Nonlinear modelling of breast tissue

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Abstract

Previous approaches to modelling the deformation of breast tissue include using linear elasticity and pseudo-nonlinear elasticity, in which the nonlinear deformation is broken up into a series of small linear isotropic deformations, with the (constant) Young's modulus of each linear deformation an exponential function of the total nonlinear strain.

Here these two approaches are compared to the solution of the full nonlinear elastic problem for tissue with an exponential relationship between stress and strain. Having formulated each model and related the coefficients between models, numerical simulations are performed on a block of incompressible material which demonstrate that the simpler models may not be appropriate when modelling deformations of the human breast under gravity.

1 Introduction

A tumour in a woman's breast is usually located using a clinical technique such as mammography (X-rays), Magnetic Resonance (MR) imaging, or ultrasound. When mammography is used, the woman stands upright and her breast is compressed between two plates. When MR imaging is used, the woman lies prone with her breasts hanging downwards into the machine used to detect the signal. For ultrasound, the woman sits upright with the breast free to move. To combine data from more than one imaging process, an accurate soft tissue model of the breast must be used to track the movement of internal tissue when the breast is moved to different positions and compressed. Should the woman require surgery, she will usually lie supine, with the breast taking yet another position. An accurate soft tissue model of the breast will make the location of the tumour in this new position a much more reliable procedure.

To develop an accurate soft tissue model of the breast, the stress–strain relationships for breast tissues must be known. The limited experimental data available (Wellman and Howe, 1998; Wellman, 1999) suggest that, for the large deformations of breast tissue that are likely to occur in the situations described above, the relationship between stress and strain is exponential, as is the case for some other biological tissues (Fung, 1993). The deformations will therefore be governed by the nonlinear equations of finite deformation elasticity theory (Malvern, 1969).

Classical linear elasticity theory was used by Schnabel *et al.* (2003) to model breast deformations. This model was used to simulate deformations of a breast whose shape was determined from an MR image. These simulated deformations were then compared with an interpolation method that also simulated deformations (Rueckert *et al.,* 1999). As no physical deformations of the breast were compared with the predictions of this model, the suitability of the linear elasticity model in this situation is unclear.

A different approach to modelling breast deformations was used by Azar *et al.* (2001, 2002). These authors modelled the whole deformation as consisting of a large number of small deformations. Classical linear elasticity theory was used to calculate each of these small deformations, with the Young's modulus being a function of the strain tensor at each step. This allows an exponential stress–strain relationship to be incorporated into the model, as described above. A similar approach was employed by Samani *et al.* (2001), who also allowed the Young's modulus of each tissue type to be a nonlinear function of strain.

In this study we investigate whether the simpler models described above are good approximations to the nonlinear equations governing incompressible finite deformation elasticity. This is achieved by comparing the governing equations for each model and performing numerical simulations using the simpler models with parameter values that have been used by other authors. The results of these simulations are then compared with the results of simulations that are calculated using finite deformation theory.

2 The models

We begin by describing the nonlinear finite deformation elasticity model. The governing equations for this model are approximated for small strains, and are then compared with the governing equations for classical linear elasticity theory. This section is concluded by describing the pseudo nonlinear model proposed by Azar *et al.* (2001, 2002) that attempts to adapt the theory of linear elasticity to model large strains.

2.1 The finite deformation model

2.1.1 Strain

Let $\mathbf{X}$ denote position in the undeformed body, and $\mathbf{x}$ denote position in the deformed body, so that the displacement is given by

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}.$$

We define the deformation gradient tensor $F = (F_{iM})$ by

$$F_{iM} = \frac{\partial x_i}{\partial X_M} \tag{1}$$

so that $\mathbf{dx} = F\mathbf{dX}$. A general deformation comprises both stretch and rigid body rotation; we may extract the stretch component by considering the Lagrange-Green strain tensor,

$$E = \frac{1}{2}\left(F^T F - I\right); \qquad E_{MN} = \frac{1}{2}\left(\sum_{i=1}^{3} F_{iM}F_{iN} - \delta_{MN}\right), \tag{2}$$

where δ_{MN} is the Kronecker delta, and I is the identity tensor. The tensor E is independent of any rigid body rotation, and is such that

$$\mathbf{dx}^2 - \mathbf{dX}^2 = 2\mathbf{dX}^T E \mathbf{dX}.$$

In terms of the displacement $\mathbf{u}$,

$$E_{MN} = \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M} + \sum_{P=1}^{3} \frac{\partial u_P}{\partial X_M}\frac{\partial u_P}{\partial X_N}\right).$$

Thus, if we linearise in $\mathbf{u}$ we find

$$E_{MN} = \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M}\right), \tag{3}$$

which is the usual expression for the strain tensor in linear elasticity.

2.1.2 Stress

We define $\sigma = (\sigma_{ij})$ to be the Cauchy stress tensor, i.e. the force measured per unit deformed area acting on the deformed body. Then applying Newton's law in the steady state the equations governing the deformation are (Malvern, 1969)

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j = 0 \qquad j = 1, 2, 3, \tag{4}$$

where ρ is the density of tissue, and g_j is the component of the gravitational force per unit mass in the X_j direction.

Typically, the boundary of the deformed body, $\partial\Omega$, must be partitioned into two non–intersecting sets. On the first set, $\partial\Omega^D$, there are Dirichlet displacement boundary conditions. On the second set, $\partial\Omega^{ND}$, there are non–Dirichlet traction boundary conditions. These boundary conditions are

$$x_j = x_{0_j} \qquad j = 1, 2, 3 \qquad \text{on } \partial\Omega^D, \tag{5}$$

$$\sum_{i=1}^{3} \sigma_{ij} n_i = s_j \qquad j = 1, 2, 3 \qquad \text{on } \partial\Omega^{ND}, \tag{6}$$

where $(x_{0_1}, x_{0_2}, x_{0_3})$ are the displacement boundary conditions, (n_1, n_2, n_3) is the outward pointing unit normal vector to the deformed body and (s_1, s_2, s_3) is the force per unit deformed area acting on the deformed body.

Equation (4) expresses the deformation with the coordinates of the deformed body as the independent variables. It is often more convenient to use a Lagrangian formulation, with the coordinates of the undeformed body as the independent variables, and the coordinates of the deformed body (or equivalently the displacement) as the dependent variables. In this case the natural stress tensor to use is $T = (T_{MN})$, the second Piola-Kirchoff stress tensor, which is the force per unit undeformed area acting on the undeformed body, and which is related to the Cauchy stress tensor by (Malvern, 1969)

$$\sigma = \frac{1}{\det F} F T F^T. \tag{7}$$

The governing equations for this formulation are (Malvern, 1969)

$$\sum_{M,N=1}^{3} \frac{\partial}{\partial X_M} \left(T_{MN} \frac{\partial x_j}{\partial X_N} \right) + \rho g_j = 0 \qquad j = 1, 2, 3. \tag{8}$$

Finally, (4) or (8) are closed by adding a constitutive relationship between stress and deformation. Typically this is accomplished by introducing a strain energy function, W, which is a function of F, and defining (Green & Adkins, 1970)

$$T_{MN} = \frac{1}{2} \left(\frac{\partial W}{\partial E_{MN}} + \frac{\partial W}{\partial E_{NM}} \right). \tag{9}$$

2.1.3 The strain energy function

For an isotropic material W must be independent of rigid body rotations and the particular choice of coordinates, and must therefore be a function only of the three invariants of the tensor $F^T F$,

$$\begin{aligned} I_1 &= \operatorname{trace}(F^T F), \\ I_2 &= \frac{1}{2} \left((\operatorname{trace}(F^T F))^2 - \operatorname{trace}((F^T F)^2) \right), \\ I_3 &= \det(F^T F). \end{aligned}$$

A typical strain energy function for biological tissue is given by

$$W = a\left(\mathrm{e}^{b(I_1-3)} - 1\right) - \frac{1}{2} f(I_3 - 1), \tag{10}$$

where $a, b > 0$ are constants and the function f is a measure of the compressibility of the material. In terms of E, the invariants I_1 and I_3 are given by

$$\begin{aligned} I_1 &= 3 + 2(E_{11} + E_{22} + E_{33}), && (11) \\ I_3 &= 1 + 2(E_{11} + E_{22} + E_{33}) + \\ & \quad 4(E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{23}^2 - E_{31}^2) + \\ & \quad 8(E_{11}E_{22}E_{33} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2 + 2E_{12}E_{23}E_{31}). && (12) \end{aligned}$$

We note that when using the strain energy function given by (10) and the definition of T_{MN} given by (9) we have

$$T_{MN} = 2(ab - f'(0))\delta_{MN} + \mathcal{O}(E_{PQ}) \qquad \text{as} \qquad E_{PQ} \to 0. \tag{13}$$

Thus if the stress in the undeformed state is zero, we must have $f'(0) = ab$.

With stress defined as in (9), equation (8) is the Euler-Lagrange equation associated with the minimisation of the strain and potential energy of the material,

$$\int_{\Omega_0} (W - \rho\mathbf{g} \cdot \mathbf{x})\, dV_0. \tag{14}$$

To see this we note that under a small perturbation $\mathbf{x} \to \mathbf{x} + \mathbf{U}$ the strain tensor becomes

$$E_{MN} \to E_{MN} + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial U_i}{\partial X_M}\frac{\partial x_i}{\partial X_N} + \frac{\partial U_i}{\partial X_N}\frac{\partial x_i}{\partial X_M}\right),$$

so that

$$\begin{aligned} \int_{\Omega_0} W\, dV_0 \to \int_{\Omega_0} W\, dV_0 + \int_{\Omega_0} \frac{1}{2}\sum_{M,N=1}^{3}\frac{\partial W}{\partial E_{MN}}\sum_{i=1}^{3}\left(\frac{\partial U_i}{\partial X_M}\frac{\partial x_i}{\partial X_N} + \frac{\partial U_i}{\partial X_N}\frac{\partial x_i}{\partial X_M}\right) dV_0 \\ = \int_{\Omega_0} W\, dV_0 - \int_{\Omega_0} \frac{1}{2}\sum_{i=1}^{3}\left[\sum_{M,N=1}^{3}\frac{\partial}{\partial X_M}\left(\frac{\partial W}{\partial E_{MN}}\frac{\partial x_i}{\partial X_N}\right) + \frac{\partial}{\partial X_N}\left(\frac{\partial W}{\partial E_{MN}}\frac{\partial x_i}{\partial X_M}\right)\right] U_i\, dV_0, \end{aligned}$$

on integrating by parts. Then, switching the labels M and N in the second term, the Euler-Lagrange equations associated with minimising the energy (14) are

$$\sum_{M,N=1}^{3}\frac{\partial}{\partial X_M}\left[\frac{1}{2}\left(\frac{\partial W}{\partial E_{MN}} + \frac{\partial W}{\partial E_{NM}}\right)\frac{\partial x_i}{\partial X_N}\right] + \rho g_i = 0,$$

which are equivalent to (8) with T_{MN} given by (9).

2.1.4 Incompressibility

Often it is a good approximation to assume that biological materials are incompressible. We can enforce incompressibility by imposing the constraint

$$\det F = 1 \tag{15}$$

on the deformation. However, we have now introduced a fourth equation, so we must introduce another unknown if we are to retain a well-posed mathematical system. This extra term is an isotropic internal force, often referred to as a pressure, so that an extra term $-p(F^T F)^{-1}$ must be incorporated into the second Piola-Kirchoff stress tensor (Spencer 1980):

$$T = \frac{1}{2}\left(\frac{\partial W}{\partial E_{MN}} + \frac{\partial W}{\partial E_{NM}}\right) - p(F^T F)^{-1}. \tag{16}$$

The pressure can be thought of as a Lagrange multiplier associated with the constraint $I_3 = \det F^T F = 1$. Suppose we try to minimise the energy under this constraint. Then

$$\min_{\det F^T F=1} \int_{\Omega_0} (W - \rho \mathbf{g} \cdot \mathbf{x})\, dV_0 = \min \int_{\Omega_0} \left(W - \rho \mathbf{g} \cdot \mathbf{x} - \frac{1}{2} p(I_3 - 1)\right) dV_0, \tag{17}$$

which gives the Euler-Lagrange equations as

$$\sum_{M,N=1}^{3} \frac{\partial}{\partial X_M}\left[\frac{1}{2}\left(\frac{\partial W}{\partial E_{MN}} + \frac{\partial W}{\partial E_{NM}} - \frac{p}{2}\left(\frac{\partial I_3}{\partial E_{MN}} + \frac{\partial I_3}{\partial E_{NM}}\right)\right)\frac{\partial x_i}{\partial X_N}\right] + \rho g_i = 0.$$

Since

$$\frac{\partial I_3}{\partial E} = \frac{\partial I_3}{\partial E^T} = 2(F^T F)^{-1},$$

(using (15)) we have

$$\sum_{M,N=1}^{3} \frac{\partial}{\partial X_M}\left[\left(\frac{1}{2}\left(\frac{\partial W}{\partial E_{MN}} + \frac{\partial W}{\partial E_{NM}}\right) - p(F^T F)^{-1}\right)\frac{\partial x_i}{\partial X_N}\right] + \rho g_i = 0,$$

which is equivalent to (8) with the new stress tensor (16). We note that from (7) the corresponding addition to the Cauchy stress tensor is simply $-pI$.

2.2 Classical linear elasticity

We may compare the theory of linear and nonlinear elasticity by taking the lowest order terms in (10) and comparing this approximate strain energy function to the strain energy function that arises in classical linear elasticity theory. It is unusual in linear elasticity to enforce incompressibility, although some authors (Azar *et al.*, 2001, 2002; Schnabel *et al.*, 2003; Zhang *et al.*, 1997) have enforced near incompressibility by setting the Poisson ratio to $0.5 - \epsilon$, with ϵ small and positive. Thus we begin by comparing the compressible finite deformation model described earlier to the classical linear elasticity model. We will then formulate the model for incompressible linear elasticity, and compare it to the incompressible finite deformation model.

2.2.1 Comparing the nonlinear model with classical linear elasticity

The strain energy function, W_{lin}, for classical linear elasticity with Lamé constants λ and μ is (Malvern, 1969)

$$\begin{aligned} W_{\text{lin}} &= \frac{1}{2}(\lambda+2\mu)\left(E_{11}^2+E_{22}^2+E_{33}^2\right)+2\mu\left(E_{12}^2+E_{23}^2+E_{31}^2\right) \\ &\quad +\lambda\left(E_{11}E_{22}+E_{22}E_{33}+E_{33}E_{11}\right). \end{aligned} \tag{18}$$

To compare the finite deformation formulation with the classical linear model for low strains, we take the lower order terms in the strain energy function for the finite deformation formulation given in (10), remembering that $f'(0)=ab$, to give

$$\begin{aligned} W &= 2(ab^2+f''(0))(E_{11}^2+E_{22}^2+E_{33}^2)+4ab(E_{12}^2+E_{23}^2+E_{31}^2) \\ &\quad +4(ab^2-ab+f''(0))(E_{11}E_{22}+E_{22}E_{33}+E_{33}E_{11})+O(E^3). \end{aligned} \tag{19}$$

Comparing equations (18) and (19) we see that (19) is a strain energy function corresponding to Lamé constants

$$\lambda = 4(ab(b-1)+f''(0)), \quad \mu = 2ab, \tag{20}$$

which correspond to Young's modulus, E_Y, and Poisson ratio, ν, given by

$$E_Y=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}=\frac{4ab(ab(3b-2)+3f''(0))}{ab(2b-1)+2f''(0)}, \quad \nu=\frac{\lambda}{2(\lambda+\mu)}=\frac{ab(b-1)+f''(0)}{ab(2b-1)+2f''(0)}. \tag{21}$$

Note that as we make the material less compressible, so that $f''(0)\to\infty$, then $\lambda\to\infty$ and $\nu\to 1/2$ as expected.

Equation (21) allows us to interchange between the parameter values used in the strain energy function given in (10) and the Young's modulus and Poisson ratio of the linearised form.

Using the definition (9) with the strain energy (18) gives

$$T_{MN}=2\mu E_{MN}+\lambda\,\text{trace}(E)\delta_{MN}.$$

If we substitute into equation (8) and linearise in the displacement we are left with the familiar equations of classical linear elasticity (see, for example, Landau & Lifshitz, 1986)

$$\sum_{M=1}^{3}\frac{\partial}{\partial X_M}\left(\mu\left(\frac{\partial u_M}{\partial X_N}+\frac{\partial u_N}{\partial X_M}\right)+\lambda\delta_{MN}\sum_{i=1}^{3}\frac{\partial u_i}{\partial X_i}\right)+\rho g_N = 0 \quad N=1,2,3, \tag{22}$$

or

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\rho\mathbf{g}=0. \tag{23}$$

2.2.2 Incorporating incompressibility into classical linear elasticity

The linearised version of the incompressibility condition (15) is

$$\text{trace}(E) = E_{11} + E_{22} + E_{33} = 0,$$

or equivalently, in terms of displacement,

$$\nabla \cdot \mathbf{u} = 0. \tag{24}$$

A linear elastic material becomes incompressible in the limit that $\lambda \to \infty$. In this limit, (23) gives that $\nabla \cdot \mathbf{u} \to 0$ in such a way that $\lambda \nabla \cdot \mathbf{u}$ tends to a finite value, which we write as $-P$. This function is unknown, and is determined by the constraint (24). Thus the equations of incompressible linear elasticity are

$$\mu \nabla^2 \mathbf{u} - \nabla P + \rho \mathbf{g} = 0, \tag{25}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{26}$$

The linear elastic energy (18) may be rewritten

$$W_{\text{lin}} = \frac{\lambda}{2} (E_{11} + E_{22} + E_{33})^2 + \mu \sum_{i,j=1}^{3} E_{ij}^2. \tag{27}$$

The function P can again be thought of as a Lagrange multiplier associated with minimising the energy subject to the constraint (24). If the material is incompressible (27) becomes

$$W_{\text{lin}} = \mu \sum_{i,j=1}^{3} E_{ij}^2,$$

and when we minimise and introduce a Lagrange multiplier we have

$$\min_{\nabla \cdot \mathbf{u} = 0} \int_{\Omega_0} (W_{\text{lin}} - \rho \mathbf{g} \cdot \mathbf{u}) \, dV = \min \int_{\Omega_0} (W_{\text{lin}} - \rho \mathbf{g} \cdot \mathbf{u} - P \nabla \cdot \mathbf{u}) \, dV, \tag{28}$$

and the Euler-Lagrange equations are (25).

2.2.3 Comparing the incompressible nonlinear model with incompressible linear elasticity

The easiest way to compare the linear and nonlinear models is to take the limit of small strain in the nonlinear energy including the Lagrange multiplier (17) and compare it to the corresponding linear energy (28). We find

$$\begin{aligned} W &= (2ab - p)(E_{11} + E_{22} + E_{33}) + 2ab^2(E_{11}^2 + E_{22}^2 + E_{33}^2) + \\ &\quad 2(2ab^2 - p)(E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}) + 2p(E_{12}^2 + E_{23}^2 + E_{31}^2) + O(E^3). \end{aligned}$$

Thus we see that the pressures are related according to

$$P = p - 2ab. \tag{29}$$

This means that in the nonlinear model, in the strain-free stress-free state, $p = 2ab$: the pressure p is not the only isotropic force in this model, there is also a contribution from the first term in (10). Substituting for p using (29) and retaining only quadratic terms in P and E gives

$$\begin{aligned} W &= 2ab^2(E_{11}^2 + E_{22}^2 + E_{33}^2) + 4ab(E_{12}^2 + E_{23}^2 + E_{31}^2) \\ &\quad + 4ab(b-1)(E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}) - P(E_{11} + E_{22} + E_{33}) \\ &= 2ab(b-1)(E_{11} + E_{22} + E_{33})^2 + 2ab\sum_{i,j=1}^{3} E_{ij}^2 - P(E_{11} + E_{22} + E_{33}). \end{aligned}$$

Thus, as before, $\mu = 2ab$.

2.3 Pseudo nonlinear models

In the model used by Azar *et al.* (2001, 2002) the compression of a breast between two plates (without gravity) is modelled by dividing the whole displacement into a sequence of small displacements. Classical linear elasticity is used to calculate each small displacement, with the Young's modulus being an exponential function of the strain before the current displacement is imposed. More precisely, the Young's modulus, E_Y, is defined as the gradient of the stress–strain curve for uniaxial tension, and an exponential relationship is assumed, so that

$$E_Y = \frac{\mathrm{d}T}{\mathrm{d}E} = c_1 \mathrm{e}^{c_2 E}, \tag{30}$$

where T is the stress, E is the strain, and c_1, c_2 are constants. For more general deformations (30) is used with E defined as the largest eigenvalue of the (nonlinear) strain tensor (again, evaluated before the current displacement is imposed). Azar *et al.* take the material to be incompressible, but in fact impose this by modelling a compressible material with a Poisson ratio $\nu = 0.49999$. We can also consider an incompressible version of the model by introducing the constraint $\nabla \cdot \mathbf{u} = 0$ for each small displacement and the associated Lagrange multiplier p. In this limit the Lamé constant $\mu = E_Y/3$.

To apply this pseudo nonlinear model to the problem of deformation due to gravity we need to divide the deformation due to gravity into a sequence of small deformations. We can do this by imagining increasing the gravitational constant g from zero to its true value in a sequence of small steps. The pseudo nonlinear model corresponds to linearising the nonlinear model about a state of finite strain to calculate the next displacement. Since the equations are solved with respect to displacements from the current deformation it is easiest to work in terms of the Eulerian formulation (4) when comparing with the full nonlinear model. If we increase the strength of gravity by a small amount by setting

$$g = g_0 + \epsilon g_1$$

and expand all variables in powers of ϵ as

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \epsilon \sigma_{ij}^{(1)} + \cdots, \tag{31}$$

$$E_{ij} = E_{ij}^{(0)} + \epsilon E_{ij}^{(1)} + \cdots, \tag{32}$$

$$X_j = X_j^{(0)} + \epsilon X_j^{(1)} + \cdots \tag{33}$$

etc., then the pseudo nonlinear model corresponds to writing down isotropic linear elasticity for the correction terms $\sigma^{(1)}$ etc., with a Young's modulus which is a function of $E^{(0)}$. We can compare this approach with the actual equations which would result from substituting the expansions (31)-(33) into equation the full nonlinear problem. In the incompressible case, (4) becomes

$$\nabla \cdot \sigma^{(1)} + \rho g^{(1)} = 0, \tag{34}$$

which simply represents a balance of forces at order ϵ. The difficulty is in relating the correction to the stress $\sigma^{(1)}$ to the correction to the strain $E^{(1)}$. If we label the increment in the displacement $X_j^{(1)}$ by u_j as would be normal in linear elasticity, then the perturbation to the deformation tensor is

$$F^{(1)} = \frac{\partial u_i}{\partial X_M} = \sum_j \frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial X_M} = DF^{(0)}, \tag{35}$$

where $D = (D_{ij})$ is the deformation tensor for linear elasticity,

$$D_{ij} = \frac{\partial u_i}{\partial x_j}.$$

The easiest way to proceed is to write

$$\sigma = FTF^T - pI,$$

where in a slight abuse of notation T is the second Piola-Kirchoff stress tensor minus the contribution from the pressure. Then the first correction to the stress tensor satisfies

$$\begin{aligned}
\sigma^{(1)} &= F^{(1)}T^{(0)}F^{(0)T} + F^{(0)}T^{(0)}F^{(1)T} + F^{(0)}T^{(1)}F^{(0)T} - p^{(1)}I \\
&= DF^{(0)}T^{(0)}F^{(0)T} + F^{(0)}T^{(0)}F^{(0)T}D^T + F^{(0)}T^{(1)}F^{(0)T} - p^{(1)}I \\
&= D\sigma^{(0)} + \sigma^{(0)}D^T + p^{(0)}(D + D^T) + F^{(0)}T^{(1)}F^{(0)T} - p^{(1)}I.
\end{aligned} \tag{36}$$

¿From (2) the first correction to the Lagrange-Green strain tensor is

$$E^{(1)} = \frac{1}{2}\left(F^{(0)T}F^{(1)} + F^{(1)T}F^{(0)}\right) = \frac{1}{2}F^{(0)T}\left(D + D^T\right)F^{(0)}. \tag{37}$$

Finally, from (9), the correction to T is

$$T_{MN}^{(1)} = \frac{1}{2}\sum_{K,L}\left(\frac{\partial^2 W}{\partial E_{MN}\partial E_{KL}} + \frac{\partial^2 W}{\partial E_{NM}\partial E_{KL}}\right)E_{KL}^{(1)}. \tag{38}$$

Now, if $W = W(I_1)$ (as in the incompressible version of (10)), then

$$\frac{\partial W}{\partial E_{MN}} = 2W'(I_1)\delta_{MN}, \qquad \frac{\partial^2 W}{\partial E_{MN}\partial E_{KL}} = 4W''(I_1)\delta_{MN}\delta_{KL}.$$

In this case

$$T^{(1)}_{MN} = 4W''(I_1)\delta_{MN} \sum_K E^{(1)}_{KK}. \tag{39}$$

Combining (36), (37) and (39) gives $\sigma^{(1)}$ as a linear function of D, which completes the specification of the linear elastic problem. The pseudo nonlinear model assumes that this relationship is isotropic, i.e. that it is of the form

$$\sigma^{(1)} = \mu(D + D^T).$$

Clearly this will not be the case in general. In Section 3 we examine the numerically the effect of this simplification on a test problem.

To compare the full nonlinear problem with the pseudo nonlinear model we need to relate the parameters c_1 and c_2 in (30) to the parameters a and b in (10). To do this let us consider the case of uniaxial stress, in which the pseudo nonlinear model should be a good approximation to the full nonlinear model.

We consider first incompressible linear elasticity in uniaxial stress, for which the displacement is $\mathbf{u} = (u_1(x_1), u_2(x_2), u_3(x_3))$, so that

$$E = D = \begin{pmatrix} \frac{du_1}{dx_1} & 0 & 0 \\ 0 & \frac{du_2}{dx_2} & 0 \\ 0 & 0 & \frac{du_3}{dx_3} \end{pmatrix}, \qquad \sigma = 2\mu \begin{pmatrix} \frac{du_1}{dx_1} & 0 & 0 \\ 0 & \frac{du_2}{dx_2} & 0 \\ 0 & 0 & \frac{du_3}{dx_3} \end{pmatrix} - p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\sigma_{22} = \sigma_{33} = 0$ implies

$$p = 2\mu \frac{du_2}{dx_2} = 2\mu \frac{du_3}{dx_3}.$$

Then the condition of incompressibility, div $\mathbf{u} = 0$, gives

$$\frac{du_2}{dx_2} = \frac{du_3}{dx_3} = -\frac{1}{2}\frac{du_1}{dx_1} = \text{constant},$$

and so

$$\sigma_{11} = 3\mu \frac{du_1}{dx_1} = E_Y E_{11},$$

since the Young's modulus $E_Y = 3\mu$. The generalisation to nonlinear elasticity is as follows. The displacement is $\mathbf{x} = (x_1(X_1), x_2(X_2), x_3(X_3))$, so that[1]

$$F = \begin{pmatrix} x_1' & 0 & 0 \\ 0 & x_2' & 0 \\ 0 & 0 & x_3' \end{pmatrix}, \qquad E = \frac{1}{2} \begin{pmatrix} (x_1')^2 - 1 & 0 & 0 \\ 0 & (x_2')^2 - 1 & 0 \\ 0 & 0 & (x_3')^2 - 1 \end{pmatrix},$$

and the condition of incompressibility is

$$x_1' x_2' x_3' = 1. \tag{40}$$

[1]To ease the notation we use a prime to denote differentiation of a function with respect to its argument. Thus $'$ may represent d/dX_1, d/dX_2 or d/dX_3 depending on whether it is attached to x_1, x_2 or x_3. Since each is a function of one variable only there is no ambiguity.

The Piola-Kirchoff stress tensor is

$$T = 2ab\exp\left[b\left((x_1')^2+(x_2')^2+(x_3')^2-3\right)\right]\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}\frac{p}{\left(x_1'\right)^2}&0&0\\0&\frac{p}{\left(x_2'\right)^2}&0\\0&0&\frac{p}{\left(x_3'\right)^2}\end{pmatrix}$$

Then $T_{22}=T_{33}=0$ implies

$$\begin{aligned}p &= (x_2')^2\,2ab\exp\left[b\left((x_1')^2+(x_2')^2+(x_3')^2-3\right)\right]\\ &= (x_3')^2\,2ab\exp\left[b\left((x_1')^2+(x_2')^2+(x_3')^2-3\right)\right].\end{aligned}$$

Incompressibility then implies

$$x_2' = x_3' = (x_1')^{-1/2} = \text{constant},$$

so that

$$\begin{aligned}p &= \frac{2ab}{x_1'}\exp\left[b\left((x_1')^2+\frac{2}{x_1'}-3\right)\right],\\ T_{11} &= 2ab\left(1-\frac{1}{(x_1')^3}\right)\exp\left[b\left((x_1')^2+\frac{2}{x_1'}-3\right)\right],\\ \sigma_{11} &= 2ab\left((x_1')^2-\frac{1}{x_1'}\right)\exp\left[b\left((x_1')^2+\frac{2}{x_1'}-3\right)\right]\end{aligned}$$

Let us examine the linearisation in the case of uniaxial tension. We find

$$E^{(1)} = \begin{pmatrix}(x_1')^2\frac{du_1}{dx_1}&0&0\\0&(x_2')^2\frac{du_2}{dx_2}&0\\0&0&(x_3')^2\frac{du_3}{dx_3}\end{pmatrix},$$

$$T^{(1)} = T^*I,$$

where

$$\begin{aligned}T^* &= 4ab^2\exp\left[b\left((x_1')^2+\frac{2}{x_1'}-3\right)\right]\left((x_1')^2\frac{du_1}{dx_1}+(x_2')^2\frac{du_2}{dx_2}+(x_3')^2\frac{du_3}{dx_3}\right)\\ &= 4ab^2\exp\left[b\left((x_1')^2+\frac{2}{x_1'}-3\right)\right]\frac{du_1}{dx_1}\left((x_1')^2-\frac{1}{x_1'}\right)\\ &= 2b\sigma_{11}^{(0)}\frac{du_1}{dx_1}.\end{aligned}$$

We find

$$\begin{aligned}\sigma^{(1)} &= \begin{pmatrix}2\frac{du_1}{dx_1}\sigma_{11}^{(0)}&0&0\\0&0&0\\0&0&0\end{pmatrix}+2p^{(0)}\begin{pmatrix}\frac{du_1}{dx}&0&0\\0&-\frac{1}{2}\frac{du_1}{dx}&0\\0&0&-\frac{1}{2}\frac{du_1}{dx}\end{pmatrix}\\ &\quad+T^*\begin{pmatrix}(x_1')^2&0&0\\0&1/x_1'&0\\0&0&1/x_1'\end{pmatrix}-p^{(1)}I.\end{aligned}$$

Therefore

$$p^{(1)} = \frac{T^*}{x_1} - p^{(0)}\frac{du_1}{dx_1},$$

giving

$$\begin{aligned}
\sigma_{11}^{(1)} &= \left(2\sigma_{11}^{(0)} + 3p^{(0)}\right)\frac{du_1}{dx_1} + T^*\left((x_1')^2 - \frac{1}{x_1'}\right) \\
&= \frac{du_1}{dx_1}\left(2\sigma_{11}^{(0)} + 3p^{(0)} + 2b\sigma_{11}^{(0)}\left((x_1')^2 - \frac{1}{x_1'}\right)\right) \\
&= \frac{du_1}{dx_1}2ab\exp\left[b\left((x_1')^2 + \frac{2}{x_1'} - 3\right)\right]\left(2\left((x_1')^2 - \frac{1}{x_1'}\right) + 2b\left((x_1')^2 - \frac{1}{x_1'}\right)^2 + \frac{3}{x_1'}\right).
\end{aligned}$$

Hence the Young's modulus E_Y is

$$E_Y = 2ab\exp\left[b\left((x_1')^2 + \frac{2}{x_1'} - 3\right)\right]\left(2\left((x_1')^2 - \frac{1}{x_1'}\right) + 2b\left((x_1')^2 - \frac{1}{x_1'}\right)^2 + \frac{3}{x_1'}\right). \quad (41)$$

Note that

$$\frac{d\sigma_{11}}{dE_{11}} = \frac{2ab}{(x_1')^2}\exp\left[b\left((x_1')^2 + \frac{2}{x_1'} - 3\right)\right]\left(2\left((x_1')^2 - \frac{1}{x_1'}\right) + 2b\left((x_1')^2 - \frac{1}{x_1'}\right)^2 + \frac{3}{x_1'}\right).$$

The difference is due to the fact that

$$E_{11}^{(1)} = (x_1')^2\frac{du_1}{dx_1},$$

so that the increment in the Lagrange-Green strain is not just du_1/dx_1.

Since $x_1' = (2E_{11}+1)^{1/2}$, in terms of the strain E_{11} (41) is

$$\begin{aligned}
E_Y &= 2ab\exp\left[b\left(2E_{11} + \frac{2}{(2E_{11}+1)^{1/2}} - 2\right)\right] \times \\
&\quad \left(4E_{11} + 2 + 2b\left(2E_{11} + 1 - \frac{1}{(2E_{11}+1)^{1/2}}\right)^2 + \frac{1}{(2E_{11}+1)^{1/2}}\right). \quad (42)
\end{aligned}$$

The Young's modulus (42) is not of the form (30), but has a similar behaviour. We estimate the constants a and b in terms of c_1 and c_2 given by Azar *et al.* (2002) by matching the principle part of the exponent. This requires setting

$$b = \frac{1}{2}c_2. \quad (43)$$

We then choose a so that the linear behaviour is the same in both models. For small strains $E_Y = 6ab = c_1$, giving

$$a = \frac{c_1}{3c_2}. \quad (44)$$

3 Numerical simulations

3.1 Description of simulations

In this section we perform numerical simulations to investigate the theoretical results of the previous section. In the absence of a function f that may be used in (10) to use as a measure of the compressibility of the body we restrict ourselves to comparing the incompressible models described earlier.

We have described two simplifications to the incompressible finite elasticity model: (i) incompressible linear elasticity; and (ii) incompressible pseudo nonlinear elasticity. Our simulations compare both of these simplifications to the incompressible finite elasticity model, using parameters that have been used by other authors. We simulate fatty tissue because this is one of the main tissue types in the breast, and parameter values for this tissue are readily available. In all simulations we use the cylindrical domain

$$X^2 + Y^2 < R^2, \qquad 0 < Z < Z_{\mathrm{max}},$$

where R and Z_{max} are constants. We apply zero displacement boundary conditions on the face $Z = 0$, and stress–free boundary conditions elsewhere. In our simulations we take $R = Z_{\mathrm{max}} = 0.1$ m, so that the domain is similar in shape and size to a human breast. We simulate a woman lying, both prone and supine, and a woman standing up. For the woman lying prone we have $g_1 = g_2 = 0$, $g_3 = g$, where g is the magnitude of the gravitational force per unit mass. For the woman lying supine we have $g_1 = g_2 = 0$, $g_3 = -g$, and for the woman standing, we have $g_1 = g$, $g_2 = g_3 = 0$. In all simulations we take $g = 9.8$ m/s^2 and $\rho = 940$ kg/m^3.

We use the parameters $c_1 = 4460$ N/m^2 and $c_2 = 7.4$ that were used in the calculation of the Young's modulus (30) by Azar *et al.* (2002). These may be related to the constants a and b in (10) by using (43) and (44). These parameters may then be related to the Lamé coefficient μ using (20), and to our approximation of the pseudo nonlinear model using (42).

When comparing one of the simplified models listed above with finite deformation elasticity theory, we simulate displacements using both models. We denote the coordinates of the deformed body predicted by the finite deformation calculation by $\mathbf{x_1}(\mathbf{X})$, and the coordinates of the model that it is being compared to by $\mathbf{x_2}(\mathbf{X})$. To compare the models, for each simulation we calculate the following quantities:

1. V_{frac}, the ratio of the volume of the body defined by $\mathbf{x_2}(\mathbf{X})$ to the volume of the body defined by $\mathbf{x_1}(\mathbf{X})$;

2. u_{max}, the maximum displacement calculated using the finite deformation calculation;

3. $D_{\mathrm{max}} = \max |\mathbf{x_1}(\mathbf{X}) - \mathbf{x_2}(\mathbf{X})|$;

4. the volume–averaged difference between the solutions given by

$$D_{\mathrm{ave}} = \frac{\int_{\Omega_0} |\mathbf{x_1}(\mathbf{X}) - \mathbf{x_2}(\mathbf{X})| \, \mathrm{d}V}{\int_{\Omega_0} \mathrm{d}V} \tag{45}$$

where Ω_0 is the volume occupied by the undeformed body; and

5. E_1, the maximum eigenvalue of the strain tensor calculated from $\mathbf{x_1}$, and E_2, the maximum eigenvalue of the strain tensor calculated from $\mathbf{x_2}$.

3.2 Numerical techniques

The governing equations for each model were solved using the finite element method: see, for example, Reddy (1993). The three–dimensional volume was discretised into elements with eight nodes, and a trilinear approximation was used for the dependent spatial variables in each of these elements. For a stable approximation to the pressure as element size is reduced, a lower order approximation must be used to calculate the pressure (in the calculations in which pressure appears) than is used to calculate the displacements (Reddy, 1993). We therefore use a piecewise constant approximation on each element to calculate pressure. The nonlinear equations arising in the finite deformation calculations were solved using Newton's method (see, for example, Atkinson, 1989) with damping.

3.3 Results of the simulations

In Fig. 1 we plot the simulations described in Section 3.1. The subplots in Fig. 1 are as follows: (a) finite deformation, $\mathbf{g} = (0, 0, g)^t$; (b) linear elasticity, $\mathbf{g} = (0, 0, g)^t$; (c) pseudo nonlinear elasticity, $\mathbf{g} = (0, 0, g)^t$; (d) finite deformation, $\mathbf{g} = (0, 0, -g)^t$; (e) linear elasticity, $\mathbf{g} = (0, 0, -g)^t$; (f) pseudo nonlinear elasticity, $\mathbf{g} = (0, 0, -g)^t$; (g) finite deformation, $\mathbf{g} = (g, 0, 0)^t$; (h) linear elasticity, $\mathbf{g} = (g, 0, 0)^t$; (i) pseudo nonlinear elasticity, $\mathbf{g} = (g, 0, 0)^t$. The values of V_{frac}, u_{max}, D_{max}, D_{ave}, E_1 and E_2 described in Section 3.1 are listed in Table 1.

We see in Table 1 that, although incompressibility is enforced when simulating the pseudo nonlinear model, the linear incompressible model does not always enforce incompressibility exactly. This is because the incompressibility constraint in this model neglects quadratic and higher order terms in the displacements. In the simulations presented here these terms are beginning to have an effect on the total volume. For the pseudo nonlinear model the size of the displacement calculated on each increment is much smaller than the displacement calculated using the linear model, and so incompressibility is enforced more strictly.

We can see, by comparing the values of D_{max} and u_{max}, that the pseudo nonlinear model is a better approximation to the finite deformation model than the linear model. For the comparison between the linear and finite deformation models the value of $D_{\text{max}}/u_{\text{max}}$ varies between 0.14 and 0.45 for the simulations carried out here: for the comparison between the pseudo nonlinear and finite deformation models $D_{\text{max}}/u_{\text{max}}$ varies between 0.043 and 0.079. There are also significant errors in the calculation of the maximum eigenvalue of the strain tensor using the linear model, most notably for the simulation with gravity in the X-direction.

The discrepancies between the linear model and the finite deformation model may be attributed to the displacements computed for the linear model being too large for linear elasticity to be valid. The discrepancies between the pseudo nonlinear model and the finite deformation model are due mainly to the assumption of isotropy in each increment, even in later increments where the body is stressed, and clearly will not be truly isotropic.

4 Discussion

We have described the finite deformation elasticity model for biological tissue, and compared it to simpler models that are based on classical linear and pseudo-nonlinear elasticity theory. For the special case of incompressible tissue we have performed numerical simulations to illustrate the differences between the models, using parameter values that have been used for fatty tissues by other authors.

It is perhaps not surprising that linear elasticity does not give good results (although this has not stopped it being used to model breast deformations in the past). An order of magnitude estimate for the strain in linear theory is given by

$$\frac{\rho g L}{\mu},$$

where L is the height of the breast (taken to be 0.1m). Using the values $\rho = 940\text{kg/m}^3$ and $\mu = 2230\text{N/m}^2$ this gives a typical strain of about 0.4, which is outside the linear range of most materials.

More surprising is the significant error in the pseudo-nonlinear model. Azar *et al.* (2001, 2002) suggest that the error in this model is due to an accumulation of discretisation error, as the large deformation is modelled as a series of small linear deformations. However, more significant is that fact that while each small deformation can accurately be modelled as a linear deformation, it should not be modelled as a deformation of an isotropic material, because of the non-isotropic nature of the underlying stress distribution. This is where most of the errors arise.

Although we have focussed on biological tissues with strain energy given by equation (10), of course with minor modifications similar comparisons can be made for other materials (for example materials with a Mooney–Rivlin strain energy function).

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Table 1: Results of simulations comparing both incompressible linear elasticity and incompressible pseudo nonlinear elasticity with finite deformation elasticity. See text for details of the simulations.

Model	Gravity	V_{frac}	u_{max} (m)	D_{max} (m)	D_{ave} (m)	E_1	E_2
Linear	$(0, 0, g)^t$	1.000	7.40×10^{-3}	1.05×10^{-3}	3.44×10^{-4}	0.156	0.196
Linear	$(0, 0, -g)^t$	1.000	6.59×10^{-3}	1.29×10^{-3}	8.05×10^{-4}	0.190	0.273
Linear	$(g, 0, 0)^t$	1.032	3.15×10^{-2}	1.42×10^{-2}	8.35×10^{-3}	0.471	1.366
Pseudo nonlinear	$(0, 0, g)^t$	1.000	7.40×10^{-3}	5.52×10^{-4}	3.21×10^{-4}	0.156	0.151
Pseudo nonlinear	$(0, 0, -g)^t$	1.000	6.59×10^{-3}	5.21×10^{-4}	1.84×10^{-4}	0.190	0.227
Pseudo nonlinear	$(g, 0, 0)^t$	1.000	3.15×10^{-2}	1.36×10^{-3}	2.82×10^{-4}	0.471	0.535

Figure Legends

Figure 1. Results of simulations comparing the incompressible models described in the text. In all cases both the deformed and undeformed body are shown. (a) finite deformation, $\mathbf{g} = (0, 0, g)^t$; (b) linear elasticity, $\mathbf{g} = (0, 0, g)^t$; (c) Azar model, $\mathbf{g} = (0, 0, g)^t$; (d) finite deformation, $\mathbf{g} = (0, 0, -g)^t$; (e) linear elasticity, $\mathbf{g} = (0, 0, -g)^t$; (f) Azar model, $\mathbf{g} = (0, 0, -g)^t$; (g) finite deformation, $\mathbf{g} = (g, 0, 0)^t$; (h) linear model, $\mathbf{g} = (g, 0, 0)^t$; (i) Azar model, $\mathbf{g} = (g, 0, 0)^t$.

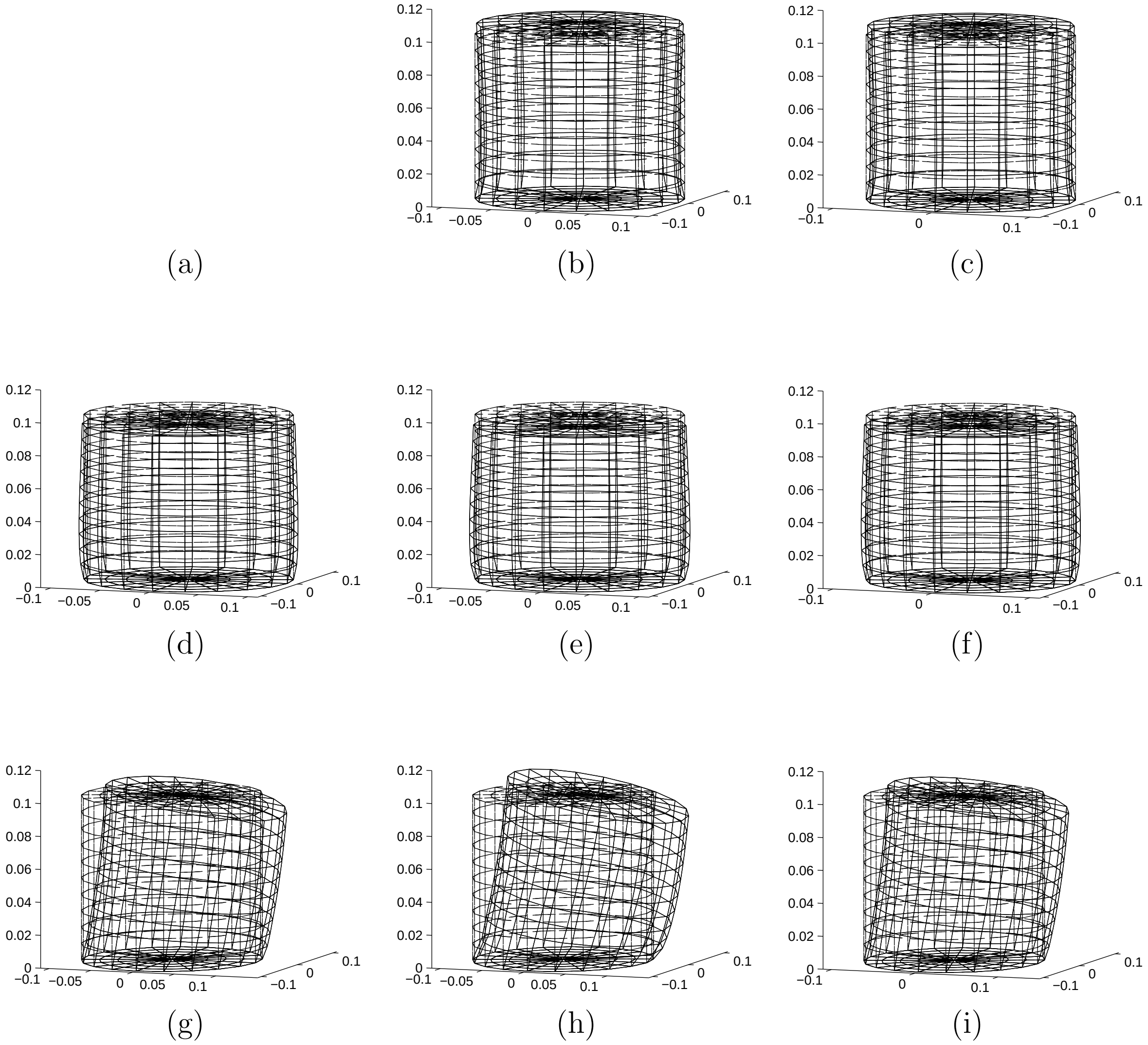

Figure 1: