{2\kappa}(\psi^* \nabla\psi - \psi\nabla\psi^*) + \left(\frac{1}{\kappa^2}\right)|\psi|^2 \mathbf{A} = 0 \quad \text{in } \Omega\,, \tag{3.2}$$

$$\nabla \times \nabla \times \mathbf{A} = \mathbf{0} \quad \text{in } \Omega_e\,, \tag{3.3}$$

$$\mathbf{n} \cdot \left(i\nabla + \frac{1}{\kappa}\mathbf{A} \right) \psi = 0 \quad \text{on } \Gamma\,, \tag{3.4}$$

$$[\mathbf{A} \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.5}$$

$$[(\nabla \times \mathbf{A}) \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.6}$$

$$\nabla \times \mathbf{A} \to \mathbf{H}_e \quad \text{as } |\mathbf{x}| \to \infty\,, \tag{3.7}$$

and

$$\mathbf{J} = -\frac{i}{2\kappa}(\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{1}{\kappa^2}|\psi|^2 \mathbf{A}\,. \tag{3.8}$$

Let $\mathbf{H}_e = \kappa \mathbf{H}_0$, where $\mathbf{H}_0$ is independent of κ. We formally expand all variables in powers of κ:

$$\psi = \sum_{j=0} \frac{1}{\kappa^{2j}} \psi_j\,, \qquad \mathbf{A} = \kappa \sum_{j=0} \frac{1}{\kappa^{2j}} \mathbf{A}_j\,, \qquad \text{and} \qquad \mathbf{J} = \kappa \sum_{j=0} \frac{1}{\kappa^{2j}} \mathbf{J}_j\,. \tag{3.9}$$

Substituting the expansions (3.9) into (3.1)-(3.8) and equating powers of κ yields, to leading order, the system for $\mathbf{A}_0$:

$$\nabla \times \nabla \times \mathbf{A}_0 = \mathbf{J}_0 = \mathbf{0} \quad \text{in } \Omega \text{ and } \Omega_e\,, \tag{3.10}$$

$$[\mathbf{A}_0 \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.11}$$

$$[(\nabla \times \mathbf{A}_0) \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.12}$$

and

$$\nabla \times \mathbf{A}_0 \to \mathbf{H}_0 \quad \text{as } |\mathbf{x}| \to \infty \tag{3.13}$$

and the system for ψ_0:

$$(i\nabla + \mathbf{A}_0)^2 \psi_0 - \psi_0 + |\psi_0|^2 \psi_0 = 0 \quad \text{in } \Omega\,, \tag{3.14}$$

and

$$\mathbf{n} \cdot (i\nabla + \mathbf{A}_0)\, \psi_0 = 0 \quad \text{on } \Gamma\,. \tag{3.15}$$

We have re-ordered the equations in the leading order system (3.10)-(3.15) in order to highlight the decoupling of the magnetic potential and the order parameter. In fact, (3.10)-(3.13) simply describes the potential of the normal state ($\psi \equiv 0$), and is easily solved. For example if the external field is aligned with the z-axis, i.e., $\mathbf{H}_0 = (0, 0, H_0)$, then $\mathbf{A}_0 = H_0(0, x, 0)$ is a solution of (3.10)-(3.13). Once $\mathbf{A}_0$ has been determined from (3.10)-(3.13), (3.14) and (3.15) may be used to determine ψ_0.

Proceeding to the next order in equations (3.1)-(3.8) with ψ, $\mathbf{A}$ and $\mathbf{J}$ given by (3.9), we find that $\mathbf{A}_1$ may be determined from

$$\nabla \times \nabla \times \mathbf{A}_1 = \mathbf{J}_1 = -\frac{i}{2}(\psi_0^* \nabla \psi_0 - \psi_0 \nabla \psi_0^*) - |\psi_0|^2 \mathbf{A}_0 \quad \text{in } \Omega\,, \tag{3.16}$$

$$\nabla \times \nabla \times \mathbf{A}_1 = \mathbf{0} \quad \text{in } \Omega_e\,, \tag{3.17}$$

$$[\mathbf{A}_1 \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.18}$$

$$[(\nabla \times \mathbf{A}_1) \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{3.19}$$

and

$$\nabla \times \mathbf{A}_1 \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \tag{3.20}$$

and that then ψ_1 may be determined from

$$\begin{aligned} &(i\nabla + \mathbf{A}_0)^2 \psi_1 - \psi_1 + 2|\psi_0|^2 \psi_1 + \psi_0^2 \psi_1^* \\ &\qquad = -(i\nabla + \mathbf{A}_0) \cdot (\psi_0 \mathbf{A}_1) - \mathbf{A}_1 \cdot (i\nabla + \mathbf{A}_0)\, \psi_0 \quad \text{in } \Omega \end{aligned} \tag{3.21}$$

and

$$\mathbf{n} \cdot (i\nabla + \mathbf{A}_0)\, \psi_1 = -\psi_0 (\mathbf{A}_1 \cdot \mathbf{n}) \quad \text{on } \Gamma\,. \tag{3.22}$$

Since the right-hand side of (3.16) is known, (3.16)-(3.20) corresponds to the problem of determining the vector potential due to a body carrying a specified current distribution $\mathbf{J}_1$. Well-known methods of magnetostatics are available for its solution. However, by the Fredholm Alternative, we see that there will only exist a solution $\mathbf{A}_1$ if a certain solvability condition is satisfied, since the homogeneous version of (3.16)-(3.20) has the non-trivial solution $\mathbf{A}_1$ = constant. Indeed, integrating (3.16) over Ω yields

$$\begin{aligned} \int_\Omega \mathbf{J}_1 \, dV &= \int_\Omega \nabla \times \nabla \times \mathbf{A}_1 \, dV \\ &= \int_{\partial\Omega} (\nabla \times \mathbf{A}_1) \times \mathbf{n} \, dV \\ &= -\int_{\Omega_e} \nabla \times \nabla \times \mathbf{A}_1 \, dV = 0. \end{aligned}$$

Thus there is a solution $\mathbf{A}_1$ if and only if

$$\int_\Omega \mathbf{J}_1 \, dV = 0. \tag{3.23}$$

However, we expect (3.10)-(3.15) to give only distinct solutions, so that there are no parameters in ψ_0 and $\mathbf{A}_0$ that we can adjust in order to satisfy (3.23). Thus (3.23) must be satisfied automatically by the leading order solution if our asymptotic expansion is valid. Fortunately, this is the case. The condition (3.23) is exactly the condition that the free energy should be minimized with respect to constants in the magnetic potential. However, because of the gauge invariant property of the free energy (see Section 4), this is equivalent to minimising the free energy with respect to linear functions in the phase of the order parameter. Since these are contained in our class of test functions, (3.23) must actually follow from (3.14) and (3.15). In one dimension, however, things are slightly different, and the solvability condition is not automatically satisfied. We consider this case separately in the next section.

Having found $\mathbf{A}_1$ from (3.16)-(3.20), ψ_1 may be determined from the inhomogeneous linear system (3.21)-(3.22).

In a similar manner, one may proceed to develop equations that determine the higher-order terms in the expansions (3.9). The simplification due to the decoupling of the problems for the vector potential and the order parameter is evident. At each stage the equations for the vector potential are reduced to standard problems in *linear* magnetostatics, namely, determining the vector potential due to a body carrying specified current distributions, or in the case of $\mathbf{A}_0$, determining the vector potential due to a body placed in a uniform external field. The only nonlinear problem left to solve is (3.14)-(3.15) for ψ_0, since the equations for the higher-order terms in ψ will all be *linear*, e.g., (3.21)-(3.22) for ψ_1. Moreover, for all ψ_j, $j \geq 1$, the differential operator is the same, i.e., that given in (3.21)-(3.22), and only the right-hand sides change with j.

3.1. One-dimensional Equations

We take the domain Ω to be given by $-d < x < d$, and the magnetic field to be directed in the z-direction. Then we may take the magnetic potential to be directed in the y-direction, $\mathbf{A} = (0, A, 0)$. We seek a one-dimensional solution of the form $A = A(x)$, $\psi = f(x)e^{-icy}$, where f may be taken to be real. Then, equations (3.1)-(3.8) become

$$f'' = f^3 - f + f\left(\frac{A}{\kappa} + c\right)^2, \tag{3.24}$$

$$A'' = \frac{f^2}{\kappa}\left(\frac{A}{\kappa} + c\right), \tag{3.25}$$

with boundary conditions

$$f'(\pm d) = 0, \tag{3.26}$$

$$A'(\pm d) = \kappa H_0, \tag{3.27}$$

where $' \equiv d/dx$. Clearly, any constant in A may be absorbed into c, so that we may fix the gauge by requiring that $A(0) = 0$. Formally expanding in powers of κ,

$$\psi = \sum_{j=0} \frac{1}{\kappa^{2j}}\psi_j\,, \qquad A = \kappa\sum_{j=0} \frac{1}{\kappa^{2j}}A_j\,, \qquad \text{and} \qquad c = \kappa\sum_{j=0} \frac{1}{\kappa^{2j}}c_j\,,$$

substituting into (3.24)-(3.27), and equating powers of κ yields at leading order

$$f_0'' = f_0^3 - f_0 + f_0 (A_0 + c_0)^2 , \tag{3.28}$$

$$A_0'' = 0, \tag{3.29}$$

$$f_0'(\pm d) = 0, \tag{3.30}$$

$$A_0'(\pm d) = H_0, \qquad A_0(0) = 0. \tag{3.31}$$

Hence $A_0 = H_0 x/d$. Then, for each c_0, (3.28) and (3.30) determine f_0; c_0 is undetermined at leading order. Proceeding to the next order in (3.24)-(3.27) we find

$$f_1'' = 3f_0^2 f_1 - f_1 + f_1 \left(\frac{H_0 x}{d} + c_0\right)^2 + 2f_0 \left(\frac{H_0 x}{d} + c_0\right)(A_1 + c_1), \tag{3.32}$$

$$A_1'' = f_0^2 \left(\frac{H_0 x}{d} + c_0\right), \tag{3.33}$$

$$f_0'(\pm d) = 0, \tag{3.34}$$

$$A_0'(\pm d) = 0, \qquad A_0(0) = 0. \tag{3.35}$$

Integrating (3.33) from $-d$ to d, using (3.35), gives the solvability condition (3.23) as

$$\int_{-d}^{d} f_0^2 \left(\frac{H_0 x}{d} + c_0\right) dx = 0. \tag{3.36}$$

This is the equation that determines c_0. Having determined c_0, (3.33) and (3.35) give a unique solution for A_1. Then for each c_1, (3.32) and (3.34) can be solved for f_1. As before, c_1 is undetermined at this order, but will be determined by a solvability condition for the second order terms.

Thus we see that in one dimension the solvability condition (3.23) is not automatically satisfied by the solution of (3.24), (3.26) , but is necessary in order to determine the constant c. The reason for this difference between one dimension and more dimensions is as follows. We have already mentioned that (3.36) follows from minimizing the free energy with respect to constants in the potential A. Since constants in A can be absorbed into c, this is the same as minimizing the free energy with respect to c, or linear functions of y in the phase of ψ. However, in one dimension, such functions are not contained in the class of test functions used in deriving (3.24) and (3.26), so that this condition is not automatically satisfied by the solution of (3.24), (3.26), but is an extra constraint.

4. Existence of minimizers

We now explore some of the notions discussed in §2. Specifically, we briefly consider the question of the existence of minimizers of the functional (2.10) and some elementary properties possessed by such minimizers.

Before continuing, we introduce some notation that will be used below. Throughout, for any non-negative integer k and domain $\mathcal{D} \subset \mathbb{R}^n$, $n = 2$ or 3, $H^k(\mathcal{D})$ will denote the Sobolev space of real-valued functions having square integrable derivatives of order up to k. The corresponding spaces of complex-valued functions, each of whose real and imaginary parts belong to $H^k(\mathcal{D})$, will be denoted by $\mathcal{H}^k(\mathcal{D})$. Corresponding spaces of vector-valued functions, each of whose components belong to $H^k(\mathcal{D})$, will be denoted by $\mathbf{H}^k(\mathcal{D})$, i.e., $\mathbf{H}^k(\mathcal{D}) = [H^k(\mathcal{D})]^n$. Norms of functions belonging to $H^k(\mathcal{D})$, $\mathbf{H}^k(\mathcal{D})$, and $\mathcal{H}^k(\mathcal{D})$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_{k,\mathcal{D}}$ or $\|\cdot\|_k$. The latter notation will be used when there is no chance of confusion. We use $\check{\mathbf{H}}^1(\mathbb{R}^3)$ to denote the completion of $\mathbf{C}_0^\infty(\mathbb{R}^3)$ under the norm

$$\|\mathbf{Q}\|_*^2 = \int_{\mathbb{R}^3} |\nabla \mathbf{Q}|^2 \, d\mathbf{x}\,.$$

For details concerning these spaces, one may consult [3]. We will also use the subspaces

$$\mathbf{H}^1_n(\mathbb{R}^3) = \{\, \mathbf{Q} \in \check{\mathbf{H}}^1(\mathbb{R}^3) \ : \ \mathbf{Q}|_\Omega \in \mathbf{H}^1(\Omega) \ , \ \ \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}\,,$$

and

$$\mathbf{H}^1_n(\mathrm{div}\,;\mathbb{R}^3) = \{\, \mathbf{Q} \in \mathbf{H}^1_n(\mathbb{R}^3) \ : \ \nabla\cdot\mathbf{Q} = 0 \text{ in } \mathbb{R}^3 \ \}\,.$$

A similar notational convention applies to the Lebesgue spaces $L^p(\mathcal{D})$, $\mathcal{L}^p(\mathcal{D})$, and $\mathbf{L}^p(\mathcal{D})$.

Since $\nabla \times \mathbf{A} \to \mathbf{H}_e$ as $|\mathbf{x}| \to \infty$, it is clear that $\mathbf{A}$ does not have finite $\check{\mathbf{H}}^1(\mathbb{R}^3)$-norm. Thus, choosing $\mathbf{A}_e$ so that $\nabla \times \mathbf{A}_e = \mathbf{H}_e$ in all of $\mathbb{R}^3$, we let $\mathbf{A} = \hat{\mathbf{A}} + \mathbf{A}_e$. (For example, if $\mathbf{H}_e = (0,0,H_e)$, we could choose $\mathbf{A}_e = H_e(0,x,0)$.) With $\xi = \ell$ and $\mathbf{A} = \hat{\mathbf{A}} + \mathbf{A}_e$ the functional (2.10) reduces to

$$\begin{aligned} \mathcal{G}(\psi,\hat{\mathbf{A}}) = \int_\Omega \left(\frac{1}{2}|\psi|^4 - |\psi|^2 + \left| \left(i\nabla + \frac{1}{\kappa}(\hat{\mathbf{A}} + \mathbf{A}_e) \right) \psi \right|^2 \right) d\mathbf{x} \\ + \int_\Omega \left|\nabla \times \hat{\mathbf{A}}\right|^2 d\mathbf{x} + \int_{\Omega_e} |\nabla \times \hat{\mathbf{A}}|^2 \, d\mathbf{x}\,. \end{aligned} \tag{4.1}$$

The functional (4.1) has an important property, namely, gauge invariance. To be specific, for any $\phi \in H^2(\mathbb{R}^3)$, let the linear transformation G_ϕ be defined by: for all $(\psi, \hat{\mathbf{A}})$ and $(\zeta, \mathbf{Q}) \in \mathcal{H}^1(\Omega) \times \check{\mathbf{H}}^1(\mathbb{R}^3)$

$$(\zeta,\mathbf{Q}) = G_\phi(\psi,\hat{\mathbf{A}}) \quad \text{if} \quad \zeta = \psi e^{i\phi} \quad \forall\, \mathbf{x} \in \Omega \quad \text{and} \quad \mathbf{Q} = \hat{\mathbf{A}} + \kappa\nabla\phi \quad \forall\, \mathbf{x} \in \mathbb{R}^3\,.$$

Then, we have:

Definition 4.1. *$(\psi, \hat{\mathbf{A}})$ and $(\zeta, \mathbf{Q})$ are said to be* gauge equivalent *if and only if there exists a function $\phi \in H^2(\Omega)$ such that $(\zeta, \mathbf{Q}) = G_\phi(\psi, \hat{\mathbf{A}})$.*

Proposition 4.2. *For all $\phi \in H^2(\mathbb{R}^3)$ and for all $(\psi, \hat{\mathbf{A}}) \in \mathcal{H}^1(\Omega) \times \check{\mathbf{H}}^1(\mathbb{R}^3)$, $\mathcal{G}(\psi, \hat{\mathbf{A}}) = \mathcal{G}(G_\phi(\psi, \hat{\mathbf{A}}))$,* i.e., *the functional $\mathcal{G}$ of (4.1) is invariant under the gauge transformation G_ϕ.*

LEMMA 4.3. *Any* $(\zeta, \mathbf{Q}) \in \mathcal{H}^1(\Omega) \times \breve{\mathbf{H}}^1(\mathbb{R}^3)$ *is gauge equivalent to an element of* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_{\mathrm{n}}(\mathrm{div}\,; \mathbb{R}^3)$. □

For proofs of the above results, and for the other results of this section, one may consult [13].

For convenience the functional (4.1) is often modified into

$$\mathcal{F}(\psi, \hat{\mathbf{A}}) = \mathcal{G}(\psi, \hat{\mathbf{A}}) + \int_\Omega |\nabla \cdot \hat{\mathbf{A}}|^2 \, d\mathbf{x} + \int_{\Omega_e} |\nabla \cdot \hat{\mathbf{A}}|^2 \, d\mathbf{x} \, . \tag{4.2}$$

One may then show that:

THEOREM 4.4. $\mathcal{G}$ *has a least one minimizer belonging to* $\mathcal{H}^1(\Omega) \times \breve{\mathbf{H}}^1(\mathbb{R}^3)$ *and* $\mathcal{F}$ *has a least one minimizer belonging to* $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_{\mathrm{n}}(\mathbb{R}^3)$. *Moreover,*

$$\min_{\mathcal{H}^1(\Omega)\times\breve{\mathbf{H}}^1(\mathbb{R}^3)} \mathcal{G} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathbb{R}^3)} \mathcal{F} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\mathbb{R}^3)} \mathcal{G} \, .$$

Thus, the inclusion of the additional terms in $\mathcal{F}$ allows one to look for minimizers in a space of vector potentials that is not required to be solenoidal. This, for example, can greatly simplify numerical calculations.

Solutions of the (nondimensional) Ginzburg-Landau equations satisfy the following maximum principle.

PROPOSITION 4.5. *If* $(\psi, \hat{\mathbf{A}})$ *is a solution of the Ginzburg-Landau equations* (3.1)-(3.7)*, then* $|\psi| \leq 1$ *almost everywhere.* □

5. Convergence to leading order solution

We now examine in more detail the relation between solutions of the Ginzburg-Landau equations (3.1)-(3.7), as $\kappa \to \infty$, and solutions of the leading order equations (3.10)-(3.15).

As was noted above, we assume that $\mathbf{H}_e = \kappa \mathbf{H}_0$, where $\mathbf{H}_0$ is independent of κ so that the applied field $\mathbf{H}_e$ is of $O(\kappa)$. Again, such values of the applied field are of interest since, for technological reasons, one is interested in large fields. We may then also set $\mathbf{A}_e = \kappa \mathbf{A}_0$.

Let $\epsilon = 1/\kappa$, $\hat{\mathbf{A}} = \kappa \hat{\mathbf{A}}_\epsilon$, and $\mathbf{A}_e = \kappa \mathbf{A}_0$ so that $\mathbf{A} = \kappa(\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)$. Then, we have from (3.1)-(3.7), for any $\epsilon > 0$,

$$(i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)^2 \psi_\epsilon - \psi_\epsilon + |\psi_\epsilon|^2 \psi_\epsilon = 0 \quad \text{in } \Omega \, , \tag{5.1}$$

$$-\nabla \times \nabla \times \hat{\mathbf{A}}_\epsilon = \epsilon^2 \left(\frac{i}{2} (\psi_\epsilon^* \nabla \psi_\epsilon - \psi_\epsilon \nabla \psi_\epsilon^*) + |\psi_\epsilon|^2 (\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) \right) \quad \text{in } \Omega \, , \tag{5.2}$$

$$\nabla \times \nabla \times \hat{\mathbf{A}}_\epsilon = 0 \quad \text{in } \Omega_e \, , \tag{5.3}$$

$$\mathbf{n} \cdot (i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) \psi_\epsilon = 0 \quad \text{on } \Gamma \, , \tag{5.4}$$

$$[\hat{\mathbf{A}}_\epsilon \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma \, , \tag{5.5}$$

$$\Big[(\nabla \times \hat{\mathbf{A}}_\epsilon) \times \mathbf{n}\Big] = 0 \quad \text{on } \Gamma\,, \tag{5.6}$$

and

$$\nabla \times \hat{\mathbf{A}}_\epsilon \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty\,. \tag{5.7}$$

As a result of Theorem 4.4, we can invoke the London gauge

$$\nabla \cdot \hat{\mathbf{A}}_\epsilon = 0 \quad \text{in } \Omega \cup \Omega_e\,, \tag{5.8}$$

and

$$\hat{\mathbf{A}}_\epsilon \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,. \tag{5.9}$$

Using the current scalings, the functional (4.1) is given by

$$\begin{aligned} \mathcal{G}_\epsilon(\zeta, \mathbf{Q}) = &\int_\Omega \left(\frac{1}{2}|\zeta|^4 - |\zeta|^2 + |(i\nabla + \mathbf{Q} + \mathbf{A}_0)\,\zeta|^2\right) d\mathbf{x} \\ &+ \frac{1}{\epsilon^2}\int_\Omega |\nabla \times \mathbf{Q}|^2 \, d\mathbf{x} + \frac{1}{\epsilon^2}\int_{\Omega_e} |\nabla \times \mathbf{Q}|^2 \, d\mathbf{x}\,. \end{aligned} \tag{5.10}$$

We are interested in solutions of (5.1)-(5.9) that are minimizers of the functional (5.10) over the space $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$. The system (5.1)-(5.9) has the following weak formulation: seek $\hat{\mathbf{A}}_\epsilon \in \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$ and $\psi_\epsilon \in \mathcal{H}^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \Big(\nabla \times \hat{\mathbf{A}}_\epsilon\Big) \cdot \nabla \times \tilde{\mathbf{A}}\, d\mathbf{x} + \int_{\Omega_e} (\nabla \times \hat{\mathbf{A}}_\epsilon) \cdot \nabla \times \tilde{\mathbf{A}}\, d\mathbf{x} \\ &\qquad = -\epsilon^2 \int_\Omega \Big(|\psi_\epsilon|^2 (\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) \cdot \tilde{\mathbf{A}} + \Re\left\{ i\psi^*_\epsilon \nabla \psi_\epsilon \right\} \cdot \tilde{\mathbf{A}} \Big)\, d\mathbf{x} \\ &\qquad\qquad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3) \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} &\int_\Omega \Re\Big\{ (i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)\psi_\epsilon \cdot (-i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)\tilde{\psi}^* \Big\}\, d\mathbf{x} \\ &\qquad + \int_\Omega (|\psi_\epsilon|^2 - 1)\Re\Big\{ \psi_\epsilon \tilde{\psi}^* \Big\}\, d\mathbf{x} = 0 \qquad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega)\,. \end{aligned} \tag{5.12}$$

Of course, (5.11) and (5.12) are the necessary conditions for $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ to be minimizers of the functional $\mathcal{G}_\epsilon$ of (5.10).

We begin our investigation of the behavior of solutions as $\kappa \to \infty$, or equivalently as $\epsilon \to 0$, by considering some uniform estimates for $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$. First, we have the following lemma.

LEMMA 5.1. *For any $\epsilon > 0$, let $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$ be a minimizer of the functional $\mathcal{G}_\epsilon$ given in* (5.10)*. Then,*

$$\mathcal{G}_\epsilon(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon) \le 0\,.$$

Proof. In (5.10), set $\zeta = 0$ and $\mathbf{Q} = \mathbf{0}$. Then, $\mathcal{G}_\epsilon(0,\mathbf{0}) = 0$. The result then follows since $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ is a minimizer of $\mathcal{G}_\epsilon$. □

One can easily deduce the following result.

LEMMA 5.2. *For $\epsilon > 0$, $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ is uniformly bounded in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,; I\!R^3)$.* □

One easily then has that there exists a sequence $\epsilon_k \to 0$, such that the sequence of corresponding solutions $\{(\psi_{\epsilon_k}, \hat{\mathbf{A}}_{\epsilon_k})\}$ converges weakly to some element $(\psi_0, \hat{\mathbf{A}}_0)$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$, and consequently, since all the nonlinearities appear in the bounded region Ω, we have the following result.

COROLLARY 5.3. *For any p such that $1 \le p \le 6$ and for any bounded region $\mathcal{D} \in \mathbb{R}^3$,*

$$\lim_{\epsilon_k \to 0} ||\psi_{\epsilon_k} - \psi_0||_{\mathcal{L}^p(\Omega)} = 0 \qquad \textit{and} \qquad \lim_{\epsilon_k \to 0} ||\hat{\mathbf{A}}_{\epsilon_k} - \mathbf{A}_0||_{\mathbf{L}^p(\mathcal{D})} = 0\,.$$

The above result implies almost everywhere convergence in the domain Ω; thus, in the weak formulation (5.11)-(5.12), we can pass to the limit and obtain the following weak formulation for the weak limit $(\psi_0, \hat{\mathbf{A}}_0)$: seek $\hat{\mathbf{A}}_0 \in \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$ and $\psi_0 \in \mathcal{H}^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \left(\nabla \times \hat{\mathbf{A}}_0\right) \cdot \nabla \times \tilde{\mathbf{A}}\, d\mathbf{x} \\ &\qquad + \int_{\Omega_e} (\nabla \times \hat{\mathbf{A}}_0) \cdot \nabla \times \tilde{\mathbf{A}}\, d\mathbf{x} = 0 \qquad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\mathrm{div}\,; \mathbb{R}^3) \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} &\int_\Omega \Re\left\{(i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)\psi_0 \cdot (-i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)\tilde{\psi}^*\right\} d\mathbf{x} \\ &\qquad + \int_\Omega (|\psi_0|^2 - 1)\Re\left\{\psi_0\tilde{\psi}^*\right\} d\mathbf{x} = 0 \qquad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega)\,. \end{aligned} \tag{5.14}$$

One easily finds that (5.13) is a weak formulation of

$$\nabla \times \nabla \times \hat{\mathbf{A}}_0 = \mathbf{0} \quad \text{in } \Omega \text{ and } \Omega_e\,, \tag{5.15}$$

$$[\hat{\mathbf{A}}_0 \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Gamma\,, \tag{5.16}$$

$$\left[(\nabla \times \hat{\mathbf{A}}_0) \times \mathbf{n}\right] = 0 \quad \text{on } \Gamma\,, \tag{5.17}$$

and

$$\nabla \times \hat{\mathbf{A}}_0 \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty\,. \tag{5.18}$$

Similarly, one finds that (5.14) is a weak formulation of

$$(i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)^2\psi_0 - \psi_0 + |\psi_0|^2\psi_0 = 0 \quad \text{in } \Omega\,, \tag{5.19}$$

and

$$\mathbf{n} \cdot (i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)\psi_0 = 0 \quad \text{on } \Gamma \,. \tag{5.20}$$

Clearly, $\hat{\mathbf{A}}_0 = \mathbf{0}$ satisfies (5.15)-(5.18). Then, we see that (5.18)-(5.19) are identical to the leading order equations (3.14)-(3.15). Thus, since as $\epsilon \to 0$, $A_\epsilon = \kappa(\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) \to \kappa\mathbf{A}_0$ and $\psi_\epsilon \to \psi_0$, we see that the limit function $(\psi_0, \mathbf{A}_0)$ satisfies the leading order equations (3.10)-(3.15).

Using standard techniques and the weak formulations, we deduce the strong convergence of the subsequence.

LEMMA 5.4. *Any convergent subsequence of $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ converges strongly to a solution $(\psi_0, \hat{\mathbf{A}}_0)$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_{\mathrm{n}}(\mathrm{div}\,; \mathbb{R}^3)$.*

Proof. For simplicity, we denote the subsequence by $\{(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)\}$. Setting $\tilde{\mathbf{A}} = \hat{\mathbf{A}}_\epsilon$ in (5.11) we have that

$$\begin{aligned} \int_\Omega \left|\nabla \times \hat{\mathbf{A}}_\epsilon\right|^2 d\mathbf{x} &+ \int_{\Omega_e} \left|\nabla \times \hat{\mathbf{A}}_\epsilon\right|^2 d\mathbf{x} \\ &= -\epsilon^2 \int_\Omega \Big(|\psi_\epsilon|^2(\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) + \Re\left\{i\psi_\epsilon^* \nabla\psi_\epsilon\right\}\Big) \cdot \hat{\mathbf{A}}_\epsilon \, d\mathbf{x} \,. \end{aligned}$$

Letting $\epsilon \to 0$, then, the uniform bound for $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ and the Sobolev imbedding lemma yield

$$\lim_{\epsilon \to 0} ||\nabla \times \hat{\mathbf{A}}_\epsilon||_0 = 0 \,.$$

Since $\mathrm{div}\, \hat{\mathbf{A}}_\epsilon = 0$, it follows that

$$\lim_{\epsilon \to 0} ||\hat{\mathbf{A}}_\epsilon||_* = 0 \,.$$

By Corollary 5.3, we get that

$$\lim_{\epsilon \to 0} ||\hat{\mathbf{A}}_\epsilon||_{1,\Omega} = 0 \,.$$

Similarly, setting $\tilde{\psi} = \psi_\epsilon - \psi_0$ in (5.12) and (5.14), we have

$$\begin{aligned} \int_\Omega \Re\Big\{(i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)\psi_\epsilon \cdot (-i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)(\psi_\epsilon - \psi_0)^*\Big\} \, d\mathbf{x} \\ + \int_\Omega (|\psi_\epsilon|^2 - 1)\Re\left\{\psi_\epsilon(\psi_\epsilon - \psi_0)^*\right\} d\mathbf{x} = 0 \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \Re\Big\{(i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)\psi_0 \cdot (-i\nabla + \hat{\mathbf{A}}_0 + \mathbf{A}_0)(\psi_\epsilon - \psi_0)^*\Big\} \, d\mathbf{x} \\ + \int_\Omega \Re\left\{(|\psi_0|^2 - 1)\psi_0 \cdot (\psi_\epsilon - \psi_0)^*\right\} d\mathbf{x} = 0 \,. \end{aligned}$$

Subtracting these results yields

$$\begin{aligned} \int_\Omega & |\nabla(\psi_\epsilon - \psi_0)|^2 \, d\mathbf{x} \\ &= -\Re \int_\Omega (|\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0|^2\psi_\epsilon - |\hat{\mathbf{A}}_0 + \mathbf{A}_0|^2\psi_0)(\psi_\epsilon - \psi_0)^* \, d\mathbf{x} \\ &\quad - 2\Re \int_\Omega i \Big(\nabla\psi_\epsilon \cdot (\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0) - \nabla\psi_0 \cdot (\hat{\mathbf{A}}_0 + \mathbf{A}_0)\Big) (\psi_\epsilon - \psi_0)^* \, d\mathbf{x} \\ &\quad - \Re \int_\Omega (|\psi_\epsilon|^2 - 1)\psi_\epsilon(\psi_\epsilon - \psi_0)^* \, d\mathbf{x} + \Re \int_\Omega (|\psi_0|^2 - 1)\psi_0(\psi_\epsilon - \psi_0)^* \, d\mathbf{x} \,. \end{aligned}$$

Letting $\epsilon \to 0$, then, the weak convergence and the almost everywhere convergence of the appropriate sequences and the Sobolev imbedding lemma yield

$$\lim_{\epsilon\to 0} \|\nabla(\psi_\epsilon - \psi_0)\|_0 = 0\,.$$

The strong convergence now follows. □

In summary, we have shown that as $\kappa \to \infty$, solutions of (5.1)-(5.7) that are minimizers of (5.10) converge strongly in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\mathbb{R}^3)$ to solutions of the leading order equations (5.15)-(5-20). Equivalently, we have shown that as $\kappa \to \infty$, solutions of the Ginzburg-Landau equations (3.1)-(3.7) converge to solutions of the leading order equations (3.10)-(3.15). (It can actually be shown that the convergence is quadratic, i.e., that $\hat{\mathbf{A}}_\epsilon = O(\epsilon^2) = O(\kappa^{-2})$ and $\psi_\epsilon - \psi_0 = O(\epsilon^2) = O(\kappa^{-2})$ as $\kappa \to \infty$. This provides partial justification for the expansions (3.9) in terms of powers of $1/\kappa^2$.)

We can also prove some additional results. For any $\psi \in \mathcal{H}^1(\Omega)$, let

$$\mathcal{G}_0(\psi) = \int_\Omega \left(\frac{1}{2}|\psi|^4 - |\psi|^2 + |(i\nabla + \mathbf{A}_0)\,\psi|^2\right)\,d\mathbf{x}\,.$$

We then have the following results.

Proposition 5.5.

$$\lim_{\epsilon\to 0} \mathcal{G}_\epsilon(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon) = \mathcal{G}_0(\psi_0)\,.$$

Proof. Using the strong convergence in $\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$ of $(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$ to $(\psi_0, \mathbf{0})$, we easily deduce that

$$\lim_{\epsilon\to 0}\int_\Omega \left(\frac{1}{2}|\psi_\epsilon|^4 - |\psi_\epsilon|^2 + \left|\left(i\nabla + \hat{\mathbf{A}}_\epsilon + \mathbf{A}_0\right)\,\psi_\epsilon\right|^2\right)\,d\mathbf{x} = \mathcal{G}_0(\psi_0)\,.$$

Setting $\tilde{\mathbf{A}} = \hat{\mathbf{A}}_\epsilon$ in (5.11), we have

$$\frac{1}{\epsilon^2}\int_\Omega |\nabla\times\hat{\mathbf{A}}_\epsilon|^2\,d\mathbf{x} = \\ -\int_\Omega \left(|\psi_\epsilon|^2(\hat{\mathbf{A}}_\epsilon + \mathbf{A}_0)\cdot\hat{\mathbf{A}}_\epsilon + \Re\left\{i\psi_\epsilon^*\nabla\psi_\epsilon\right\}\cdot\hat{\mathbf{A}}_\epsilon\right)\,d\mathbf{x}$$

so that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_\Omega |\nabla\times\hat{\mathbf{A}}_\epsilon|^2\,d\mathbf{x} = 0\,.$$

This completes the proof. □

Proposition 5.6. *ψ_0 is a global minimizer of $\mathcal{G}_0$ in $\mathcal{H}^1(\Omega)$.*

Proof. Suppose the result is not true. Then, there exists $\zeta \in \mathcal{H}^1(\Omega)$ such that

$$\mathcal{G}_0(\zeta) < \mathcal{G}_0(\psi_0)\,.$$

But then, for sufficiently small ϵ,

$$\mathcal{G}_\epsilon(\zeta, \mathbf{A}_0) < \mathcal{G}_\epsilon(\psi_\epsilon, \hat{\mathbf{A}}_\epsilon)$$

which, of course, is a contradiction. □

6. Two-dimensional boundary value and periodic value problems

The computational results reported on in [14] and in §7 below deal with two-dimensional periodic value and boundary value models, respectively. Thus, in this section, we consider high-κ models in these two settings. Since many of the details are similar to those given above for three-dimensional boundary value problems, our presentation here will be somewhat brief.

6.1. Two-dimensional boundary value problems

It is well-known (see, e.g., [2]), that if the superconductor occupies an infinite cylinder having bounded cross-section Ω and if the applied field $\mathbf{H}_e$ is perpendicular to this cross-section, then the field in the region exterior to the cylinder is constant and equal to the applied field. As a result, in forming the Gibbs free energy, we need only consider the bounded region $\Omega \in \mathbb{R}^2$. Using the nondimensionalizations of §3, the Ginzburg-Landau equations are given by

$$\left(i\nabla + \frac{1}{\kappa}\mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{6.1}$$

$$\nabla \times \nabla \times \mathbf{A} = \mathbf{J} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \left(\frac{1}{\kappa^2}\right)|\psi|^2\mathbf{A} \quad \text{in } \Omega\,, \tag{6.2}$$

$$\mathbf{n}\cdot\left(i\nabla + \frac{1}{\kappa}\mathbf{A}\right)\psi = 0 \quad \text{on } \Gamma\,, \tag{6.3}$$

and

$$\nabla \times \mathbf{A} = H_e \quad \text{on } \Gamma\,. \tag{6.4}$$

The nondimensional Gibbs free energy is given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(\frac{1}{2}|\psi|^4 - |\psi|^2 + \left|\left(i\nabla + \frac{1}{\kappa}\mathbf{A}\right)\psi\right|^2 + |\nabla \times \mathbf{A} - H_e|^2\right) d\mathbf{x}\,. \tag{6.5}$$

In (6.1)-(6.5) we have adopted the convention that $\mathbf{H}_e = (0, 0, H_e)$ and that the curl of a two-component vector may be viewed as a scalar.

On substituting in the expansions (3.9), one easily finds that the equations for the leading order terms are given by

$$\nabla \times \nabla \times \mathbf{A}_0 = \mathbf{J}_0 = \mathbf{0} \quad \text{in } \Omega \tag{6.6}$$

and

$$\nabla \times \mathbf{A}_0 = H_0 \quad \text{on } \Gamma \tag{6.7}$$

for $\mathbf{A}_0$ and

$$(i\nabla + \mathbf{A}_0)^2\,\psi_0 - \psi_0 + |\psi_0|^2\psi_0 = 0 \quad \text{in } \Omega \tag{6.8}$$

and

$$\mathbf{n}\cdot(i\nabla+\mathbf{A}_0)\,\psi_0=0 \quad \text{on } \Gamma \tag{6.9}$$

for ψ_0, where again $H_e=\kappa H_0$. Note that (6.6) and (6.7) imply that we may choose $\mathbf{A}_0$ such that $\nabla\times\mathbf{A}_0=H_0$.

The next terms in the expansions in (3.9) are determined from

$$\nabla\times\nabla\times\mathbf{A}_1=\mathbf{J}_1=-\frac{i}{2}(\psi_0^*\nabla\psi_0-\psi_0\nabla\psi_0^*)-|\psi_0|^2\mathbf{A}_0 \quad \text{in}\Omega$$

and

$$\nabla\times\mathbf{A}_1=0 \quad \text{on } \Gamma$$

for $\mathbf{A}_1$ and

$$\begin{aligned}&(i\nabla+\mathbf{A}_0)^2\,\psi_1-\psi_1+2|\psi_0|^2\psi_1+\psi_0^2\psi_1^*\\&\qquad=-\,(i\nabla+\mathbf{A}_0)\cdot(\psi_0\mathbf{A}_1)-\mathbf{A}_1\cdot(i\nabla+\mathbf{A}_0)\,\psi_0 \quad \text{in } \Omega\end{aligned}$$

and

$$\mathbf{n}\cdot(i\nabla+\mathbf{A}_0)\,\psi_1=-\psi_0(\mathbf{A}_1\cdot\mathbf{n}) \quad \text{on } \Gamma$$

for ψ_1.

All the results of §4, properly restated for the current two-dimensional setting, are still valid; see, e.g., [13]. Likewise, all the results of §5, again properly restated, are also still valid. Indeed, their proofs are somewhat simpler than those given in those sections since one does not have to worry about infinite domains. In particular, if for any $\epsilon=1/\kappa$, we let $\mathbf{A}=\kappa\hat{\mathbf{A}}_\epsilon$ it can be rigorously shown that solutions of the Ginzburg-Landau equations for $(\psi_\epsilon,\hat{\mathbf{A}}_\epsilon)$, i.e.,

$$(i\nabla+\hat{\mathbf{A}}_\epsilon)^2\psi_\epsilon-\psi_\epsilon+|\psi_\epsilon|^2\psi_\epsilon=0 \quad \text{in } \Omega\,,$$

$$-\nabla\times\nabla\times\hat{\mathbf{A}}_\epsilon=\epsilon^2\left(\frac{i}{2}(\psi_\epsilon^*\nabla\psi_\epsilon-\psi_\epsilon\nabla\psi_\epsilon^*)+|\psi_\epsilon|^2\hat{\mathbf{A}}_\epsilon\right) \quad \text{in } \Omega\,,$$

$$\mathbf{n}\cdot(i\nabla+\hat{\mathbf{A}}_\epsilon)\psi_\epsilon=0 \quad \text{on } \Gamma\,,$$

and

$$\nabla\times\hat{\mathbf{A}}_\epsilon=H_0 \quad \text{on } \Gamma$$

converge, as $\kappa\to\infty$, to solutions $(\psi_0,\mathbf{A}_0)$ of the leading order equations (6.6)-(6.9).

6.2. Two-dimensional periodic value problems

We now consider the case in which the region occupied by the superconductor occupies is infinite in extent. We choose the (x,y)-plane to be perpendicular to the applied field $\mathbf{H}_e$. In this case, it is well-known that the solutions of the Ginzburg-Landau equations are periodic with repect to a lattice in the plane. This model has proved popular in analytical and computational studies of superconductivity; see, e.g., [1], [7], [12], [17], [21], [22], [25], and [26]. Many of the details about the background material of this section may be found in these references and in [14], [15], and [16].

Given two arbitrary vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ that span $\mathbb{R}^2$, we say that a function $f(\mathbf{x})$ is *periodic with respect to the lattice determined by* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ if

$$f(\mathbf{x}+\mathbf{t}_k) = f(\mathbf{x}) \qquad \text{for } k=1,2 \text{ and } \forall\, \mathbf{x} \in \mathbb{R}^2\,. \tag{6.10}$$

Here, f may be scalar or vector-valued and may be real or complex-valued. The vectors $\mathbf{t}_k$, $k=1,2$, will be referred to as *lattice vectors*; without loss of generality, we assume that the counter-clockwise angle between $\mathbf{t}_1$ and $\mathbf{t}_2$ is less than π and that $\mathbf{t}_1$ is aligned with the x-axis. For the sake of brevity, we will often refer to functions satisfying (6.10) merely as being *periodic.* (The most interesting choice of lattice vectors are those for an equilateral triangular lattice.) Given any point $\mathbf{P} \in \mathbb{R}^2$, a *cell of the lattice with respect to the point* $\mathbf{P}$ is the open parallelogram $\Omega_P \subset \mathbb{R}^2$ having a vertex at $\mathbf{P}$ and bounded on two sides by the lattice vectors emanating from $\mathbf{P}$.

Again using the nondimensionalizations of §3, The Gibbs free energy of a lattice cell is given, to within an unimportant constant, by

$$\mathcal{G}(\psi,\mathbf{A}) = \int_{\Omega_P} \left(\frac{1}{2}|\psi|^4 - |\psi|^2 + \left| \left(i\nabla + \frac{1}{\kappa}\mathbf{A} \right) \psi \right|^2 + |\nabla\times\mathbf{A}|^2 \right) d\mathbf{x}\,.$$

A basic assumption of the periodic Ginzburg-Landau model for superconductivity concerns the periodic nature of the physical attributes of the superconductor, i.e., the density of superconducting charge carriers, the magnetic field, and the free energy density are periodic with respect to the lattice vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. In terms of the magnetic potential $\mathbf{A}$ and order parameter ψ, these infer that

$$\nabla\times\mathbf{A}\,, \quad |\psi|\,, \quad \text{and} \quad \left(\nabla\phi - \frac{1}{\kappa}\mathbf{A}\right) \quad \text{are periodic}\,, \tag{6.11}$$

where ϕ denotes the phase of the order parameter, i.e., $\psi = |\psi|e^{i\phi}$. The current is also periodic and the value of the Gibbs free energy in a lattice cell is the same for any cell, i.e., it is independent of the choice of the point $\mathbf{P}$. Note that the magnetic potential and order parameter themselves are not assumed to be periodic.

Another assumption concerns the notion of *fluxoid quantization.* In the present context, we have that

$$\kappa\bar{B}|\Omega| = 2\pi n \tag{6.12}$$

where $|\Omega|$ denotes the area of the lattice cell, n is an integer, and $\bar{B}$ is the average magnetic field over a cell Ω_P, i.e.,

$$\bar{B} = \frac{1}{|\Omega|}\int_{\Omega_P} \nabla\times\mathbf{A}\, d\mathbf{x}\,.$$

Note that $|\Omega|$ and $\bar{B}$ are independent of the choice of $\mathbf{P}$.

Let

$$\mathbf{A}_0 = \frac{\bar{B}}{2\kappa}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\,.$$

and

$$g_k(\mathbf{x}) = -\frac{\bar{B}}{2\kappa}(\mathbf{x}\times\mathbf{t}_k)\,, \quad k=1,2\,.$$

Then, if $\psi = |\psi|e^{i\phi}$ and $\mathbf{A}$ satisfy (6.11), it can be shown that $(\psi, \mathbf{A})$ is gauge equivalent to $(\zeta, \mathbf{Q} + \kappa\mathbf{A}_0)$ where

$$|\zeta| \quad \text{and} \quad |\mathbf{Q}| \quad \text{are periodic}\,, \qquad \nabla\cdot\mathbf{Q} = 0 \quad \text{in } \Omega\,,$$

and

$$\omega(\mathbf{x} + \mathbf{t}_k) - \omega(\mathbf{x}) = g_k(\mathbf{x})\,, \quad k = 1, 2\,,$$

where ω denotes the phase of ζ, i.e., $\zeta = |\zeta|e^{i\omega}$. Furthermore, the magnetic field H and the density of superconducting charge carriers N_s may be recovered from ζ and $\mathbf{Q}$ through the relations

$$H = \nabla\times\mathbf{A} = \nabla\times\mathbf{Q} + \bar{B} \quad \text{and} \quad N_s = |\psi|^2 = |\zeta|^2\,.$$

In terms of ζ and $\mathbf{Q}$, we have the energies $\mathcal{F}$ and $\mathcal{G}$ defined by

$$\begin{aligned}\mathcal{F}(\zeta,\mathbf{Q}) =& \mathcal{G}(\zeta,\mathbf{Q}) + \int_{\Omega_P} (\nabla\cdot\mathbf{Q})^2\,d\mathbf{x} = \\ & \int_{\Omega_P}\left(\frac{1}{2}|\zeta|^4 - |\zeta|^2 + \left|\left(i\nabla + \mathbf{A}_0 + \frac{1}{\kappa}\mathbf{Q})\right)\zeta\right|^2\right)\,d\mathbf{x}\,. \\ & + \int_{\Omega_P}\Big((\nabla\cdot\mathbf{Q})^2 + |\nabla\times(\mathbf{Q}+\kappa\mathbf{A}_0)|^2\Big)\,d\mathbf{x}\end{aligned}$$

The model is completely specified by choosing the Ginzburg-Landau parameter κ, the number of fluxoids in a lattice cell n, the position of the point $\mathbf{P}$, the average magnetic field $\bar{B}$, the direction of the lattice vector $\mathbf{t}_2$ (recall that we have chosen the direction of $\mathbf{t}_1$ to coincide with the x-axis), and the relative magnitudes of the lattice vectors. (The absolute magnitude of the lattice vectors is determined from the above data and the fluxoid quantization condition (6.12)). Note that it is convenient not to specify the external filed H_e, but rather to specify the average field $\bar{B}$. The applied field H_e may be recovered from the solution $\mathbf{A}$ and ψ (or $\mathbf{Q}$ and ζ) in various ways. One such method is discussed at the end of this section.

For the periodic model, the Ginzburg-Landau equations corresponding to the minimization of the free energy are given by

$$\nabla\times\nabla\times\mathbf{Q} + \frac{1}{\kappa}\Re\left\{\zeta^*\left(i\nabla + \mathbf{A}_0 + \frac{1}{\kappa}\mathbf{Q}\right)\hat{\zeta}\right\} = \mathbf{0} \quad \text{in } \Omega_P\,,$$

$$\left(i\nabla + \mathbf{A}_0 + \frac{1}{\kappa}\mathbf{Q}\right)^2\zeta + (|\zeta|^2 - 1)\zeta = 0 \quad \text{in } \Omega_P\,,$$

$$\nabla\times\mathbf{Q} = 0 \quad \text{in } \Omega_P\,,$$

$$\mathbf{Q}(\mathbf{x}+\mathbf{t}_k) = \mathbf{Q}(\mathbf{x}) \quad \forall\,\mathbf{x}\in\Gamma_k,\ k = 1, 2\,,$$

$$\zeta(\mathbf{x}+\mathbf{t}_k) = \zeta(\mathbf{x})e^{ig_k(\mathbf{x})} \quad \forall\,\mathbf{x}\in\Gamma_k,\ k = 1, 2\,,$$

$$\Big(\nabla|\zeta|\Big)\Big|_{\mathbf{x}+\mathbf{t}_k} = \Big(\nabla|\zeta|\Big)|_{\mathbf{x}} \quad \forall\,\mathbf{x}\in\Gamma_k,\ k = 1, 2\,,$$

$$\nabla\times\mathbf{Q}\big|_{\mathbf{x}+\mathbf{t}_k} = \nabla\times\mathbf{Q}\big|_{\mathbf{x}} \quad \forall\,\mathbf{x}\in\Gamma_k,\ k = 1, 2\,,$$

and

$$|\zeta|\Big(\nabla\omega - \mathbf{A}_0\Big)\Big|_{\mathbf{x}+\mathbf{t}_k} = |\zeta|\Big(\nabla\omega - \mathbf{A}_0\Big)\Big|_{\mathbf{x}} \qquad \forall\, \mathbf{x} \in \Gamma_k,\ k = 1, 2,$$

where ω denotes the phase of ζ. Here, Γ_1 and Γ_2 respectively denote the left and bottom sides of the parallelogram Ω_P.

If we assume that $\bar{B} = \kappa\hat{B}$, where $\hat{B}$ is independent of κ, and introduce expansions for ζ and $\mathbf{Q}$ analogous to the ones of ψ and $\mathbf{A}$ in (3.9), we have that the leading order term $\mathbf{Q}_0$ satisfies

$$\nabla\times\nabla\times\mathbf{Q}_0 = \mathbf{0} \quad \text{and} \quad \nabla\times\mathbf{Q}_0 = 0 \quad \text{in } \Omega_P\,, \tag{6.13}$$

and

$$\mathbf{Q}_0(\mathbf{x}+\mathbf{t}_k) = \mathbf{Q}_0(\mathbf{x}) \quad \text{and} \quad \nabla\times\mathbf{Q}_0|_{\mathbf{x}+\mathbf{t}_k} = \nabla\times\mathbf{Q}_0|_{\mathbf{x}} \qquad \forall\, \mathbf{x}\in\Gamma_k,\ k=1,2. \tag{6.14}$$

These equations imply that $\mathbf{Q}_0 = \mathbf{0}$. Note that then, to leading order, the magnetic potential $\mathbf{A} = \mathbf{Q} + \kappa\mathbf{A}_0$ is given by $\mathbf{A} \approx \kappa\mathbf{A}_0$. The corresponding equations for the leading order term ζ_0 are given by

$$\left(i\nabla + \mathbf{A}_0\right)^2\zeta_0 + (|\zeta_0|^2 - 1)\zeta_0 = 0 \quad \text{in } \Omega_P\,. \tag{6.15}$$

$$\zeta_0(\mathbf{x}+\mathbf{t}_k) = \zeta_0(\mathbf{x})e^{ig_k(\mathbf{x})} \qquad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2, \tag{6.16}$$

and

$$\Big(\nabla|\zeta_0|\Big)\Big|_{\mathbf{x}+\mathbf{t}_k} = \Big(\nabla|\zeta_0|\Big)|_{\mathbf{x}} \quad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2. \tag{6.17}$$

We see that again (6.13)-(6.14), the equations for $\mathbf{Q}_0$, uncouple from (6.15)-(6.17), the equations for ζ_0. Once it is determined that $\mathbf{Q}_0 = \mathbf{0}$, one may solve (6.15)-(6.17) for ζ_0.

All the results of §4, properly restated for the current periodic setting, are still valid; see, e.g., [16]. Likewise, all the results of §5, again properly restated, are also still valid. In particular, if for any $\epsilon = 1/\kappa$, we let $\mathbf{Q} = \kappa\hat{\mathbf{Q}}_\epsilon$ it can be rigorously shown that solutions of the Ginzburg-Landau equations for $(\zeta_\epsilon, \hat{\mathbf{Q}}_\epsilon)$, i.e.,

$$-\nabla\times\nabla\times\hat{\mathbf{Q}}_\epsilon = \epsilon^2\Re\Big\{|\zeta_\epsilon|^2\hat{\mathbf{Q}}_\epsilon + \zeta_\epsilon^*(i\nabla+\mathbf{A}_0)\zeta_\epsilon\Big\} \quad \text{in } \Omega_P\,,$$

$$(i\nabla + \mathbf{A}_0 + \hat{\mathbf{Q}}_\epsilon)^2\zeta_\epsilon + (|\zeta_\epsilon|^2 - 1)\zeta_\epsilon = 0 \quad \text{in } \Omega_P\,,$$

$$\operatorname{div}\hat{\mathbf{Q}}_\epsilon = 0 \quad \text{in } \Omega_P\,,$$

$$\hat{\mathbf{Q}}_\epsilon(\mathbf{x}+\mathbf{t}_k) = \hat{\mathbf{Q}}_\epsilon(\mathbf{x}) \qquad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2,$$

$$\zeta_\epsilon(\mathbf{x}+\mathbf{t}_k) = \zeta_\epsilon(\mathbf{x})e^{ig_k(\mathbf{x})} \qquad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2,$$

$$\Big(\nabla|\zeta_\epsilon|\Big)\Big|_{\mathbf{x}+\mathbf{t}_k} = \Big(\nabla|\zeta_\epsilon|\Big)\Big|_{\mathbf{x}} \quad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2,$$

$$\nabla\times\hat{\mathbf{Q}}_\epsilon|_{\mathbf{x}+\mathbf{t}_k} = \nabla\times\hat{\mathbf{Q}}_\epsilon|_{\mathbf{x}} \qquad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2,$$

and

$$|\zeta_\epsilon|\Big(\nabla\omega_\epsilon + \mathbf{A}_0\Big)\Big|_{\mathbf{x}+\mathbf{t}_k} = |\zeta_\epsilon|\Big(\mathbf{grad}\,\mathbf{x}_\epsilon + \mathbf{A}_0\Big)\Big|_{\mathbf{x}} \qquad \forall\,\mathbf{x}\in\Gamma_k,\ k=1,2,$$

where $\zeta_\epsilon = |\zeta_\epsilon| e^{i\omega_\epsilon}$, converge, as $\kappa \to \infty$, to solutions $(\zeta_\epsilon, \hat{\mathbf{Q}}_\epsilon) = (\zeta_\epsilon, \mathbf{0})$ of the leading order equations (6.13)-(6.17).

As was mentioned above, in the the periodic Ginzburg-Landau model one usually specifies the average field $\bar{B}$ (or $\hat{B} = \bar{B}/\kappa$) and not the applied magnetic field H_e. After solving for the magnetic potential and order parameter for specified values of the input parameters, one often would like to recover the applied field. The best method for determining H_e is developed in [11]. For any $\epsilon = 1/\kappa$, in terms of ζ_ϵ and $\hat{\mathbf{Q}}_\epsilon$, the applied field is given by

$$H_e(\epsilon_n) = \frac{1}{\bar{B}|\Omega|} \int_{\Omega_P} \left(\frac{1}{2} \left| \left(i\nabla + \hat{\mathbf{Q}}_\epsilon + \mathbf{A}_0 \right) \zeta_\epsilon \right|^2 + \frac{1}{\epsilon^2} |\nabla \times \hat{\mathbf{Q}}_\epsilon + \hat{B}|^2 \right) d\mathbf{x} \, .$$

Similar to the proof of the Proposition 5.5, we have the following result.

Proposition 6.1.

$$\lim_{\epsilon_n \to 0} \left(\frac{1}{\kappa} H_e(\epsilon_n) \right) = \hat{H}_e^0 = \frac{1}{2\kappa \bar{B}|\Omega|} \int_{\Omega_P} \left(|i\nabla + \mathbf{Q}_0 + \mathbf{A}_0) \zeta_0 \right|^2 d\mathbf{x} \, . \quad \Box$$

Note that the fluxoid quantization condition (6.12) implies that $\bar{B}|\Omega| = O(\kappa^{-1})$ so that, as expected, $\hat{H}_e^0 = O(1)$.

7. Computational results

In this section, we examine some computational results for both the full Ginzburg-Landau model and for the leading order equations. We first briefly discuss some results that were given in [14] for the periodic model discussed in §6.2 and then proceed to some new results of computational experiments for the two-dimensional boundary value model of §6.1. All the results presented and discussed here were obtained using quadratic finite element discretizations.

7.1. The periodic model

In [14], two sets of computational results relevant to our current considerations were given. We refer the reader to that paper for the figures of interest. In one of these figures, the maximum modulus of the order parameter was plotted vs. the applied field scaled by κ. Graphs for three different values of κ were provided. The graphs for $\kappa = 5$ and $\kappa = 20$ were indistinguishable, and this pair differed subtantially from the graph for $\kappa = 1$. Another pair of plots displayed the level lines of the magnitude of the order parameter. For both these plots the number of fluxoids per lattice cell was chosen to be unity, i.e., $n = 1$, and the lattice vectors were chosen to be those for an equilateral triangular lattice. One plot was for the choices $\kappa = 20$ and $\bar{B} = 2\pi/5$ and the other was for $\kappa = 5$ and $\bar{B} = \pi/10$. Note that in both cases $\bar{B} = (\pi/50)\kappa$. As a result of the fluxoid quantization condition (6.12), the linear dimension of a lattice cell in the first case was one-fourth of that for the second case. However, the level curves of the magnitude of the order parameter were, except for the uniform stretching in all directions, virtually identical in the two plots.

These results all support the contention that if the various inputs and outputs of the periodic Ginzburg-Landau model are properly scaled, then for "large" values of κ solutions of the Ginzburg-Landau model are largely independent of κ. This exact conclusion may be reached from the analyses of §6.2. Indeed, if $\bar{B} = \hat{B}\kappa$ with $\hat{B}$ independent of κ, as was asssumed in that section, then we have the leading order equations (6.13)-(6.17) which are independent of κ and also, we have, from Proposition 6.3, that to leading order, the applied field is given by $H_e^0 = \kappa\hat{H}_e^0$, where $\hat{H}_e^0$ is independent of κ. Thus, for "large" κ, we expect that the graphs of the maximum modulus of the order parameter vs. H_e/κ to approach the graph of $\max|\zeta_0|$ vs. $\hat{H}_e^0$. The first plot discussed above suggests that this is indeed true. Moreover, for "large" κ, we expect that the level lines of the magnitude of the order paramater to approach those for $|\zeta_0|$. The second set of plots discussed above are suggestive of this observation.

The results presented in [14] suggest that in practice "large" values of κ means $\kappa \approx 5$ or greater in the sense that already for $\kappa = 5$ the results of the periodic model are practically indistinguishable from the corresponding results for larger values of κ.

7.2. The boundary value model

We now present some computational results for the two-dimensional boundary value model of §6.1.

First, we examine the magnetic field in the superconductor for different values of κ and the applied field H_e. In particular, we look at the maximum and minimum values of the magnetic field. According to the discussion of §6.1, for large values of κ, we have that the magnetic field in the superconductor should be nearly constant and equal to the applied field. An examination of Table 1 shows that this is indeed the case for $\kappa = 20$ for which the computed values of H_{max} and H_{min} are both very nearly equal to H_e. Indeed, the difference between the the first two and the latter are well within the discretization error of the computational simulations. On the other hand, for $\kappa = 1.5$, we see a large discrepancy between H_{max} and H_e, and an even greater one between H_{min} and H_e; certainly, H is not a constant and not very close to H_e. We also see that even for the moderate value of $\kappa = 5$ that H_{max} is very nearly equal to H_e and that H_{min} is reasonably close to H_e as well. In fact, the results confirm the conclusion that the difference between H/κ and H_e/κ is of $O(\kappa^{-2})$.

κ	H_e/κ	H_{max}/κ	H_{min}/κ
1.5	.6	.549	.166
	.8	.740	.530
	.9	.880	.779
5	.5	.500	.480
	.7	.700	.687
	.9	.900	.893
20	.5	.500	.499
	.7	.700	.699
	.9	.900	.899

TABLE 1. *Maximum and minimum values of the magnetic field for dif-*

ferent values of κ and of the applied field H_e.

Figures 1-3 show the level lines of the magnitude of the order parameter for different values of the κ and the applied field H_e. In all cases the side of the square domain is ten times the coherence length. In Figure 1, $H_e/\kappa = 0.5$ (in nondimensional units) and the values of κ are as indicated on the individual plots. The plot labeled $\kappa = \infty$ corresponds to the solution obtained using the leading order equations (6.6)-(6.9); all the other plots were obtained using the full Ginzburg-Landau equations (6.1)-(6.4). It is evident from the plots that the there is very little difference between the results for the full Ginzburg-Landau equations for values of $\kappa \geq 5$, and that these are also indistinguishable from the results obtained using the leading order equations. The same can be said from an examination of Figures 2 and 3 which correspond to $H_e/\kappa = 0.7$ and $H_e/\kappa = 0.9$. All of these observations can be expected from the discussion of §6.1.

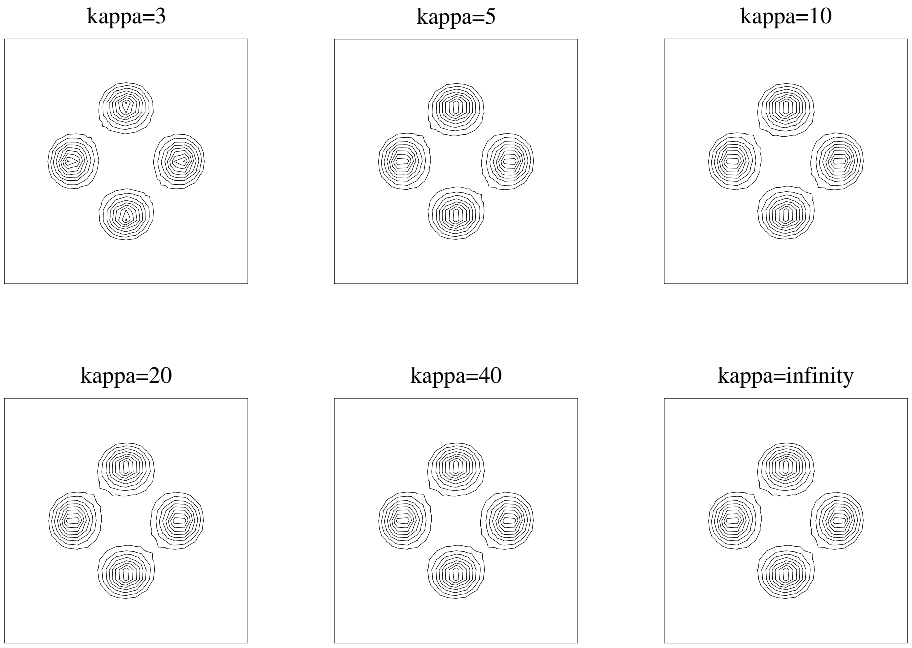

FIGURE 1. *Level curves of the magnitude of the order parameter for $H_e/\kappa = 0.5$ and different values of κ.*

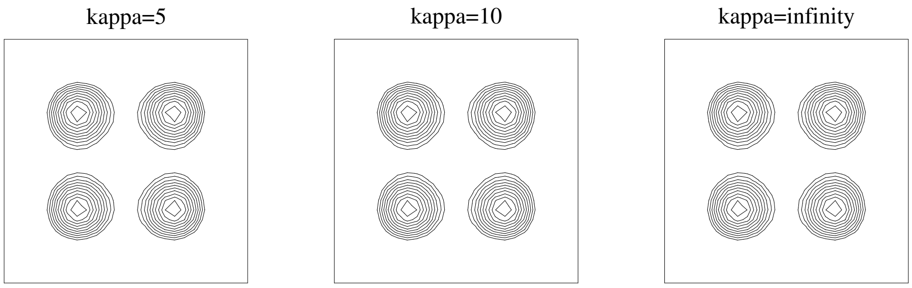

FIGURE 2. *Level curves of the magnitude of the order parameter for* $H_e/\kappa = 0.7$ *and different values of* κ.

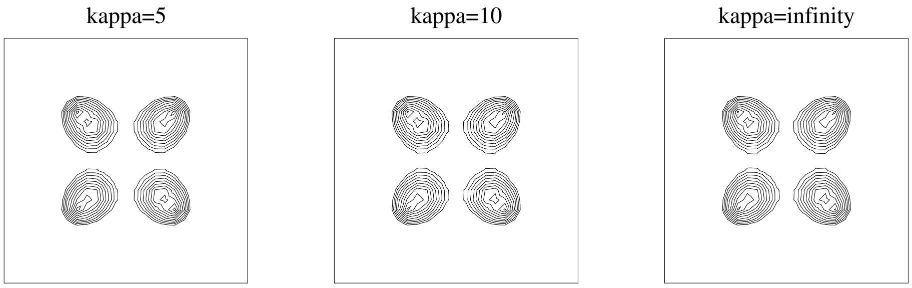

FIGURE 3. *Level curves of the magnitude of the order parameter for* $H_e/\kappa = 0.9$ *and different values of* κ.

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