Elastic Membranes and Thin Elastic Materials

1 The Elastic Membrane

1.1 Membrane Assumptions and Nonlinear Strain

We consider a one-dimensional membrane (i.e. a string) which is stretched from $\{ [0,1] \times [0] \}$ to $\{\mathbf{u}(s) = (u(s), v(s)) \ : \ s \in [0,1] \}$, with the endpoints fixed, so that $\mathbf{u}(0) = \mathbf{u}(1) = 0$. Consider the arclength τ along the deformed curve. We have

$$\begin{aligned} \mathrm{d}\tau^2 &= \left(u(s+\delta s) - u(s)\right)^2 + \left(v(s+\delta s) - v(s)\right)^2 \\ &= \left(u'^2 + v'^2\right) \mathrm{d}s^2 \end{aligned}$$

and therefore

$$\text{strain} = \frac{\mathrm{d}\tau - \mathrm{d}s}{\mathrm{d}s} = (u'^2 + v'^2)^{1/2} - 1$$

This is the quantity which needs to be minimised in an energy minimisation formulation.

For the linear membrane we assume $u(s) \approx s$ and $v(s) \ll 1$, so that the strain can be approximated by

$$\text{strain} \approx (1 + v_x^2)^{1/2} - 1 \approx \frac{1}{2} v_x^2$$

1.2 The Linear Membrane with Constant Pressure

Consider a stretched string that is subjected a pressure P from the underneath. The minimisation formulation for this problem is (why?)

$$\min \int_0^1 \frac{1}{2} v_x^2 - Pv \,\mathrm{d}x$$

subject to boundary conditions $v(0) = v(1) = 0$.

The Euler-Lagrange equation is

$$v_{xx} = P,$$

so the solution, satisfying the boundary conditions, is

$$v = \frac{P}{2} x(1-x),$$

which is a simple quadratic.

1.3 The Linear Membrane Enclosing a Fluid

We now consider the problem of a linear membrane enclosing a incompressible fluid. Since the fluid is incompressible, the area under the membrane is fixed. We need to minimise the strain, subject to the area under the curve being equal to a constant, A say. The minimisation problem is

$$\min \int_0^1 \frac{1}{2} v_x^2 + \lambda v \, \mathrm{d}x,$$

where λ is a Lagrange multiplier. The boundary conditions $v(0) = v(1) = 0$, and the area integral condition is $\int_0^1 v \, \mathrm{d}x = A$.

The Euler-Lagrange equation is

$$v_{xx} = \lambda,$$

so the solution is $v = -\frac{\lambda}{2} x(1-x)$, which gives, on using the area condition

$$v = 6Ax(1-x).$$

We see, on comparing this result with the solution of the linear membrane with constant pressure, that the Lagrange multiplier can be interpreted as an internal pressure, $P = -\lambda$, which, in this case is equal to $12A$.

1.4 The Linear Membrane with Gravity

Next, we consider the problem of a membrane enclosing an incompressible fluid, with the fluid acted on by gravity. The minimisation problem now has a term representing the gravitational potential energy of the fluid:

$$\min \int_0^1 \frac{1}{2} v_x^2 + \lambda v \, \mathrm{d}x \quad + \quad \iint_\Omega \rho \mathbf{g} \cdot \mathbf{x} \, \mathrm{d}V,$$

where Ω is the deformed volume and ρ is the fluid density.

Now,

$$\iint_\Omega \rho g_1 x \, \mathrm{d}V = \rho g_1 \int_{x=0}^{x=1} \int_{y=0}^{y=v(x)} x \, \mathrm{d}y \mathrm{d}x = \rho g_1 \int_0^1 x v(x) \, \mathrm{d}x$$

and

$$\iint_\Omega \rho g_2 y \, \mathrm{d}V = \rho g_2 \int_{x=0}^{x=1} \int_{y=0}^{y=v(x)} y \, \mathrm{d}y \mathrm{d}x = \rho g_2 \int_0^1 \frac{v^2}{2} \, \mathrm{d}x$$

so that the problem can be written as

$$\min \int_0^1 \frac{1}{2} v_x^2 + \lambda v + \rho g_1 x v + \rho g_2 \frac{v^2}{2} \, \mathrm{d}x,$$

The corresponding Euler-Lagrange equation is

$$v_{xx} = \lambda + \rho g_1 x + \rho g_2 v$$

The general solution to this takes the form of cosh/sinh or cos/sin curves (depending the sign of g_2) added to a linear term if $g_2 \neq 0$, or a cubic if $g_2 = 0$ but $g_1 \neq 0$, or a quadratic if $g_1 = g_2 = 0$.

1.5 The Linear Membrane with Contact

Consider a linear membrane enclosing an incompressible fluid which is subjected to compression from above

1.6 The Nonlinear Membrane Enclosing a Fluid

We now move to full nonlinear deformations of the membrane. Here we have to use the nonlinear version of strain. Since the area $\int v \, \mathrm{d}x = \int v u_s \, \mathrm{d}s$, we have

$$\min \int_0^1 \left(u_s^2 + v_s^2\right)^{1/2} + \lambda u_s v \, \mathrm{d}s,$$

subject to $u(0) = v(0) = u(1) = v(1) = 0$, and $\int u_s v \, \mathrm{d}s = A$.

The Euler-Lagrange equations for this problem are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}}{\partial u_s}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \quad &\Rightarrow \quad \frac{u_s}{\left(u_s^2 + v_s^2\right)^{1/2}} + \lambda v = \text{constant} \\
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}}{\partial v_s}\right) - \frac{\partial \mathcal{L}}{\partial v} = 0 \quad &\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{v_s}{\left(u_s^2 + v_s^2\right)^{1/2}}\right) - \lambda u_s = 0
\end{aligned}$$

It can be shown that these equations are equivalent (this is because a variation in the tangent direction gives no information as any reparameterisation is valid, only variations in the normal direction give an Euler-Lagrange equation), and that both are equivalent to

$$\frac{u_s v_{ss} - v_s u_{ss}}{\left(u_s^2 + v_s^2\right)^{3/2}} + \lambda = 0$$

Now, the curvature is $\kappa = v_\tau u_{\tau\tau} - u_\tau v_{\tau\tau} = (v_s u_{ss} - u_s v_{ss})/(u_s^2 + v_s^2)^{3/2}$, and the Lagrange multiplier is equal to the negative of a constant pressure P, so that

$$\kappa = P$$

We see that the curvature is constant, so the solutions are arcs of circles.

1.7 The Nonlinear Membrane with Gravity

Here, we consider the nonlinear membrane enclosing an incompressible fluid that is acted upon by gravity. We have to determine the gravitational potential energy of the fluid:

$$\text{GPE} = \iint_\Omega \rho \mathbf{g} \cdot \mathbf{x} \, \mathrm{d}V$$

Now,

$$\begin{aligned}
\nabla \cdot \left(\frac{1}{n+1}\left(\mathbf{g} \cdot \mathbf{x}\right)\mathbf{x}\right) &= \frac{\partial}{\partial x_j}\left(\frac{1}{n+1} g_i x_i x_j\right) \\
&= \frac{1}{n+1} g_i \left(\frac{\partial x_i}{\partial x_j} x_j + x_i \frac{\partial x_j}{\partial x_j}\right) \\
&= \mathbf{g} \cdot \mathbf{x}
\end{aligned}$$

where n is the dimensional of the space we are in, and therefore, in 2D,

$$\text{GPE} = \iint_{\Omega} \rho \mathbf{g} \cdot \mathbf{x} \, \mathrm{d}V = \frac{1}{3} \oint \rho \left(\mathbf{g} \cdot \mathbf{x}\right) \left(\mathbf{x} \cdot \mathbf{n}\right) d\tau$$

On $y = 0$, $\mathbf{x} = (x, 0), \mathbf{n} = (0, -1)$, so that $\mathbf{x} \cdot \mathbf{n} = 0$. Elsewhere,

$$\mathbf{x} = \mathbf{u}, \quad \mathbf{n} = \begin{bmatrix} -v_s \\ u_s \end{bmatrix} \frac{1}{\left(u_s^2 + v_s^2\right)^{1/2}} \quad \text{and} \quad \mathrm{d}\tau = -\left(u_s^2 + v_s^2\right)^{1/2} \mathrm{d}s$$

(the negative sign coming from the fact that in this case we are taking the arclength τ to be in the opposite to s on the curve). The gravitational potential energy is then

$$\begin{aligned} GPE &= \int_0^1 -\frac{1}{3}\rho(\mathbf{g} \cdot \mathbf{x})\mathbf{u} \cdot \begin{bmatrix} -v_s \\ u_s \end{bmatrix} \mathrm{d}s \\ &= \int_0^1 \frac{1}{3}\rho \left(\mathbf{g} \cdot \mathbf{u}\right) \left(u v_s - v u_s\right) \mathrm{d}s \end{aligned}$$

The minimisation problem is now

$$\min \int_0^1 \left(u_s^2 + v_s^2\right)^{1/2} + \lambda u_s v + \frac{1}{3}\rho(g_1 u + g_2 v)(u v_s - v u_s) \, \mathrm{d}s,$$

subject to $u(0) = v(0) = u(1) = v(1) = 0$, and $\int u_s v \, \mathrm{d}s = A$.

Again, the Euler-Lagrange equations for u and v are identical. For u, we obtain (writing c_i for $\rho g_i / 3$ to simplify the algebra)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{u_s}{\left(u_s^2 + v_s^2\right)^{1/2}} + \lambda v\right] + \frac{\mathrm{d}}{\mathrm{d}s}\left[-c_1 u v - c_2 v^2\right] - 2c_1 u v_s + c_1 v u_s - c_2 v v_s &= 0 \\ v_s \kappa + \lambda v_s - c_1 u_s v - c_1 u v_s - 2c_2 v v_s - 2c_1 u v_s + c_1 v u_s - c_2 v v_s &= 0 \\ \kappa + \lambda - 3c_1 u - 3c_2 v &= 0 \\ \kappa &= -\lambda + \rho \mathbf{g} \cdot \mathbf{u} \end{aligned}$$

Thus we obtain the expected curvature equation, the curvature is equal to the effective pressure caused by the gravity loaded fluid.

(This could be derived directly by considering balance of forces on a small segment: balancing the tensions acting tangentially on either side with the force provided by the pressure acting normally on the segment gives

$$\begin{aligned} T(\tau + \mathrm{d}\tau)\mathbf{t}(\tau + \mathrm{d}\tau) - T(\tau)\mathbf{t}(\tau) + P\mathbf{n}\delta\tau &= 0 \\ \frac{\partial\left(T\mathbf{t}\right)}{\partial\tau} + P\mathbf{n} &= 0 \\ \frac{\partial T}{\partial \tau}\mathbf{t} + T\frac{\partial \mathbf{t}}{\partial \tau} + P\mathbf{n} &= 0 \\ \frac{\partial T}{\partial \tau}\mathbf{t} + (P - T\kappa)\mathbf{n} &= 0 \end{aligned}$$

(using $\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\tau} = -\kappa\mathbf{n}$ as $\mathbf{n}$ is the outward-facing normal), which gives the two equations $T =$ constant and $T\kappa = P = P_0 + \rho \mathbf{g} \cdot \mathbf{u}$).

Numerical Solution

To solve this equation we introduce the angle $\theta = \theta(\tau)$ as the angle between $\mathbf{u}(\tau)$ and the horizontal. It follows that $u_\tau = \cos(\theta)$ and $v_\tau = \sin(\theta)$. We differentiate ** with respect to τ to remove the unknown λ, obtaining

$$\kappa_\tau = \rho \mathbf{g} \cdot \mathbf{u}_\tau,$$

which is equivalent to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = \rho \mathbf{g} \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix},$$

or, introducing the constants g and χ such that $\mathbf{g} = g \begin{bmatrix} \cos(\chi) \\ \sin(\chi) \end{bmatrix}$,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = \rho g \cos(\theta(\tau) - \chi),$$

The equation determining the shape of a nonlinear membrane enclosing a gravity loading fluid is thus the same as the pendulum equation.

... numerics: formulate as 2D shooting prob (θ_0,$\theta_{\tau 0}$ to (area,length), use newton's method, approximating Jacobean with divided diff...

..pics..

1.8 The Nonlinear Membrane with Contact

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2 The Thin Elastic Solid in Two Dimensions

We now consider the deformation of a thin elastic solid in two dimensions. We assume the undeformed solid is rectangular and given by $(X, Y) \in \{[0, L] \times [-\epsilon, \epsilon]\}$. We will rescale in the thin Y direction using the new variable $\hat{Y}$ satisfying $\epsilon\hat{Y} = Y$.

picture

We will need to use the first two terms of the asymptotic expansion of $\mathbf{x}$:

$$\mathbf{x}(X, \hat{Y}) = \mathbf{x}^{(0)}(X, \hat{Y}) + \epsilon\mathbf{x}^{(1)}(X, \hat{Y}) + \dots$$

The first result that needs to be proven is that the leading order term of the deformation is independent of the thin direction.

CLAIM: $\mathbf{x}^{(0)}$ satisfies $\mathbf{x}^{(0)}(X) \equiv \mathbf{x}^{(0)}(X)$ ******everywhere except near the boundaries.*******????

PROOF: $\square$

2.1 Deformation and Strain

We will now define the deformation gradient and Lagrange tensors.

The deformation gradient is given by

$$F_{iM} = \frac{\partial x_i}{\partial X_M},$$

which to leading order (O(1)) is

$$F^{(0)} = \begin{bmatrix} x_X^{(0)} & x_{\hat{Y}}^{(1)} \\ y_X^{(0)} & y_{\hat{Y}}^{(1)} \end{bmatrix}$$

We define following vectors to simplify the notation.

$$\mathbf{t} = \frac{\partial \mathbf{x}^{(0)}}{\partial X} \qquad \text{and} \qquad \mathbf{h} = \frac{\partial \mathbf{x}^{(1)}}{\partial \hat{Y}}$$

$\mathbf{t}$ is the tangent vector of the image of the middle line. If we define $\tau \equiv \tau(X)$ to be arclength along this curve then we can write the tangent as $\mathbf{t} = \tau'\hat{\mathbf{t}}$, where $\hat{\mathbf{t}}$ is the normalised tangent vector. τ' represents the local stretching of the middle line.

Now, using $\mathbf{t}$ and $\mathbf{h}$, $F^{(0)}$ has the form

$$F^{(0)} = \begin{bmatrix} \hat{t}_1\tau'(X) & h_1 \\ \hat{t}_2\tau'(X) & h_2 \end{bmatrix} = \begin{bmatrix} \mathbf{t}(X) & \mathbf{h}(X, \hat{Y}) \end{bmatrix}$$

Consider the deformed thin elastic solid, Ω. We can use $\mathbf{t}$ and τ to define a curvilinear coordinate system on the Ω. Let $\hat{mbn}$ be the unit outward normal of $\mathbf{x}^{(0)}$, perpendicular to $\mathbf{t}$. $\hat{mbt}$ and $\hat{mbn}$ define an orthonormal basis on Ω. The curvilinear coordinate system (τ, n) is defined by

$$\Omega = \{\mathbf{x}(\tau, n) = \mathbf{x}^{(0)}(\tau) + n\mathbf{n}(\tau) \, : \, \tau \in [0, l], \, n \in [\epsilon d^{[1]}(\tau), \epsilon d^{[2]}(\tau)] \, \}$$

where $\epsilon d^{[1]}(< 0)$ is the negative of the normal distance from the centre line to the bottom surface, and $\epsilon d^{[2]}$ the distance from the centre line to the top surface, and l is the length of the curve $\mathbf{x}^{(0)}$.

(assuming the deformed shape Ω can be parameterised like this means assumptions that the image of the centre line is not too curved)

There are two important derived tensors to consider. The left Green-Lagrange tensor, $C = F^T F$ is independent of the curvilinear coordinate system, and to leading order,

$$C^{(0)} = \begin{bmatrix} \|\mathbf{t}\|^2 & \mathbf{t}\cdot\mathbf{h} \\ \mathbf{t}\cdot\mathbf{h} & \|\mathbf{h}\|^2 \end{bmatrix}$$

The right Green-Lagrange tensor, $B = FF^T$ has the following form to leading order, in the curvilinear coordinates

$$B^{(0)} = \begin{bmatrix} \|\mathbf{t}\|^2 + (\mathbf{t}\cdot\mathbf{h})^2 & (\mathbf{t}\cdot\mathbf{h})(\mathbf{n}\cdot\mathbf{h}) \\ (\mathbf{t}\cdot\mathbf{h})(\mathbf{n}\cdot\mathbf{h}) & \|\mathbf{h}\|^2 \end{bmatrix}$$

The Lagrangian strain is defined as $\frac{1}{2}(C - I)$. We won't ever explicitly use the Lagrangian strain, since nonlinear stress-strain laws derived from strain-energy functions can be written in terms of C instead of E.

We will only ever consider isotropic materials, in which case material laws can be written as a function of the principle invariants of C, $I_1 = \mathrm{tr}(C)$ and $I_2 = \det(C)$. To leading order

$$\begin{aligned} I_1^{(0)} &= \|\mathbf{t}\|^2 + \|\mathbf{h}\|^2 \\ I_2^{(0)} &= \|\mathbf{t}\|^2\|\mathbf{h}\|^2 - (\mathbf{t}\cdot\mathbf{h})^2 \end{aligned}$$

Assuming a material strain energy function W, we are now ready to write down the Cauchy stress tensor. Since $\sigma = \frac{1}{J}FTF^{\mathrm{T}}$, where the Jacobean $J = \det F = \sqrt{I_2}$ and T is the 2nd Piola-Kirchoff stress, $T = \frac{\partial W}{\partial E} = 2\frac{\partial W}{\partial C}$, we have

$$\begin{aligned} \sigma &= \frac{1}{J}FTF^{\mathrm{T}} \\ &= \frac{2}{J}F\frac{\partial W}{\partial C}F^{\mathrm{T}} \\ &= \frac{2}{J}F\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C}\right)F^{\mathrm{T}} \\ &= \frac{2}{J}F\left(\frac{\partial W}{\partial I_1}I + \frac{\partial W}{\partial I_2}I_2(C^{-1})\right)F^{\mathrm{T}} \\ &= \frac{2}{J}\frac{\partial W}{\partial I_1}B + 2J\frac{\partial W}{\partial I_2}I \end{aligned}$$

So, to highest order, the Cauchy stress is

$$\sigma^{(r)} = \frac{2}{J}\frac{\partial W}{\partial I_1}\begin{bmatrix} \|\mathbf{t}\|^2 + (\mathbf{t}\cdot\mathbf{h})^2 & (\mathbf{t}\cdot\mathbf{h})(\mathbf{n}\cdot\mathbf{h}) \\ (\mathbf{t}\cdot\mathbf{h})(\mathbf{n}\cdot\mathbf{h}) & \|\mathbf{h}\|^2 \end{bmatrix} + 2J\frac{\partial W}{\partial I_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here r is the (as yet unknown) order of σ, which will be shown to be -1 in the next section.

2.2 Analysis of Cauchy Stress

In this section we study the force balance equations in terms of the Cauchy stress and the curvilinear coordinate system, and equate it with the known form of σ to derive the equations determining the deformed shape.

Recall that the curvilinear coordinate system is given by

$$\Omega = \{\mathbf{x}(\tau, n) = \mathbf{x}^{(0)}(\tau) + n\mathbf{n}(\tau) \,:\, \tau \in [0, l],\, n \in [\epsilon d^{[1]}(\tau), \epsilon d^{[2]}(\tau)]\,\}$$

Firstly, note that the curvilinear tangent vectors are

$$\begin{aligned} \mathbf{x}_\tau &= \mathbf{f}_\tau + n\mathbf{n}_\tau = (1 + \kappa n)\mathbf{t} \\ \mathbf{x}_n &= \mathbf{n} \end{aligned}$$

(using $\frac{\mathrm{d}\mathbf{n}_{\text{inward}}}{\mathrm{d}\tau} = -\kappa\mathbf{t}$). We write $n = \epsilon\nu$, and wish to asymptotically expand the stress about ϵ.

Euler's equation for elastic equilibrium of the solid is $\sigma_{ij,j} = \rho^{(s)} g_i$, where $\rho^{(s)}$ is the density of the thin solid. Writing this in the new coordinate system, we have

$$\begin{aligned} \frac{\partial}{\partial\tau}(\sigma_{\tau\tau}) + \frac{1}{\epsilon}\frac{\partial}{\partial\nu}((1 + \kappa\epsilon\nu)\sigma_{\tau\nu}) + \kappa\sigma_{\tau\nu} &= \rho^{(s)} g_\tau \\ \frac{\partial}{\partial\tau}(\sigma_{\tau\nu}) + \frac{1}{\epsilon}\frac{\partial}{\partial\nu}((1 + \kappa\epsilon\nu)\sigma_{\nu\nu}) - \kappa\sigma_{\tau\tau} &= \rho^{(s)} g_\nu \end{aligned}$$

The boundary conditions on the surface of the solid are

$$\begin{aligned} \sigma\mathbf{n}^{[1]} + P\mathbf{n}^{[1]} &= 0 \qquad \text{on } \nu = d^{[1]} \\ \sigma\mathbf{n}^{[2]} &= 0 \qquad \text{on } \nu = d^{[2]} \end{aligned}$$

In order to obtain compatible equations we must asymptotically expand the stress as

$$\begin{aligned} \sigma_{\tau\tau} &= \frac{1}{\epsilon}\sigma_{\tau\tau}^{(-1)} + \sigma_{\tau\tau}^{(0)} + \dots \\ \sigma_{\tau\nu} &= \sigma_{\tau\nu}^{(0)} + \epsilon\sigma_{\tau\nu}^{(1)} + \dots \\ \sigma_{\nu\nu} &= \sigma_{\nu\nu}^{(0)} + \epsilon\sigma_{\nu\nu}^{(1)} + \dots \end{aligned}$$

It is easy to show that the order of $\sigma_{\tau\tau}$ must be one lower than $\sigma_{\tau\nu}$ and $\sigma_{\nu\nu}$. But from this simple point we can make a number of important observations. Since the highest order of σ is -1, i.e. $r = -1$, we see that $\frac{\partial W}{\partial I_1}$ and $\frac{\partial W}{\partial I_1}$ must be $O(1/\epsilon)$ (i.e. any dimensional material constants are $O(1/\epsilon)$). Equating *** with

$$\sigma = \frac{1}{\epsilon}\begin{bmatrix} \sigma_{\tau\tau}^{(-1)} & 0 \\ 0 & 0 \end{bmatrix} + \dots$$

we can say by looking at the diagonal terms that

$$\mathbf{t} \cdot \mathbf{h} = 0$$

(as clearly $\mathbf{n} \cdot \mathbf{h} = 0$ is not possible for a valid deformation). Therefore the whole solid deforms smoothly without shearing, and C, B, E, T and σ are all diagonal in this coordinate system to leading order. Equating $\sigma_{\nu\nu}$ with zero gives $\frac{2}{J}\frac{\partial W}{\partial I_1}\|\mathbf{h}\|^2 + 2J\frac{\partial W}{\partial I_2} = 0$, a relationship between $\|\mathbf{t}\|$ and $\|\mathbf{h}\|$. Simplifying this, we obtain

$$\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\|\mathbf{t}\|^2 = 0$$

The relationship between $\|\mathbf{t}\|$ and $\|\mathbf{h}\|$ holds independently of the gross deformation or boundary conditions.

Now, since we know that $\mathbf{h}$ is orthogonal to $\mathbf{t}$, and $normh$ is determined by $\|\mathbf{t}\|$, and $\mathbf{t}$ is a function of X only, we can also say that $\mathbf{h}$ *is also a function of* X *only.* We have thus determined the form of $\mathbf{x}^{(1)}$

$$\mathbf{x} = \mathbf{x}^{(0)} + \epsilon \hat{Y} \mathbf{h}(\mathbf{X}) + \dots$$

where the direction and magnitude of $\mathbf{h}$ has been derived in terms of $\mathbf{t}$ (i.e. in terms of $\mathbf{x}^{(0)}$. We can also say that the normal distances from the image of the centre-line to the upper surface and lower surfaces are equal, i.e. $\epsilon\|\mathbf{h}\| = d^{[2]} = -d^{[1]}$ $(= \frac{d}{2}$, say).

Finally, note that since $\sigma_\tau^{(-1)}\tau$ a function of $\mathbf{t}$ and $\mathbf{h}$, which are independent of $\hat{Y}$, and $\mathbf{h}$ is in the $\mathbf{n}$ direction, we can say that $\sigma_\tau^{(-1)}\tau$ when written as a function of τ and ν, is independent of ν.

It remains to determine the leading order deformation. For this, we need to solve Cauchy's equations. Substituting **** into the Euler equations gives leading order equations of

$$\frac{\partial \sigma_{\tau\tau}^{(-1)}}{\partial \tau} + \frac{\partial \sigma_{\tau\nu}^{(0)}}{\partial \nu} = 0$$

and

$$\frac{\partial \sigma_{\nu\nu}^{(0)}}{\partial \nu} = \kappa \sigma_{\tau\tau}^{(-1)}$$

We will shortly integrate these equations; we first need to consider the boundary conditions. The upper and lower surfaces are given in parameterised form by

$$\mathbf{x}^{[k]} = \mathbf{f}(\tau) + \epsilon d^{[k]}(\tau)\mathbf{n}(\tau)$$

(for $k = 1, 2$), so that the tangents along the upper and lower surfaces are given by

$$\mathbf{t}^{[k]} = \mathbf{f}_\tau + \epsilon d_\tau^{[k]}\mathbf{n} + \epsilon d^{[k]}\mathbf{n}_\tau = (1 + \epsilon d^{[k]}\kappa)\mathbf{t} + \epsilon d_\tau^{[k]}\mathbf{n}$$

The (un-normalised) normal directions on the upper and lower are therefore given by

$$\mathbf{n}^{[k]} = (1 + \epsilon d^{[k]}\kappa)\mathbf{n} - \epsilon d_\tau^{[k]}\mathbf{t}$$

The boundary equations are

$$\begin{bmatrix} \sigma_{\tau\tau} & \sigma_{\tau\nu} \\ \sigma_{\tau\nu} & \sigma_{\nu\nu} \end{bmatrix} \begin{bmatrix} -\epsilon d_\tau^{[2]} \\ 1 + \epsilon d^{[2]}\kappa \end{bmatrix} = 0 \qquad \text{on } \nu = d^{[2]}$$

and

$$\begin{bmatrix} \sigma_{\tau\tau} & \sigma_{\tau\nu} \\ \sigma_{\tau\nu} & \sigma_{\nu\nu} \end{bmatrix} \begin{bmatrix} -\epsilon d_\tau^{[1]} \\ 1 + \epsilon d^{[1]}\kappa \end{bmatrix} + P \begin{bmatrix} -\epsilon d_\tau^{[1]} \\ 1 + \epsilon d^{[1]}\kappa \end{bmatrix} = 0 \qquad \text{on } \nu = d^{[1]}$$

The leading order boundary equations are then

$$\begin{aligned} -d_\tau^{[1]}\sigma_{\tau\tau}^{(-1)} + \sigma_{\tau\nu}^{(0)} &= 0 \qquad \text{on } \nu = d^{[1]} \\ -d_\tau^{[2]}\sigma_{\tau\tau}^{(-1)} + \sigma_{\tau\nu}^{(0)} &= 0 \qquad \text{on } \nu = d^{[2]} \end{aligned}$$

and

$$\begin{aligned} \sigma_{\nu\nu}^{(0)} + P &= 0 \qquad \text{on } \nu = d^{[1]} \\ \sigma_{\nu\nu}^{(0)} &= 0 \qquad \text{on } \nu = d^{[2]} \end{aligned}$$

Integrating (**), using the fact that $\sigma_{\tau\tau}^{(-1)}$ is independent of ν, gives

$$0 = \int_{d^{[1]}}^{d^{[2]}} \frac{\partial \sigma_{\tau\tau}^{(-1)}(\tau)}{\partial \tau} + \frac{\partial \sigma_{\tau\nu}^{(0)}(\tau,\nu)}{\partial \nu}\, \mathrm{d}\nu = (d^{[2]} - d^{[1]})\frac{\partial \sigma_{\tau\tau}^{(-1)}}{\partial \tau} + \left[\sigma_{\tau\nu}^{(0)}\right]_{d^{[1]}}^{d^{[2]}}$$

Using the boundary condition, this becomes

$$(d^{[2]} - d^{[1]})\frac{\partial \sigma_{\tau\tau}^{(-1)}}{\partial \tau} + (d_\tau^{[2]} - d_\tau^{[1]})\sigma_{\tau\tau}^{(-1)} = 0,$$

or, writing the thickness as $d(\tau) = d^{[2]} - d^{[1]}$,

$$\frac{\partial}{\partial \tau}\left(d(\tau)\sigma_{\tau\tau}^{(-1)}(\tau)\right) = 0$$

Thus we have $d\sigma_{\tau\tau}^{(-1)}$ is equal to a constant, which we call T (the tension):

$$d\sigma_{\tau\tau}^{(-1)} = T$$

Also, integrating (**), we have

$$0 = \int_{d^{(1)}}^{d^{(2)}} \kappa\sigma_{\tau\tau}^{(-1)}(\tau) - \frac{\partial \sigma_{\nu\nu}^{(0)}(\tau,\nu)}{\partial \nu}\,\mathrm{d}\nu = \kappa d\sigma_{\tau\tau}^{(-1)} - \left[\sigma_{\nu\nu}^{(0)}\right]_{d^{(1)}}^{d^{(2)}},$$

so that $\kappa d\sigma_{\tau\tau}^{(-1)} - P = 0$, so

$$\kappa T = P$$

(prob should be $\kappa^{(0)}T = P$)

Finally, we need to equate the two equations for $\sigma_{\tau\tau}^{(-1)}$.

$$\frac{T}{d} = \sigma_{\tau\tau}^{(-1)} = \frac{2}{J}\frac{\partial W}{\partial I_1}\|\mathbf{t}\|^2 + 2J\frac{\partial W}{\partial I_2}$$

Simplifying, using $d = 2\epsilon\|\mathbf{h}\|$, and grouping $\epsilon\frac{\partial W}{\partial I_i}$ as $O(1)$ quantities, gives

$$\left(4\epsilon\frac{\partial W}{\partial I_1}\right)\|\mathbf{t}\| + \left(4\epsilon\frac{\partial W}{\partial I_2}\right)\|\mathbf{t}\|\|\mathbf{h}\|^2 = T$$

We now have two algebraic equations for two unknowns $\|\mathbf{t}\|$ and $\|\mathbf{h}\|$. These equations are not explicitly dependent on X, so we can also say $\|\mathbf{t}\|$ and $\|\mathbf{h}\|$ are independent of X as well, i.e. the solid stretches uniformly.

2.3 The Minimisation Formulation

We now derive the same results using the nonlinear energy minimisation formulation, and show that the energy minimisation for the thin solid deformation is equivalent, when taken to leading order, to the membrane energy minimisation problem.

We begin again from $\mathbf{x}(X,\hat{Y}) = \mathbf{x}^{(0)}(X) + \epsilon\mathbf{x}^{(1)}(X,\hat{Y}) + \ldots$, and write

$$F^{(0)} = [\mathbf{x}_X^{(0)}\mathbf{x}_{\hat{Y}}^{(1)}] = [\mathbf{t} \quad \mathbf{h}]$$

and

$$C^{(0)} = \begin{bmatrix} \|\mathbf{t}\|^2 & \mathbf{t}\cdot\mathbf{h} \\ \mathbf{t}\cdot\mathbf{h} & \|\mathbf{h}\|^2 \end{bmatrix}$$

$$\begin{aligned} I_1^{(0)} &= \|\mathbf{t}\|^2 + \|\mathbf{h}\|^2 \\ I_2^{(0)} &= \|\mathbf{t}\|^2\|\mathbf{h}\|^2 - (\mathbf{t}\cdot\mathbf{h})^2 \end{aligned}$$

There are a number of energy minimisation problems we could consider. In general they take the form

$$\begin{aligned}
&\underset{\mathbf{x}}{\text{minimise}} \iint_{\Omega_0} \text{Strain Energy } \mathrm{d}S_0 + \text{ Remaining Energy} \\
\Rightarrow\ &\underset{\mathbf{x}}{\text{minimise}} \iint_{\Omega_0} W(I_1(\|\mathbf{t}\|^2, \|\mathbf{h}\|^2), I_2(\|\mathbf{t}\|^2, \|\mathbf{h}\|^2))\, \mathrm{d}S_0 + \text{ Remaining Energy} \\
\Rightarrow\ &\underset{\mathbf{x}}{\text{minimise}} \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W(I_1, I_2)\, \mathrm{d}\hat{Y} \epsilon\, \mathrm{d}X + \mathcal{J}\left(\mathbf{x}, \frac{\partial \mathbf{x}}{\partial X_M}\right)
\end{aligned}$$

In the asymptotic analysis, we will take the leading order components of each term, and split the minimisation as a minimisation over $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ independently.

$$\underset{\mathbf{x}^{(0)},\, \mathbf{x}^{(1)}}{\text{minimise}} \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W(I_1^{(0)}, I_2^{(0)})\, \mathrm{d}\hat{Y} \epsilon\, \mathrm{d}X + \mathcal{J}^{(0)}\left(\mathbf{x}^{(0)}, \frac{\partial \mathbf{x}^{(0)}}{\partial X_M}, \dots\right)$$

First we prove that the variation of $\mathbf{x}^{(1)}$ results in the equations determining $\mathbf{h}$.

CLAIM: Assuming $\mathcal{J}^{(0)}$ is independent of $\mathbf{x}^{(1)}$, the variation of $\mathbf{x}^{(1)}$ gives $\frac{\partial W}{\partial \mathbf{h}} = 0$ everywhere.

PROOF: Assuming the only term in the leading order energy which contains $\mathbf{x}^{(1)}$ is the strain energy, and considering a variation of $\mathbf{x}^{(1)}$ given by $\delta \mathbf{x}^{(1)}$

$$\begin{aligned}
\delta E &= \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W\left(\mathbf{h} + \frac{\partial(\delta \mathbf{x}^{(1)})}{\partial \hat{Y}}\right) - W(\mathbf{h})\, \mathrm{d}\hat{Y} \epsilon \mathrm{d}X \\
&= \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} \frac{\partial W}{\partial \mathbf{h}} \cdot \frac{\partial(\delta \mathbf{x}^{(1)})}{\partial \hat{Y}}\, \mathrm{d}\hat{Y} \epsilon \mathrm{d}X \\
&= \int_{X=0}^{X=L} \left[\frac{\partial W}{\partial \mathbf{h}} \cdot \delta \mathbf{x}^{(1)}\right]_{\hat{Y}=-1}^{\hat{Y}=1} \mathrm{d}X - \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} \left(\frac{\partial W}{\partial \mathbf{h}}\right)_{,\hat{Y}} \cdot \delta \mathbf{x}^{(1)}\, \mathrm{d}\hat{Y} \epsilon \mathrm{d}X
\end{aligned}$$

Since $\delta \mathbf{x}^{(1)}$ is completely arbitrary for $X \neq 0, L$ (there are no boundary conditions imposed on $\mathbf{x}^{(1)}$, so no boundary conditions on $\delta \mathbf{x}^{(1)}$), it follows that

$$\begin{aligned}
\frac{\partial}{\partial \hat{Y}}\left(\frac{\partial W}{\partial \mathbf{h}}\right) &= 0 \qquad \forall X, \quad \forall \hat{Y} \\
\frac{\partial W}{\partial \mathbf{h}} &= 0 \qquad \forall X, \quad \hat{Y} = 1 \\
\frac{\partial W}{\partial \mathbf{h}} &= 0 \qquad \forall X, \quad \hat{Y} = -1
\end{aligned}$$

The solution of this trivial differential equation is $\frac{\partial W}{\partial \mathbf{h}} = 0$ everywhere.

□

Now, $0 = \frac{\partial W}{\partial \mathbf{h}} = 2\frac{\partial W}{\partial I_1}\mathbf{h} + 2\frac{\partial W}{\partial I_2}(\|\mathbf{t}\|^2\mathbf{h} - (\mathbf{t} \cdot \mathbf{h})\mathbf{t})$. This is two equations, which determine $\mathbf{h}$ in terms $\mathbf{t}$.

$$\begin{aligned}
\frac{\partial W}{\partial \mathbf{h}} \cdot \mathbf{t} = 0 \quad &\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{h} \cdot \mathbf{t} + 2\frac{\partial W}{\partial I_2}(\|\mathbf{t}\|^2\mathbf{h} \cdot \mathbf{t} - (\mathbf{t} \cdot \mathbf{h})\mathbf{t} \cdot \mathbf{t}) = 0 \quad \Rightarrow \quad \mathbf{t} \cdot \mathbf{h} = 0 \\
\frac{\partial W}{\partial \mathbf{h}} \cdot \mathbf{h} = 0 \quad &\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{h} \cdot \mathbf{h} + 2\frac{\partial W}{\partial I_2}(\|\mathbf{t}\|^2\mathbf{h} \cdot \mathbf{h}) = 0 \quad \Rightarrow \quad \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\|\mathbf{t}\|^2 = 0
\end{aligned}$$

Thus we have rederived the equations for $\mathbf{h}$, again without having to consider the bulk deformation. Again, we note that since $\mathbf{h}$ is orthogonal to $\mathbf{t}$ and has norm determined by $\|\mathbf{t}\|$, we can say $\mathbf{h} \equiv \mathbf{h}(X)$ only. It follows that

$$\int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W(I_1^{(0)}, I_2^{(0)}) \, \mathrm{d}\hat{Y} \epsilon \, \mathrm{d}X = 2\epsilon \int_{X=0}^{X=L} W(I_1^{(0)}, I_2^{(0)}) \, \mathrm{d}X$$

Now we consider some specific energy minimisations:

Constant pressure:

$$\min \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W(I_1, I_2) \, \mathrm{d}\hat{Y} \epsilon \, \mathrm{d}X + ???????$$

Constant area underneath the solid (incompressible fluid):

Here we have an elastic solid containing an incompressible fluid, which introduces a constraint that the area contained under the thin elastic solid is constant. Using a Lagrange multiplier λ, the full minimisation problem is

$$\min \int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} W(I_1, I_2) \, \mathrm{d}\hat{Y} \epsilon \, \mathrm{d}X + \lambda \int_{X=0}^{X=L} \frac{\partial x(X,-1)}{\partial X} y(X,-1) \, \mathrm{d}X$$

To leading order this is

$$\min 2\epsilon \int_{X=0}^{X=L} W(I_1^{(0)}, I_2^{(0)}) \, \mathrm{d}X + \lambda \int_{X=0}^{X=L} \frac{\partial x^{(0)}}{\partial X} y^{(0)} \, \mathrm{d}X$$

The Euler-Lagrange equation for this is

$$\begin{aligned} \frac{\partial}{\partial X}\left(2\epsilon \frac{\partial W}{\partial \mathbf{t}}\right) + \frac{\partial}{\partial X}\begin{bmatrix} \lambda y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \lambda \frac{\partial x}{\partial X} \end{bmatrix} &= 0 \\ \Rightarrow \qquad \frac{\partial}{\partial X}\left(2\epsilon \frac{\partial W}{\partial \mathbf{t}}\right) + \lambda \|\mathbf{t}\| \hat{\mathbf{n}} &= 0 \end{aligned}$$

Since $\frac{1}{\|\mathbf{t}\|}\frac{\partial}{\partial X} = \frac{\partial X}{\partial \tau}\frac{\partial}{\partial X}$, this is

$$\frac{\partial}{\partial \tau}\left(2\epsilon \frac{\partial W}{\partial \mathbf{t}}\right) + \lambda \hat{\mathbf{n}} = 0$$

Now, $2\epsilon \frac{\partial W}{\partial \mathbf{t}} = 4\epsilon \frac{\partial W}{\partial I_1} \mathbf{t} + 4\epsilon \frac{\partial W}{\partial I_2} \|\mathbf{h}\|^2 \mathbf{t} = \alpha \hat{\mathbf{t}}$, where $\alpha = 4\epsilon \frac{\partial W}{\partial I_1} \|\mathbf{t}\| + 4\epsilon \frac{\partial W}{\partial I_2} \|\mathbf{h}\|^2 \|\mathbf{t}\|$. Then

$$\frac{\partial}{\partial \tau}\left(\alpha \hat{\mathbf{t}}\right) + \lambda \hat{\mathbf{n}} = 0 \qquad \Rightarrow \qquad \frac{\partial \alpha}{\partial \tau} \hat{\mathbf{t}} + \alpha \kappa \hat{\mathbf{n}} + \lambda \hat{\mathbf{n}} = 0$$

Taking the tangential and normal components, we see that

$$\begin{aligned} \frac{\partial \alpha}{\partial \tau} = 0 \quad \Rightarrow \quad \alpha = \text{ constant (in } X\text{)}, T \text{ say,} \quad &\Rightarrow \quad 4\epsilon \frac{\partial W}{\partial I_1} \|\mathbf{t}\| + 4\epsilon \frac{\partial W}{\partial I_2} \|\mathbf{h}\|^2 \|\mathbf{t}\| = T \\ \alpha \kappa + \lambda = 0 \quad &\Rightarrow \quad T\kappa = P \end{aligned}$$

We have recovered the same set of equations as in Section ??

Gravity acting on the solid:

If gravity is acting on the thin elastic solid, we need to add the following term to the energy

$$\mathrm{GPE}^{(\mathrm{s})} = \iint_{\Omega} \rho_0^{(\mathrm{s})} \mathbf{g} \cdot \mathbf{x} \, \mathrm{d}S_0$$

To leading order this is

$$\int_{X=0}^{X=L} \int_{\hat{Y}=-1}^{\hat{Y}=1} \rho^{(\mathrm{s})} \mathbf{g} \cdot \mathbf{x}^{(0)} \, \mathrm{d}\hat{Y} \epsilon \, \mathrm{d}X = 2\epsilon \int_{X=0}^{X=L} \rho^{(\mathrm{s})} \mathbf{g} \cdot \mathbf{x}^{(0)} \, \mathrm{d}X$$

since $\mathbf{x}^{(0)}$ is independent of $\hat{Y}$. But this is $O(\epsilon)$, which is higher order than the strain energy integral, and therefore we can say that the gravity acting on the solid is a higher-order effect and does not effect the leading order solution.

Gravity acting on the fluid:

If the thin elastic solid contains a fluid which is gravity-loaded, we need to add to the energy the gravitational potential energy of the fluid, which is an integral over the volume below the thin solid.

** pic **

$$\mathrm{GPE}^{(\mathrm{f})} = \iint_{fluid} \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{z} \, \mathrm{d}V$$

Now, using the results shown in Section 1.7, and taking the leading order component, we can say that

$$\mathrm{GPE}^{(\mathrm{f})} = \frac{1}{3} \oint \rho^{(\mathrm{f})} \left(\mathbf{g} \cdot \mathbf{x}\right) \left(\mathbf{x} \cdot \hat{\mathbf{n}}\right) \, \mathrm{d}\tau = -\frac{1}{3} \int_0^L \rho^{(\mathrm{f})} \left(\mathbf{g} \cdot \mathbf{x}^{(0)}\right) \left(\mathbf{x}^{(0)} \cdot \mathbf{n}\right) \, \mathrm{d}X$$

Here we have neglected the term which is an integral over the lower fixed surface, which is constant and does not affect the variational formulation.

Now, it can be shown after some algebra that, where $\mathcal{G} = (\mathbf{g} \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{n})$,

$$\frac{\partial}{\partial X}\left(\frac{\partial \mathcal{G}}{\partial \mathbf{t}}\right) - \frac{\partial \mathcal{G}}{\partial \mathbf{x}} = -3(\mathbf{g} \cdot \mathbf{x})\mathbf{n}$$

so that the term added to the Euler-Lagrange equation is $\rho^{(\mathrm{f})}(\mathbf{g} \cdot \mathbf{x}^{(0)})\mathbf{n}$:

$$\begin{aligned} \frac{\partial}{\partial X}\left(2\epsilon \frac{\partial W}{\partial \mathbf{t}}\right) + \lambda \|\mathbf{t}\| \hat{\mathbf{n}} + \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{x}^{(0)} \|\mathbf{t}\| \hat{\mathbf{n}} &= 0 \\ \Rightarrow \qquad \frac{\partial}{\partial \tau}\left(2\epsilon \frac{\partial W}{\partial \mathbf{t}}\right) + \left(\lambda + \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{x}^{(0)}\right) \hat{\mathbf{n}} &= 0 \end{aligned}$$

The solution is easily seen to be

$$\begin{aligned} 4\epsilon \frac{\partial W}{\partial I_1} \|\mathbf{t}\| + 4\epsilon \frac{\partial W}{\partial I_2} \|\mathbf{h}\|^2 \|\mathbf{t}\| &= T \qquad \text{as before} \\ T\kappa &= P + \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{x}^{(0)} \end{aligned}$$

sign error?

2.4 Incompressibility

If the thin elastic solid is incompressible we have a new constraint $I_2 = 1$ and a new unknown p, the pressure. The new strain energy is $W(I_1) - \frac{p}{2}(I_2 - 1)$. Note that this means $p = O(\frac{1}{\epsilon})$, so we write $p = \frac{1}{\epsilon}p^{(-1)} + \ldots$. It can easily be shown using the argument as before that we still must have $\mathbf{t} \cdot \mathbf{h} = 0$. The previous two equations determining $\|\mathbf{t}\|$ and $\|\mathbf{h}\|$, together with the constraint $I_2 = \|\mathbf{t}\|^2\|\mathbf{h}\|^2 = 1$, now become three equations for the three unknowns $\|\mathbf{t}\|$, $\|\mathbf{h}\|$ and $p^{(-1)}$.

The following equations hold for any deformation:

$$\begin{aligned} \mathbf{t} \cdot \mathbf{h} &= 0 \\ \left(2\epsilon \frac{\partial W}{\partial I_1}\right) \|\mathbf{h}\|^2 - p^{(-1)} &= 0 \\ \|\mathbf{t}\|\|\mathbf{h}\| &= 1 \end{aligned}$$

(Note that this means $p = O(\frac{1}{\epsilon})$). In the case of constant pressure boundary conditions, the remaining equations are

$$\begin{aligned} \left(4\epsilon \frac{\partial W}{\partial I_1}\right) \|\mathbf{t}\| - 2p^{(-1)}\|\mathbf{h}\| &= T \\ T\kappa &= P \end{aligned}$$

2.5 Higher Order Effects

2.6 Summary

Writing $\mathbf{x}$ as $\mathbf{x} = \mathbf{x}^{(0)} + \epsilon\mathbf{x}^{(0)}$, setting $\mathbf{t} = \frac{\partial \mathbf{x}^{(0)}}{\partial X}$ and $\mathbf{h} = \frac{\partial \mathbf{x}^{(1)}}{\partial \hat{Y}}$, we can say:

For any deformation:

$$\begin{aligned} \mathbf{t} \cdot \mathbf{h} &= 0 \\ \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\|\mathbf{t}\|^2 &= 0 \\ \mathbf{h} &= \mathbf{h}(X) \quad \text{only} \end{aligned}$$

With a constant pressure P the remaining equations are

$$\begin{aligned} 4\epsilon \frac{\partial W}{\partial I_1}\|\mathbf{t}\| + 4\epsilon \frac{\partial W}{\partial I_2}\|\mathbf{h}\|^2\|\mathbf{t}\| &= T \\ T\kappa &= P \end{aligned}$$

It follows that $\|\mathbf{t}\|$ (and so, also $\|\mathbf{h}\|$) are constant.

With a gravity loaded fluid the remaining equations are

$$\begin{aligned} 4\epsilon \frac{\partial W}{\partial I_1}\|\mathbf{t}\| + 4\epsilon \frac{\partial W}{\partial I_2}\|\mathbf{h}\|^2\|\mathbf{t}\| &= T \\ T\kappa &= P + \rho^{(f)}\mathbf{g} \cdot \mathbf{x}^{(0)} \end{aligned}$$

For incompressible materials the equations can be obtained by introducing an internal pressure p, replacing $\epsilon\frac{\partial W}{\partial I_2}$ with $\frac{1}{2}p^{(-1)}$, and using the constraint equation $\|\mathbf{t}\|\|\mathbf{h}\| = 1$.

3 The Thin Elastic Solid in Three Dimensions

Now we consider a three-dimensional thin elastic solid, whose undeformed shape is given by $\Omega_0 = \Omega_0^{\text{surf}} \times [-\epsilon, \epsilon]$, where Ω_0^{surf} is contained in the (X, Y) plane. We use the 2D analysis, in particular the 2D energy analysis, to motivate 3D analysis and derive the leading order governing equations for the 3D thin solid.

Let us define the rescaled thin direction by $\epsilon\hat{Z} = Z$, and expand the deformed position asymptotically in ϵ:

$$\mathbf{x}(X, Y, \hat{Z}) = \mathbf{x}^{(0)}(X, Y, Z) + \epsilon\mathbf{x}^{(1)}(X, Y, \hat{Z}) + \dots,$$

Before any analysis can be performed, we have to prove that the leading order deformation is independent of $\hat{Z}$.

CLAIM: $\mathbf{x}^{(0)}$ satisfies $\mathbf{x}^{(0)}(X, Y, \hat{Z}) \equiv \mathbf{x}^{(0)}(X, Y)$

PROOF: $\square$

Now, we know that

$$\mathbf{x}(X, Y, \hat{Z}) = \mathbf{x}^{(0)}(X, Y) + \epsilon\mathbf{x}^{(1)}(X, Y, \hat{Z}) + \dots$$

We define tangent vectors (not necessarily orthonormal) and the surface normal by

$$\mathbf{t}_1 = \frac{\partial \mathbf{x}^{(0)}}{\partial X} \qquad \mathbf{t}_2 = \frac{\partial \mathbf{x}^{(0)}}{\partial Y} \qquad \mathbf{n} = \frac{\mathbf{t}_1 \wedge \mathbf{t}_2}{\|\mathbf{t}_1 \wedge \mathbf{t}_2\|}$$

and, using $\mathbf{h} = \frac{\partial \mathbf{x}^{(1)}}{\partial \hat{Z}}$ again,

$$F^{(0)} = \begin{bmatrix} \frac{\partial \mathbf{x}^{(0)}}{\partial X} & \frac{\partial \mathbf{x}^{(0)}}{\partial Y} & \frac{\partial \mathbf{x}^{(1)}}{\partial \hat{Z}} \end{bmatrix} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{h}]$$

and

$$C^{(0)} = \begin{bmatrix} \|\mathbf{t}_1\|^2 & \mathbf{t}_1 \cdot \mathbf{t}_2 & \mathbf{t}_1 \cdot \mathbf{h} \\ \mathbf{t}_1 \cdot \mathbf{t}_2 & \|\mathbf{t}_2\|^2 & \mathbf{t}_2 \cdot \mathbf{h} \\ \mathbf{t}_1 \cdot \mathbf{h} & \mathbf{t}_2 \cdot \mathbf{h} & \|\mathbf{h}\|^2 \end{bmatrix}$$

The principal invariants are

$$\begin{aligned} I_1^{(0)} &= \|\mathbf{t}_1\|^2 + \|\mathbf{t}_2\|^2 + \|\mathbf{h}\|^2 \\ I_2^{(0)} &= \|\mathbf{t}_1\|^2\|\mathbf{t}_2\|^2 + \|\mathbf{t}_1\|^2\|\mathbf{h}\|^2 + \|\mathbf{t}_2\|^2\|\mathbf{h}\|^2 - (\mathbf{t}_1 \cdot \mathbf{t}_2)^2 - (\mathbf{t}_1 \cdot \mathbf{h})^2 - (\mathbf{t}_2 \cdot \mathbf{h})^2 \\ I_3^{(0)} &= \det(C^{(0)}) \end{aligned}$$

Recall that in 2D we needed to use the fact that $\frac{\partial I_2}{\partial \mathbf{h}} = \frac{\partial}{\partial \mathbf{h}}\left(\|\mathbf{t}\|^2\|\mathbf{h}\|^2 - (\mathbf{t} \cdot \mathbf{h})^2\right) = 2\|\mathbf{t}\|^2\mathbf{h} - 2(\mathbf{t} \cdot \mathbf{h})\mathbf{t}$ is orthogonal to $\mathbf{t}$. We now prove the corresponding result in 3D.

CLAIM: The third invariant I_3 satisfies $\frac{\partial I_3}{\partial \mathbf{h}} \cdot \mathbf{t}_\alpha = 0$, $\alpha = 1, 2$. Also, $\frac{\partial I_3}{\partial \mathbf{h}} \cdot \mathbf{h} = 2I_3$

PROOF: $I_3 = \det(C) = \det(F)^2$, so $\frac{\partial I_3}{\partial \mathbf{h}} = 2\det(F)\frac{\partial(\det(F))}{\partial \mathbf{h}}$. But $\frac{\partial(\det(F))}{\partial h_i} = \frac{\partial(\det(F))}{\partial F_{i3}} = \det(F)F_{3i}^{-1}$. Therefore, $\frac{\partial I_3}{\partial \mathbf{h}} \cdot \mathbf{t}_\alpha = 2(\det(F))^2 F_{3i}^{-1}(\mathbf{t}_\alpha)_i = 2(\det(F))^2 F_{3i}^{-1}F_{i\alpha} = 2(\det(F))^2\delta_{3\alpha} = 0$. Similarly, $\frac{\partial I_3}{\partial \mathbf{h}} \cdot \mathbf{h} = 2(\det(F))^2\delta_{33} = 2I_3$.

Now, consider a general energy minimisation problem of the form

$$\begin{aligned}
&\underset{\mathbf{x}}{\text{minimise}} \iiint_{\Omega_0} \text{ Strain Energy } \mathrm{d}V_0 + \text{ Remaining Energy} \\
\Rightarrow\ &\underset{\mathbf{x}}{\text{minimise}} \iiint_{\Omega_0} W(I_1(\|\mathbf{t}_\alpha\|^2, \|\mathbf{h}\|^2), I_2(\|\mathbf{t}_\alpha\|^2, \|\mathbf{h}\|^2), I_3(\|\mathbf{t}_\alpha\|^2, \|\mathbf{h}\|^2))\, \mathrm{d}V_0 + \text{ Remaining Energy} \\
\Rightarrow\ &\underset{\mathbf{x}}{\text{minimise}} \iint_{(X,Y)\in\Omega_0^{\text{surf}}} \int_{\hat{Z}=-1}^{\hat{Z}=1} W(I_1, I_2, I_3)\, \epsilon \mathrm{d}\hat{Z}\, \mathrm{d}X\, \mathrm{d}Y + \mathcal{J}\left(\mathbf{x}, \frac{\partial \mathbf{x}}{\partial X_M}\right)
\end{aligned}$$

For the asymptotic analysis, as in 2D we will take the leading order components of each term, and split the minimisation as a minimisation over $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ independently.

$$\underset{\mathbf{x}^{(0)},\, \mathbf{x}^{(1)}}{\text{minimise}} \iint_{(X,Y)\in\Omega_0^{\text{surf}}} \int_{\hat{Z}=-1}^{\hat{Z}=1} W(I_1^{(0)}, I_2^{(0)}, I_3^{(0)})\, \epsilon \mathrm{d}\hat{Z}\, \mathrm{d}X\, \mathrm{d}Y + \mathcal{J}^{(0)}\left(\mathbf{x}^{(0)}, \frac{\partial \mathbf{x}^{(0)}}{\partial X_M}, \dots\right)$$

We now show that the tangent vectors are both orthogonal to $\mathbf{h}$, so that $\mathbf{h}$ is in the normal direction by varying $\mathbf{x}^{(1)}$. Using an identical argument to that used in 2D, it can be shown that

$$\frac{\partial W}{\partial \mathbf{h}} = 0 \qquad \text{everywhere}$$

This equation determines $\mathbf{h}$ in terms of the tangent vectors. Consider the tangential components of the equation. Using the forms for I_1 and I_2, and the claim above, we have

$$\begin{aligned}
\frac{\partial W}{\partial \mathbf{h}} \cdot \mathbf{t}_1 = 0 \quad &\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{h}\cdot\mathbf{t}_1 + 2\frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_1\|^2\mathbf{h}\cdot\mathbf{t}_1 + \|\mathbf{t}_2\|^2\mathbf{h}\cdot\mathbf{t}_1 - (\mathbf{t}_1\cdot\mathbf{h})\mathbf{t}_1\cdot\mathbf{t}_1 - (\mathbf{t}_2\cdot\mathbf{h})\mathbf{t}_2\cdot\mathbf{t}_1\right) = 0 \\
&\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{t}_1\cdot\mathbf{h} + 2\frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_2\|^2\mathbf{t}_1\cdot\mathbf{h} - (\mathbf{t}_2\cdot\mathbf{h})(\mathbf{t}_1\cdot\mathbf{t}_2)\right) = 0 \\
\frac{\partial W}{\partial \mathbf{h}} \cdot \mathbf{t}_2 = 0 \quad &\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{t}_2\cdot\mathbf{h} + 2\frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_1\|^2\mathbf{h}\cdot\mathbf{t}_2 + \|\mathbf{t}_2\|^2\mathbf{h}\cdot\mathbf{t}_2 - (\mathbf{t}_1\cdot\mathbf{h})\mathbf{t}_1\cdot\mathbf{t}_2 - (\mathbf{t}_2\cdot\mathbf{h})\mathbf{t}_2\cdot\mathbf{t}_2\right) = 0 \\
&\Rightarrow \quad 2\frac{\partial W}{\partial I_1}\mathbf{t}_2\cdot\mathbf{h} + 2\frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_1\|^2\mathbf{t}_2\cdot\mathbf{h} - (\mathbf{t}_1\cdot\mathbf{h})(\mathbf{t}_1\cdot\mathbf{t}_2)\right) = 0
\end{aligned}$$

CLAIM: $\mathbf{t}_1\cdot\mathbf{h} = 0$ and $\mathbf{t}_2\cdot\mathbf{h} = 0$

PROOF: ** is two simultaneous equations for the two unknowns $\mathbf{t}_1\cdot\mathbf{h}$, $\mathbf{t}_2\cdot\mathbf{h}$. Suppose we write these as

$$\alpha\mathbf{t}_1\cdot\mathbf{h} + \beta\mathbf{t}_2\cdot\mathbf{h} = 0 \qquad \beta\mathbf{t}_1\cdot\mathbf{h} + \delta\mathbf{t}_2\cdot\mathbf{h} = 0$$

where $\alpha = \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\|\mathbf{t}_2\|^2$, $\delta = \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\|\mathbf{t}_1\|^2$ and $\beta = -\frac{\partial W}{\partial I_2}\mathbf{t}_1\cdot\mathbf{t}_2$. The claim must be true unless the determinant $\alpha\delta - \beta^2 = 0$. Suppose we divide through by $\frac{\partial W}{\partial I_2}$ and consider this as a quadratic equation in $\frac{\partial W}{\partial I_1}/\frac{\partial W}{\partial I_2}$. (If $\frac{\partial W}{\partial I_2} = 0$ then the claim holds trivially). Rescaling position so that $\|\mathbf{t}_1\| = 1$, setting $c = \|\mathbf{t}_2\|$, $\mathbf{t}_1\cdot\mathbf{t}_2 = c\cos\theta$ and the unknown as $\xi = \frac{\partial W}{\partial I_1}/\frac{\partial W}{\partial I_2}$, then

$$\alpha\delta - \beta^2 = 0 \qquad \Rightarrow \qquad (\xi + 1)(\xi + c^2) - c^2\cos^2\theta = 0$$

The solutions of this quadratic are $2\xi = -(c^2+1) \pm \sqrt{(c^2+1)^2 - 4c^2(1-\cos^2\theta)}$. These two solutions always exist, but, since $-4c^2(1-\cos^2\theta) \leq 0$, are always negative. Therefore, if a solution of the $\alpha\delta - \beta^2 = 0$ exists, we can say we must have $\frac{\partial W}{\partial I_1}$ and $\frac{\partial W}{\partial I_2}$ are different signs.

but they must both be positive???????**

The third component of $\frac{\partial W}{\partial \mathbf{h}}$ gives the relationship between $\|\mathbf{h}\|$ and $\|\mathbf{t}_1\|$ and $\|\mathbf{t}_2\|$.

$$\begin{aligned}\frac{\partial W}{\partial \mathbf{h}} \cdot \mathbf{h} = 0 \quad & \Rightarrow \quad 2\frac{\partial W}{\partial I_1}\|\mathbf{h}\|^2 + 2\frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_1\|^2\|\mathbf{h}\|^2 + \|\mathbf{t}_2\|^2\|\mathbf{h}\|^2\right) + 2\frac{\partial W}{\partial I_3} I_3 = 0 \\ & \Rightarrow \quad \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\left(\|\mathbf{t}_1\|^2 + \|\mathbf{t}_2\|^2\right) + \frac{\partial W}{\partial I_3}\left(\|\mathbf{t}_1\|^2\|\mathbf{t}_2\|^2 - (\mathbf{t}_1 \cdot \mathbf{t}_2)^2\right) = 0\end{aligned}$$

Alternatively, if we write $C^{(0)}$ as

$$C^{(0)} = \begin{bmatrix} \|\mathbf{t}_1\|^2 & \mathbf{t}_1 \cdot \mathbf{t}_2 & 0 \\ \mathbf{t}_1 \cdot \mathbf{t}_2 & \|\mathbf{t}_1\|^2 & 0 \\ 0 & 0 & \|\mathbf{h}\|^2 \end{bmatrix} = \begin{bmatrix} C^{(\mathrm{surf})} & 0 \\ 0 & \|\mathbf{h}\|^2 \end{bmatrix}$$

then

$$\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\mathrm{tr}(C^{(\mathrm{surf})}) + \frac{\partial W}{\partial I_3}\det(C^{(\mathrm{surf})}) = 0$$

So again we have shown that $\mathbf{h}$ is in the $\mathbf{n}$ direction, and derived a relationship between $\|\mathbf{h}\|$ and $\|\mathbf{t}_1\|$ and $\|\mathbf{t}_2\|$ that must hold for all deformations. Again, since $\mathbf{h}$ is completely determined by $\mathbf{t}_1$ and $\mathbf{t}_2$, we can say that $\mathbf{h} \equiv \mathbf{h}(X, Y)$ only. It follows that

$$\iiint_{\Omega_0} W(I_1^{(0)}, I_2^{(0)}, I_3^{(0)})\,\mathrm{d}V_0 = 2\epsilon \iint_{\Omega_0^{\mathrm{surf}}} W(I_1^{(0)}, I_2^{(0)}, I_3^{(0)})\,\mathrm{d}X\,\mathrm{d}Y$$

We are now ready to study some specific problems.

Elastic solid containing an incompressible fluid

To impose the constraint that the volume under the membrane is constant we need to add a constraint term $\lambda \int z\,\mathrm{d}x\,\mathrm{d}y$ to the energy, where in this equation x, y and z are evaluated at the lower surface of the of the solid. Transforming back to reference coordinates and taking the leading order contribution the constraint is

$$\lambda \iint_{\Omega_0^{\mathrm{surf}}} z^{(0)}\left(\frac{\partial x^{(0)}}{\partial X}\frac{\partial y^{(0)}}{\partial Y} - \frac{\partial x^{(0)}}{\partial Y}\frac{\partial y^{(0)}}{\partial X}\right)\mathrm{d}X\,\mathrm{d}Y$$

so that the leading order energy is

$$2\epsilon \iint_{\Omega_0^{\mathrm{surf}}} W(I_1^{(0)}, I_2^{(0)}, I_3^{(0)})\,\mathrm{d}X\,\mathrm{d}Y + \lambda \iint_{\Omega_0^{\mathrm{surf}}} z^{(0)}\left(\frac{\partial x^{(0)}}{\partial X}\frac{\partial y^{(0)}}{\partial Y} - \frac{\partial x^{(0)}}{\partial Y}\frac{\partial y^{(0)}}{\partial X}\right)\mathrm{d}X\,\mathrm{d}Y$$

The Euler-Lagrange equations are (if $\mathcal{I}$ is the integrand for the total energy)

$$\frac{\partial}{\partial X}\left(\frac{\partial \mathcal{I}}{\partial \mathbf{t}_1}\right) + \frac{\partial}{\partial Y}\left(\frac{\partial \mathcal{I}}{\partial \mathbf{t}_2}\right) - \frac{\partial \mathcal{I}}{\partial \mathbf{x}^{(0)}} = 0$$

After a little algebra, it can be shown that the Euler-Lagrange equations for this case are

$$\frac{\partial}{\partial X}\left(2\epsilon\frac{\partial W}{\partial \mathbf{t}_1}\right) + \frac{\partial}{\partial Y}\left(2\epsilon\frac{\partial W}{\partial \mathbf{t}_2}\right) - \lambda \mathbf{t}_1 \wedge \mathbf{t}_2 = 0$$

Elastic solid containing a gravity-loaded incompressible fluid

Here, we need to add to the energy the gravitational potential energy of the fluid, which is an integral over the volume below the thin solid.

$$\mathrm{GPE}^{(\mathrm{f})} = \iiint_{fluid} \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{z} \, \mathrm{d}V$$

Using the divergence theorem, and taking the leading order component,

$$\mathrm{GPE}^{(\mathrm{f})} = \frac{1}{4} \iint_{\Omega^{(\mathrm{surf})}} \rho^{(\mathrm{f})} \left(\mathbf{g} \cdot \mathbf{x}^{(0)}\right) \left(\mathbf{x}^{(0)} \cdot \hat{\mathbf{n}}\right) \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 = \frac{1}{4} \iint_{\Omega^{(\mathrm{surf})}} \rho^{(\mathrm{f})} \left(\mathbf{g} \cdot \mathbf{x}^{(0)}\right) \left(\mathbf{x}^{(0)} \cdot \mathbf{n}\right) \mathrm{d}X \, \mathrm{d}Y$$

Again, we neglect the term which is an integral over the lower fixed surface, as it is constant and does not affect the variational formulation.

After a lot of algebra it can be shown that, where $\mathcal{G} = (\mathbf{g} \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{n})$,

$$\frac{\partial}{\partial X} \left(\frac{\partial \mathcal{G}}{\partial \mathbf{t}_1} \right) + \frac{\partial}{\partial Y} \left(\frac{\partial \mathcal{G}}{\partial \mathbf{t}_2} \right) - \frac{\partial \mathcal{G}}{\partial \mathbf{x}} = -4(\mathbf{g} \cdot \mathbf{x})\mathbf{n}$$

so that the new Euler-Lagrange equations are

$$\frac{\partial}{\partial X} \left(2\epsilon \frac{\partial W}{\partial \mathbf{t}_1} \right) + \frac{\partial}{\partial Y} \left(2\epsilon \frac{\partial W}{\partial \mathbf{t}_2} \right) - \left(\lambda + \rho^{(\mathrm{f})} \mathbf{g} \cdot \mathbf{x}^{(0)} \right) \mathbf{t}_1 \wedge \mathbf{t}_2 = 0$$