## 5. The Jacobson Radical

We want to study finite-dimensional algebras $R$ which are not semi-simple. There are no such general structure theorems as Wedderburn's Theorem. But as a 'first approximation' one can ask how close is $R$ from being semi-simple. A natural question: Is there a largest factor algebra of $R$ which is semi-simple? Answer- yes, and the ideal to factor out is the 'Jacobson Radical'.

**5.1 Definition** The *Jacobson radical* (or just 'radical') of $R$ is defined to be $J(R) :=$ the intersection of all maximal left ideals of $R$. This is then a left ideal but it is even an ideal, by the following theorem.

Every left ideal $I$ of $R$ with $I \neq R$ is contained in some maximal left ideal $M$ of $R$ (by JH for $R$ finite-dimensional, even in general for $R$ with identity, by Zorn's Lemma). So $J(R) \neq R$.

**5.2 Theorem** *We have*

$$J(R) = \{a \in R : aS = 0 \textit{ for all simple } R\textit{-modules } S\}.$$

*In particular $J(R)$ is a 2-sided ideal of $R$.*

*Proof* $\supseteq$ Suppose $a \in R$ and $aS = 0$ for all simple $R$-modules $S$.
Take a maximal left ideal $M$ of $R$, then $R/M$ is a simple $R$-module and hence $a(R/M) = 0$. That is $a(1 + M) = 0 + M$ and $a \in M$. So $a \in J(R)$.

Assume for a contradiction that the RHS is a proper subset of $J(R)$. Then there is a simple module $S$ such that $J(R)S \neq 0$. Pick $s \in S$ such that $J(R)s \neq 0$. But $J(R)s$ is a submodule of $S$ and $S$ is simple and it follows that $J(R)s = S$. Then

$$xs = s$$

for some $x \in J(R)$. So $(x - 1)s = 0$ and $(x - 1)$ belongs to the left ideal $\mathrm{Ann}(s)$ of $R$. This is not equal to $R$ and therefore

$$\mathrm{Ann}(s) \subseteq M \subset R$$

for some maximal left ideal $M$. We also have $x \in J(R) \subseteq M$ and therefore $1 = x-(x-1) \in M$, and $M = R$, a contradiction.

**5.2.1 Example** Let $R = KG$, where $G$ is a group of order $p^n$ and $K$ is a field of size $q = p^k$.
*Then the trivial module is the only simple $R$-module* :
Suppose $V$ is a simple $R$-module. View the set $V$ as a $G$-set (by $g(x) := v_g x$). Then $V$ is a disjoint union of orbits. The size of an orbit is a power of $p$ and also $|V| = q^d = p^{kd}$ where $d = \dim(V)$. So the number of orbits of size 1 is divisible by $p$.
There is at least one such orbit, namely $\{0\}$. So there is some $0 \neq v \in V$ with $v_g x = v$ for all $g \in G$. Then $v$ spans a 1-dimensional trivial submodule of $V$ and hence $V = \langle v \rangle$.

$J(R) = \mathrm{span}\{v_g - 1 : 1 \neq g \in G\}$:
Each $v_g - 1$ annihilates the trivial module, so by 5.3 lies in $J(R)$ and since $J(R)$ is an ideal the span is contained in $J(R)$.
The set $\{v_g - 1 : 1 \neq g \in G\}$ is linearly independent and has $|G| - 1$ elements in it. And $\dim(R) = |G|$. But $J(R) \neq R$, so we must have equality.

**5.3 'Construction' of** $J(R)$ Recall $R$ is finite-dimensional. Then one can express $J(R)$ as the intersection of finitely many maximal left ideals:
We aim to construct maximal left ideals $M_i$ of $R$ with

$$(*) \qquad M_1 \supset M_1 \cap M_2 \supset M_1 \cap M_2 \cap M_3 \supset \dots$$

Suppose $M_1, \dots, M_r$ are found and let $N_r := \cap_{i=1}^r M_i$. Suppose there is some maximal left ideal $M$ with $N_r \not\subset M$. Then $M \cap N_r \subset N_r$. Set $M_{r+1} = M$ and continue $(*)$.
Since $R$ is finite-dimensional, this must stop. So for some $n$ the intersection $N_n$ contains all maximal left ideals $M$ of $R$. And then $N_n$ is automatically the intersection of *all* maximal left ideals of $R$, ie $\cap_{i=1}^n M_i = N_n = J(R)$.

**5.4** (Reminder) Suppose $I$ is a left ideal of $R$. Then
(a) $R/I$-modules are the same as $R$-modules $M$ such that $IM = 0$ (see qu. 1 sheet 1). This only changes point of view. So 'simple' remains 'simple', ie the simple $R/I$-modules are precisely the simple $R$-modules $S$ with $IS = 0$. Same with 'semi-simple'.
(b) Recall the left ideal correspondence. Submodules of $R/I$ are in 1-1 correspondence with submodules of $R$ containing $I$. In particular $M/I$ maximal in $R/I$ iff $I \subseteq M$ and $M$ maximal in $R$.

**5.5 Theorem** *(i) The algebra $R/J(R)$ is semi-simple.*
*(ii) If $I$ is a 2-sided ideal of $R$ such that $R/I$ is semi-simple then $J(R) \subseteq I$.*

**5.5.1 Corollary** *$R$ is semi-simple if and only if $J(R) = 0$.*
(Proof of the Corollary: $\Leftarrow$ by (i). $\Rightarrow$ Take $I = \{0\}$ in (ii).)

*Proof of 5.5* (i) To show that $R/J(R)$ is semi-simple as $R/J(R)$-module. By 5.4(a) we must show that it is semisimple as an $R$-module.
Take $M_1, \ldots, M_n$ as in 5.3. Define $\phi : R/J(R) \to R/M_1 \oplus R/M_2 \oplus \ldots \oplus R/M_n$ by $\phi(x + J(R)) := (x + M_1, x + M_2, \ldots, x + M_n)$. This is a (well-defined) $R-$homomorphism and it is 1-1 (check, this is because of the construction 5.3).
Both sides have the same composition factors:
$M_r \subset W := \cap_{i=1}^{r-1} M_i + M_r \subseteq R$ and $M_r$ is maximal but not $= W$, so the middle is $R$ and then

$$R/M_r = W/M_r \cong \cap_{i=1}^{r-1} M_i / \cap_{i=1}^{r} M_i.$$

by some isomorphism theorem. It follows that $\phi$ is an isomorphism.

(ii) By the hypothesis $R/I = L_1 \oplus L_2 \oplus \ldots \oplus L_n$ with $L_i$ simple $R/I-$modules and then simple $R$-modules. Let $a \in J(R)$, by 5.3 we know $aL_i = 0$ for all $i$ and then $a(R/I) = 0$ and $a(1 + I) = 0 + I$ and $a \in I$. So $J(R) \subseteq I$.

**5.6 Corollary** *An $R$-module $M$ is semisimple if and only if $J(R)M = 0$.*

*Proof* $\Rightarrow$ By 5.2 we know $J(R)M = 0$.
$\Leftarrow$ If $J(R)M = 0$ then we can view $M$ as a module for the algebra $R/J(R)$. This algebra is semi-simple and hence every $R/J(R)$-module is semi-simple and so is $M$. Then $M$ is also simple as an $R$-module (see 5.4).

**5.7** What is a 'smallest possible' module which is **not** semi-simple?
Consider $M$ with two composition factors. When is $M$ not semi-simple? It must have a simple submodule (take first step of a comp-series).
Can $M$ have two simple submodules $S_1 \neq S_2$?
If so then $S_1 \cap S_2 = \{0\}$ and then $M = S_1 \oplus S_2$ and is semi-simple! !!!
So if $M$ is not semi-simple then it can only have one simple submodule and hence $M$ has a unique composition series. [Conversely if $M$ so then $M$ is even indecomposable and not simple and therefore not semi-simple.]

Corollary 5.7 says that $M$ is not semi-simple if and only if $J(R)M \neq 0$. But $J(R)M \neq M$ (if $N$ is a maximal submodule of $M$ then $J(R)(M/N) = 0$ since $M/N$ is simple and so $J(R)M \subseteq N$).
So for $M$ with two composition factors which is not semi-simple we must have that $0 \neq J(R)M \subset M$ is a composition series, and it is the only composition series.

**5.9 More Examples** (1) Supppose $R$ is local. Then $U^c(R)$ is the unique maximal submodule
See next sheet.
(2) $R = T_n(K)$, then $J(R) = N$, the strict upper triangular matricesi. Proof later.
(3) $R = K\mathcal{Q}$, the path algebra of a quiver $\mathcal{Q}$ (It is finite-dimensional if and only if no oriented cycles). Then $J(R)$ is the span of all paths of length $\geq 1$. Proof later.