Lecture 1: Introduction to finite difference methods

Mike Giles

University of Oxford

Parabolic heat equation

We start with the simple 1D parabolic PDE which describes the change in non-dimensional temperature of a 1D rod

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$$

to be solved on $0 < x < 1$, subject to some initial conditions $V_0(x)$ at time $t{=}0$, and $V(0,t){=}V(1,t){=}0$ on the two ends.

Parabolic heat equation

The exact solution can be expressed as a combination of Fourier modes:

$$V(x,t) = \sum_{m>0} A_m \sin(m\pi x) \exp\left(-m^2\pi^2 t\right)$$

in which the amplitudes are given by

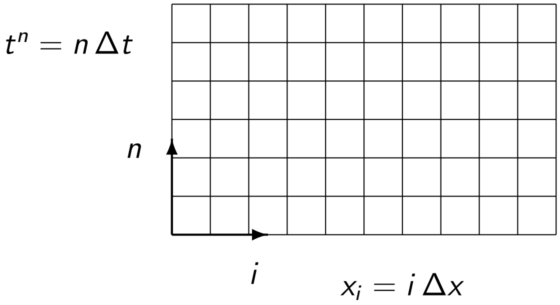
$$A_{m'} = 2 \int_0^1 V_0(x) \sin(m'\pi x)\, \mathrm{d}x$$

since

$$\int_0^1 \sin(m\pi x)\ \sin(m'\pi x)\, \mathrm{d}x = \begin{cases} \frac{1}{2}, & \text{if } m{=}m' \\ 0, & \text{otherwise} \end{cases}$$

Finite Difference Approximation

Suppose we use a computational grid with spacing $\Delta x = 1/K$ and timestep Δt:



Finite Difference Approximation

We want to construct an approximation $V_i^n \approx V(x_i, t^n)$.

To do this, we note that

$$V(x_i, t^n + \Delta t) \approx V(x_i, t^n) + \Delta t \left.\frac{\partial V}{\partial t}\right|_{(x_i,t^n)}$$

$$\Longrightarrow \quad \frac{\partial V}{\partial t} \approx \frac{1}{\Delta t}\left(V_i^{n+1} - V_i^n\right)$$

and also

$$V(x_i \pm \Delta x, t^n) \approx V(x_i, t^n) \pm \Delta x \left.\frac{\partial V}{\partial x}\right|_{(x_i,t^n)} + \tfrac{1}{2}\Delta x^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{(x_i,t^n)}$$

$$\Longrightarrow \quad \frac{\partial^2 V}{\partial x^2} \approx \frac{1}{\Delta x^2}\left(V_{i+1}^n - 2\, V_i^n + V_{i-1}^n\right)$$

Finite Difference Approximation

Putting these together gives the explicit forward-time central-space approximation of the heat equation:

$$\frac{1}{\Delta t}\left(V_i^{n+1} - V_i^n\right) = \frac{1}{(\Delta x)^2}\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right)$$

which, setting $\lambda = \Delta t/(\Delta x)^2$, can be re-arranged to give

$$\begin{aligned} V_i^{n+1} &= V_i^n + \lambda\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) \\ &= (1-2\lambda)V_i^n + \lambda\left(V_{i+1}^n + V_{i-1}^n\right) \end{aligned}$$

together with zero values for V_i^n on the two ends.

Finite Difference Approximation

The finite difference solution also has a Fourier mode decomposition of the form

$$V_i^n = \sum_{0<m<1/\Delta x} A_m^n \sin(m\pi x_i)$$

where the amplitudes A_m^n satisfy the equation

$$A_m^{n+1} = \left(1 - 4\lambda \sin^2(\tfrac{1}{2} m\Delta x)\right) A_m^n$$

We know the amplitudes should decay exponentially – the condition for this to happen is

$$-1 < \left(1 - 4\lambda \sin^2(\tfrac{1}{2} m\Delta x)\right) < 1$$

which requires

$$4\,\lambda \leq 2 \quad \Longrightarrow \quad \Delta t \leq \tfrac{1}{2}(\Delta x)^2$$

Finite Difference Approximation

This timestep stability limit also follows directly from

$$V_i^{n+1} = (1-2\lambda) V_i^n + \lambda \left(V_{i+1}^n + V_{i-1}^n \right)$$

since for $\lambda \leq 1/2$ we get

$$\begin{aligned} |V_i^{n+1}| &\leq (1-2\lambda)\, |V_i^n| + \lambda\, |V_{i+1}^n| + \lambda\, |V_{i-1}^n| \\ &\leq (1-2\lambda)\, \max_{i'} |V_{i'}^n| + \lambda\, \max_{i'} |V_{i'}^n| + \lambda\, \max_{i'} |V_{i'}^n| \\ &\leq \max_{i'} |V_{i'}^n| \end{aligned}$$

so

$$\max_i |V_i^{n+1}| \;\leq\; \max_i |V_i^n|$$

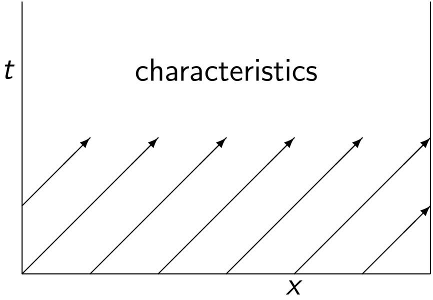
Hyperbolic equation

Another simple example is the 1D hyperbolic PDE which models convection:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} = 0$$

to be solved again on $0<x<1$, subject to some initial conditions $V_0(x)$ at time $t=0$, and $V(0,t)=0$ on the left-hand end.

The exact solution has the form $V(x,t) = V_0(x-t)$ with $V_0(-t) = 0$.



Finite Difference Approximation

Using

$$\frac{\partial V}{\partial t} \approx \frac{1}{\Delta t}\left(V_i^{n+1} - V_i^n\right)$$

and

$$\frac{\partial V}{\partial x} \approx \frac{1}{\Delta x}\left(V_i^n - V_{i-1}^n\right)$$

gives the explicit upwind discretisation

$$\frac{1}{\Delta t}\left(V_i^{n+1} - V_i^n\right) + \frac{1}{\Delta x}\left(V_i^n - V_{i-1}^n\right) = 0$$

Finite Difference Approximation

Setting $\lambda = \Delta t / \Delta x$, this can be re-arranged to give

$$\begin{aligned} V_i^{n+1} &= V_i - \lambda \left(V_i^n - V_{i-1}^n \right) \\ &= (1-\lambda)\, V_i^n + \lambda\, V_{i-1}^n \end{aligned}$$

together with $V_0^n = 0$.

If $\lambda \leq 1$ this is stable since we again get

$$\max_i |V_i^{n+1}| \;\leq\; \max_i |V_i^n|$$

Parabolic heat equation

As a 2D model problem, we consider the simple parabolic PDE which describes the change in non-dimensional temperature of a 2D plate

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

to be solved on the unit square $0<x<1$, $0<y<1$ subject to some initial conditions $V_0(x, y)$ at time $t=0$, and $V=0$ on the boundaries

Parabolic heat equation

The exact solution can be expressed as a combination of Fourier modes:

$$V(x,y,t) = \sum_{m,n>0} A_{m,n} \sin(m\pi x) \sin(n\pi y) \exp\left(-(m^2{+}n^2)\pi^2 t\right)$$

in which the amplitudes are given by

$$A_{m',n'} = 4 \int_0^1 \int_0^1 V_0(x,y)\, \sin(m'\pi x)\, \sin(n'\pi y)\, \mathrm{d}x\, \mathrm{d}y$$

since

$$\int_0^1 \int_0^1 \sin(m\pi x)\, \sin(n\pi y)\ \sin(m'\pi x)\, \sin(n'\pi y)\ \mathrm{d}x\, \mathrm{d}y$$

$$= \begin{cases} \frac{1}{4}, & \text{if } m{=}m', n{=}n' \\ 0, & \quad \text{otherwise} \end{cases}$$

Laplace equation

Suppose $U(x,y)$ is the solution to the 2D Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

subject to specified values on the boundary.

If $W(x,y,t)$ is the solution of the parabolic PDE subject to those same boundary values, then $V \equiv W - U$ satisfies the parabolic PDE with zero boundary conditions.

Since $V(x,y,t) \to 0$ as $t \to \infty$, $W(x,y,t) \to U(x,y)$ as $t \to \infty$, which gives us one approach to approximating solutions of the Laplace equation.

Finite Difference Approximation

Using a similar explicit forward-time central-space approximation of the heat equation, with $\Delta x = \Delta y$, gives:

$$\begin{aligned}
\frac{1}{\Delta t}\left(V_{i,j}^{n+1} - V_{i,j}^{n}\right) \;&=\; \frac{1}{(\Delta x)^2}\left(V_{i+1,j}^{n} - 2V_{i,j}^{n} + V_{i-1,j}^{n}\right) \\
&\quad + \frac{1}{(\Delta x)^2}\left(V_{i,j+1}^{n} - 2V_{i,j}^{n} + V_{i,j-1}^{n}\right)
\end{aligned}$$

which, setting $\lambda = \Delta t/(\Delta x)^2$, can be re-arranged to give

$$\begin{aligned}
V_{i,j}^{n+1} \;&=\; V_{i,j} + \lambda\left(V_{i+1,j}^{n} - 2V_{i,j}^{n} + V_{i-1,j}^{n}\right) \\
&\quad + \lambda\left(V_{i,j+1}^{n} - 2V_{i,j}^{n} + V_{i,j-1}^{n}\right) \\
&=\; (1-4\lambda)V_{i,j}^{n} + \lambda\left(V_{i+1,j}^{n} + V_{i-1,j}^{n} + V_{i,j+1}^{n} + V_{i,j-1}^{n}\right)
\end{aligned}$$

together with zero values for $V_{i,j}^{n}$ on the boundary.

Finite Difference Approximation

It can be shown that the finite difference solution also has a Fourier mode decomposition of the form

$$V_{i,j}^n = \sum_{0<k,m<1/\Delta x} A_{k,m}^n \sin(k\pi x_i) \sin(m\pi y_j)$$

where the amplitudes $A_{k,m}^n$ satisfy the equation

$$A_{k,m}^{n+1} = \left(1 - 4\lambda \sin^2(\tfrac{1}{2}k\Delta x) - 4\lambda \sin^2(\tfrac{1}{2}m\Delta x)\right) A_{k,m}^n$$

We know the amplitudes should decay exponentially – the condition for this to happen is

$$-1 < \left(1 - 4\lambda \sin^2(\tfrac{1}{2}k\Delta x) - 4\lambda \sin^2(\tfrac{1}{2}m\Delta x)\right) < 1$$

which requires

$$8\,\lambda \leq 2 \quad \Longrightarrow \quad \Delta t \leq \tfrac{1}{4}(\Delta x)^2$$

Finite Difference Approximation

When trying to solve the Laplace equation it is best to make λ (and hence Δt) as big as possible, while remaining stable.

Putting $\lambda = 1/4$ gives the Jacobi iteration

$$V_{i,j}^{n+1} = \frac{1}{4}\left(V_{i+1,j}^{n} + V_{i-1,j}^{n} + V_{i,j+1}^{n} + V_{i,j-1}^{n}\right)$$

This is applied to interior points; the values of the boundary points are fixed so for those we use simply

$$V_{i,j}^{n+1} = V_{i,j}^{n}$$

(There are other more efficient iterative methods, such as Conjugate Gradient and Multigrid, but we won't cover those here.)

Black-Scholes PDE in finance

The Black-Scholes PDE in mathematical finance for the value of a European option based on two financial assets each modelled by Geometric Brownian Motion is

$$
\begin{aligned}
\frac{\partial V}{\partial t} \;=\;& r\,V - r\,S_1 \frac{\partial V}{\partial S_1} - r\,S_2 \frac{\partial V}{\partial S_2} \\
& -\sigma^2 \left(\tfrac{1}{2} S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \tfrac{1}{2} S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right)
\end{aligned}
$$

where r is the risk-free interest rate, σ is the volatility, and ρ is the correlation between the motion of the two assets

This is solved backwards in time on $(0, S_{max}) \times (0, S_{max}) \times (0, T)$ with the final value $V(S_1, S_2, T)$ equal to the payoff function, and subject to some boundary conditions at $S_1 = S_{max}$, $S_2 = S_{max}$, to get the value $V(S_1, S_2, 0)$ at the initial time $t = 0$.

Black-Scholes PDE

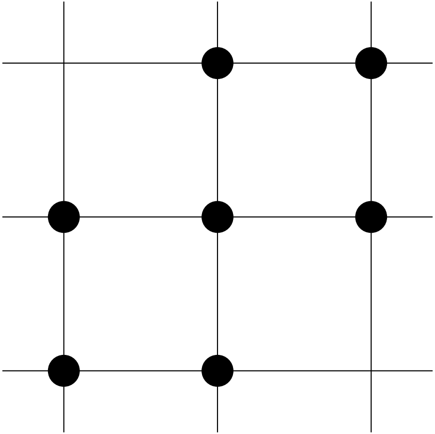
If $\rho > 0$ then a simple explicit Euler central space discretisation on a uniform grid is

$$
\begin{aligned}
V_{i,j}^{n+1} \;=\;& (1 - r\Delta t) V_{i,j}^{n} \\
&+ \frac{r\Delta t}{2\Delta S} \left(S_{1,i} \left(V_{i+1,j}^{n} - V_{i-1,j}^{n} \right) + S_{2,j} \left(V_{i,j+1}^{n} - V_{i,j-1}^{n} \right) \right) \\
&+ \frac{\sigma^2 S_{1,i}^2 \Delta t}{2\Delta S^2} \left(V_{i+1,j}^{n} - 2V_{i,j}^{n} + V_{i-1,j}^{n} \right) \\
&+ \frac{\sigma^2 S_{2,j}^2 \Delta t}{2\Delta S^2} \left(V_{i,j+1}^{n} - 2V_{i,j}^{n} + V_{i,j-1}^{n} \right) \\
&+ \frac{\rho\sigma^2 S_{1,j} S_{2,i}^2 \Delta t}{2\Delta S^2} \left(\left(V_{i+1,j+1}^{n} - V_{i+1,j}^{n} - V_{i,j+1}^{n} + V_{i,j}^{n} \right) \right. \\
&\qquad\qquad\qquad \left. + \left(V_{i,j}^{n} - V_{i-1,j}^{n} - V_{i,j-1}^{n} + V_{i-1,j-1}^{n} \right) \right)
\end{aligned}
$$

with time going backwards so that $t^n \equiv T - n\Delta t$.

Black-Scholes PDE

This gives a 7-point stencil:



Black-Scholes PDE

No boundary conditions are need when $i{=}0$ or $j{=}0$, because $V_{-1,j}$ and $V_{i,-1}$ are not used.

The boundary conditions on $i{=}I$ and $j{=}J$ are trickier, and a little unusual.

On $i{=}I$ usually omit term $(V^n_{I+1,j} - 2V^n_{I,j} + V^n_{I-1,j})$ on the basis that $\dfrac{\partial^2 V}{\partial S_1^2} \approx 0$, for most European options, and replace

$(V^n_{I+1,j+1} - V^n_{I+1,j} - V^n_{I,j+1} + V^n_{I,j})$ by $(V^n_{I,j} - V^n_{I-1,j} - V^n_{I,j-1} + V^n_{I-1,j-1})$,

and $V^n_{I+1,j} - V^n_{I,j}$ with $V^n_{I,j} - V^n_{I-1,j}$ with similar changes when $j{=}J$.