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Analytic Adjoint Solutions for the Quasi-1D Euler Equations

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The analytic properties of adjoint solutions are examined for the quasi-1D Euler equations. For shocked flow, the derivation of the adjoint problem reveals that the adjoint variables are continuous with zero gradient at the shock, and that an internal adjoint boundary condition is required at the shock. A Green's function approach is used to derive the analytic adjoint solutions corresponding to supersonic, subsonic, isentropic and shocked transonic flows in a converging-diverging duct of arbitrary shape. This analysis reveals a logarithmic singularity at the sonic throat and confirms the expected properties at the shock.

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1 Introduction

Adjoint equations arise naturally in the formulation of methods for optimal aerodynamic design. A single adjoint solution provides the linear sensitivities of an objective function, such as lift or drag, to perturbations in the multiple design variables which parameterise the aerodynamic shape. These sensitivities can then be used to drive a gradient-based optimisation procedure.

To outline the approach, we start with a system of nonlinear partial differential equations (e.g. the Euler equations or compressible Navier-Stokes equations) describing a steady flow within some given computational domain. When calculating the two-dimensional flow around an aerofoil, one technique is to use curvilinear coordinates (ξ, η) in which the aerofoil surface corresponds to $\eta = 0$ (Jameson 1995). Using these coordinates, the p.d.e. can be written as

$$R(U) = 0, \tag{1.1}$$

where the coefficients of terms within the differential operator R depend on both U and the mapping from (ξ, η) to the Cartesian coordinates (x, y), which in turn depends on the geometry of the aerofoil. Perturbing the geometry changes the mapping, and hence the coefficients. Linearising the operator $R(U)$ then leads to the linear p.d.e.

$$Lu = f, \tag{1.2}$$

where f is due to the change in the mapping, and u is the resultant linear perturbation to the flow field.

In design optimisation, one is interested in the consequential change to some objective function which is to be minimised. Usually, this objective involves an integral over the boundary of the domain, as in the case of drag minimisation. However, to simplify this exposition, we will take the objective function $J(U)$ to be an integral over the whole domain Ω, whose linear perturbation $I(u)$ can then be written as an inner product over the domain,

$$I(u) = (g, u),$$

for some given function $g(\xi, \eta)$.

Using a direct approach to design, $I(u)$ is determined separately for each design variable by defining the appropriate geometric perturbation f and solving Equation (1.2) for u. In the adjoint approach, one evaluates the perturbed functional without explicitly calculating the perturbed flow field u. This is achieved by using by using an augmented functional

$$I = (g, u) - (v, Lu - f),$$

in which the continuous Lagrange multipliers v have been introduced to enforce the constraint that u must satisfy Equation (1.2). The adjoint linear operator L^* is defined by the identity

$$(v, Lu) = (L^* v, u),$$

for all u, v satisfying the appropriate homogeneous boundary conditions. Using this identity, one obtains

$$I = (v, f) - (L^*v - g, u) = (v, f),$$

provided v is the solution of the adjoint equation

$$L^*v = g. \tag{1.3}$$

The adjoint approach provides exactly the same final answer as the direct linear perturbation analysis. The benefit of the adjoint approach is that the computational cost can be significantly lower. If there are N design variables, then a direct approach requires N solutions of Equation (1.2), each with a different function F, to obtain the linear flow perturbations u. On the other hand, with the adjoint approach, Equation (1.3) has to be solved only once for the function g corresponding to the objective function of interest. Since solving Equations (1.2) and (1.3) requires roughly equal computational effort, the overall savings become substantial as the number of design variables increases.

In the last ten years, considerable effort has been devoted to the development of optimal design methods based on the adjoint approach. Some methods use curvilinear coordinates and the differential adjoint, as outlined above (see e.g. Jameson 1988, 1995, 1999; Reuther *et al.* 1996, 1999a,b; Jameson, Pierce & Martinelli 1998). Other methods first discretise the nonlinear p.d.e. and then use the adjoint (transpose) of the linear discrete matrix operator (Elliot & Peraire 1997, Anderson & Bonhaus 1999). For a more comprehensive introduction to adjoint methods in aerodynamic design and a discussion of the relative advantages of the two main approaches, see Giles & Pierce (2000). For a review of the latest developments in design optimisation using adjoint equations, see Newman *et al.* (1999).

Recently, adjoint solutions have been recognised as providing a means of computing and minimising errors in fluid dynamics simulations, and in particular the errors in integral outputs such as lift and drag. Suppose U_h is an approximate numerical solution of Equation (1.1). Defining u to be the numerical error (the difference between the numerical and analytic solutions) gives

$$R(U_h - u) = 0.$$

Linearisation about the numerical solution then yields

$$Lu = f, \quad f \equiv R(U_h).$$

Defining the adjoint solution in the same way as before, the leading order error in the integral objective function is given by

$$(g, u) = (v, f) = (v, R(U_h)).$$

This result can be used in grid adaptation, for example by refining any cell in which an estimate of the local product $v^T R(U_h)$ multiplied by the cell area exceeds some threshold, to try to achieve the maximum reduction in the magnitude of the error for a given computational effort (Johnson *et al.* 1995; Paraschivoiu, Peraire & Patera 1997; Becker & Rannacher 1998; Süli 1998) Alternatively, this error term can be carefully evaluated and used to correct the value of the objective function given by the calculated flow field. For the 2D Poisson equation and the quasi-1D Euler equations, this has been shown to lead to corrected values of twice the order of accuracy of the flow field solution (Giles & Pierce 1998, 1999; Pierce & Giles 1998, 2000).

While significant effort has been dedicated to developing methods for calculating adjoint solutions to compressible flow equations, there has been little discussion of the properties of the adjoint solutions themselves (see Giles & Pierce 1997, 1998). The present work investigates the analytic properties of adjoint solutions for the quasi-1D Euler equations. The standard formulation of the adjoint equations using Lagrange multipliers (Jameson 1995) is extended to include the analysis of a shock. Explicit enforcement of the steady Rankine–Hugoniot conditions through an additional Lagrange multiplier leads to the result that at the shock, the adjoint variables are continuous and there is an internal adjoint boundary condition. This is consistent with a characteristic viewpoint which indicates that one internal adjoint b.c. is needed due to the disparity in the number of adjoint characteristics entering and leaving the shock. However, the conclusions differ from those of previous investigators (see Iollo, Salas & Ta'asan 1993; Iollo & Salas 1996; Cliff, Heinkenschloss & Shenoy 1996, 1998).

The analytic adjoint solutions are then derived in closed form for all Mach regimes. This is accomplished by constructing the Green's functions for the linearised Euler equations, including the linearised Rankine–Hugoniot conditions, using an extension of the approach developed by Giles and Pierce (1997) for shock-free quasi-1D flows. These solutions confirm the expected behavior at the shock and reveal a logarithmic singularity in the adjoint variables at the sonic point. These insights are helpful in understanding the requirements for developing effective numerical methods (Giles & Pierce 1998). In this regard, it is hoped that the analytic solutions will also serve as a useful set of test cases for researchers developing adjoint numerical methods.

2 Adjoint problem formulation

The quasi-1D Euler equations for steady flow in a duct of cross-section $h(x)$, on the interval $-1 \leq x \leq 1$, may be written as

$$R(U,h) \equiv \frac{d}{dx}(hF) - \frac{dh}{dx}\,P = 0\,,$$

where

$$U = \begin{pmatrix} \rho \\ \rho q \\ \rho E \end{pmatrix}, \qquad F = \begin{pmatrix} \rho q \\ \rho q^2 + p \\ \rho q H \end{pmatrix}, \qquad P = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}.$$

Here, ρ is the density, q is the velocity, p is the pressure, E is the total energy and H is the stagnation enthalpy. The system is closed by the equation of state for an ideal gas

$$H = E + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \tfrac{1}{2}q^2.$$

If the solution contains a shock at x_s, the Rankine-Hugoniot jump condition

$$[F]_{x_s^-}^{x_s^+} = 0$$

connects the smooth solutions on either side.

For design applications, linearisation of R with respect to perturbations in the flow solution, u, and the geometry, $\widetilde{h}$, produces

$$Lu - f \equiv \left(\frac{d}{dx}(hAu) - \frac{dh}{dx}\, Bu \right) - \left(\frac{d\widetilde{h}}{dx} P - \frac{d}{dx}(\widetilde{h}F) \right) = 0, \tag{2.1}$$

where $A = (\partial F/\partial U)$ and $B = (\partial P/\partial U)$.

We choose the objective function to be the integral of pressure along the duct,

$$J = \int_{-1}^{1} p\ dx = \int_{-1}^{x_s} p\ dx \ + \ \int_{x_s}^{1} p\ dx\,,$$

since this mimics the lift integral which is of importance in aeronautical applications. Other objective functions could also be considered with only minor changes to the analysis to be presented. The perturbation to this 'lift' integral due to changes in the flow is

$$I = \int_{-1}^{x_s} g^T u\ dx \ + \int_{x_s}^{1} g^T u\ dx \ - [p]_{x_s^-}^{x_s^+}\, \delta\,, \tag{2.2}$$

where $g = (\partial p/\partial U)^T$, and the third term includes the effect of a linearised displacement δ in the shock location.

Using continuous Lagrange multipliers v to enforce the differential flow constraints on either side of the shock, and a Lagrange multiplier v_s to enforce the Rankine–Hugoniot conditions at the shock, the augmented nonlinear objective function is

$$J = \int_{-1}^{x_s} p\ dx \ + \ \int_{x_s}^{1} p\ dx - \int_{-1}^{x_s^-} v^T R\ dx - \int_{x_s^+}^{1} v^T R\ dx - \ h_s v_s^T\, [F]_{x_s^-}^{x_s^+}\,,$$

where $h_s \equiv h(x_s)$. Linearising this with respect to perturbations in the geometry $\widetilde{h}$, the shock location δ and the flow solution u gives

$$\begin{aligned} I \ \ = \ \ & \int_{-1}^{x_s} g^T u\ dx \ + \int_{x_s}^{1} g^T u\ dx \ - [p]_{x_s^-}^{x_s^+}\, \delta \\ & - \int_{-1}^{x_s^-} v^T (Lu - f)\ dx \ \ - \ \int_{x_s^+}^{1} v^T (Lu - f)\ dx \\ & - \ h_s v_s^T\, [Au]_{x_s^-}^{x_s^+} - \ h_s v_s^T \left[\frac{dF}{dx} \right]_{x_s^-}^{x_s^+} \delta\ . \end{aligned}$$

After integration by parts and rearrangement, this yields

$$
\begin{aligned}
I \quad = \quad & \int_{-1}^{x_s} v^T f \, dx \; + \int_{x_s}^{1} v^T f \, dx \\
& - \int_{-1}^{x_s^-} (L^* v - g)^T u \, dx \; - \int_{x_s^+}^{1} (L^* v - g)^T u \, dx \\
& - \delta \left(h_s v_s^T \left[\frac{dF}{dx} \right]_{x_s^-}^{x_s^+} + [p]_{x_s^-}^{x_s^+} \right) \\
& - \; h_s (v_s - v(x_s^+))^T \, Au|_{x_s^+} + \; h_s (v_s - v(x_s^-))^T \, Au|_{x_s^-} \\
& - \; \left[h v^T A u \right]_{-1}^{1} ,
\end{aligned}
$$

where the adjoint operator L^* is defined by

$$
L^* v \equiv -h A^T \frac{dv}{dx} - \frac{dh}{dx} \, B^T v \, .
$$

The idea of the adjoint approach is to define the adjoint problem so as to eliminate the explicit dependence of I on u and δ, giving the adjoint form of the objective function

$$
I \; = \; \int_{-1}^{x_s} v^T f \, dx \; + \int_{x_s}^{1} v^T f \, dx \; = \; \int_{-1}^{1} v^T f dx \, . \tag{2.3}
$$

To eliminate the dependence on u, v must satisfy the adjoint o.d.e.

$$
L^* v - g = 0 \, , \tag{2.4}
$$

and at the shock v and v_s must satisfy

$$
v(x_s^-) = v_s = v(x_s^+) \, ,
$$

proving that the adjoint variables are continuous across the shock. Removing the dependence of I on δ then requires that

$$
h_s v^T(x_s) \left[\frac{dF}{dx} \right]_{x_s^-}^{x_s^+} = - \, [p]_{x_s^-}^{x_s^+} \, ,
$$

which is an internal boundary condition at the shock. Noting that

$$
\left[\frac{dF}{dx} \right]_{x_s^-}^{x_s^+} = \left[\frac{1}{h} \, \frac{dh}{dx} P \right]_{x_s^-}^{x_s^+} ,
$$

this reduces to the simple b.c.

$$
v_2(x_s) = - \left(\frac{dh}{dx}(x_s) \right)^{-1} . \tag{2.5}
$$

Finally, the inlet and exit boundary conditions for the adjoint problem are defined so as to remove the explicit dependence of

$$\left[h v^T A u\right]_{-1}^{1} \tag{2.6}$$

on u. At a boundary where the flow equations have n incoming characteristics, and hence n imposed boundary conditions, the adjoint equations will thus have $(3-n)$ b.c.'s corresponding to an equal number of incoming adjoint characteristics (Giles & Pierce 1997).

The need for an adjoint boundary condition at the shock can be understood by considering the characteristics of the hyperbolic system. For the adjoint problem, information travels along characteristics in the opposite direction as for the flow problem. Thus, at the shock, there are three outgoing characteristics on the upstream side and one outgoing characteristic on the downstream side. Continuity of the adjoint variables across the shock provides three conditions and the additional shock boundary condition provides a fourth, ensuring that all outgoing characteristics are fully determined.

In Iollo *et al.* (1993) it is suggested that one could impose $v=0$ at the shock, but this over-constrains the adjoint problem, in addition to contradicting (2.5). Cliff *et al.* (1996,1998) conclude that there is a "shock" in the adjoint variables at the shock location, having proved that the adjoint variables undergo a change of sign across the shock. However, as this change of sign is entirely due to the non-standard coordinate system they employ in formulating the augmented Lagrangian, the conclusion that the adjoint variables are discontinuous at the shock is misleading.

A final observation is that the adjoint equation (2.4) and the adjoint shock b.c. (2.5) together cause the gradient of the adjoint variables to vanish at the shock. This may be seen by writing (2.4) using Jacobians based on the non-conservative flow variables $U_p = (\rho, q, p)^T$, so that the adjoint equation becomes,

$$h \begin{pmatrix} q & q^2 & \frac{1}{2}q^3 \\ \rho & 2\rho q & \frac{\gamma}{\gamma-1}p + \frac{3}{2}\rho q^2 \\ 0 & 1 & \frac{\gamma}{\gamma-1}q \end{pmatrix} \frac{dv}{dx} = - \left(\begin{array}{c} 0 \\ 0 \\ 1 + \frac{dh}{dx} v_2 \end{array} \right),$$

and the adjoint shock b.c. produces $(dv/dx) = 0$ at the shock. This feature is important in understanding the success of certain numerical discretisations in producing the correct adjoint behavior at the shock, without explicit enforcement of the internal adjoint boundary condition (Giles & Pierce 1998).

3 Green's function approach

To verify the properties of the adjoint solutions and to provide a reference for comparison with numerical results, the analytic adjoint solutions are now derived for both isentropic and shocked transonic flows.

The derivation uses a Green's function approach (Giles & Pierce 1997) in which we consider the linearised problem with point source terms

$$L u_j(x, \xi) = f_j(\xi) \delta(x - \xi), \tag{3.1}$$

where $\delta(x)$ is the Dirac delta function. Using the adjoint form of the objective function (2.3), the corresponding linearised objective is

$$I_j(\xi) = \int_{-1}^{1} v^T(x) f_j(\xi) \delta(x-\xi)\, dx = v^T(\xi) f_j(\xi).$$

Given three linearly independent vectors $f_j(\xi)$, the three simultaneous equations can then be solved for the adjoint variables

$$v^T(\xi) = \Big(I_1(\xi) | I_2(\xi) | I_3(\xi) \Big) \Big(f_1(\xi) | f_2(\xi) | f_3(\xi) \Big)^{-1}. \tag{3.2}$$

The approach is then to choose $f_j(\xi)$, solve the linearised flow equations to obtain the flow perturbation $u_j(x,\xi)$ and the shock displacement δ, evaluate $I_j(\xi)$ using (2.2) and finally obtain $v(\xi)$ from (3.2).

The key to carrying out the procedure described above is to choose a set of source vectors $f_j(\xi)$ which lead to relatively simple solutions to the linearised flow equations. We begin by considering isentropic flow through a converging-diverging duct with inlet, throat and outlet located at $x=-1,0,+1$, respectively. The nonlinear equations ensure that mass flux $mh \equiv \rho q h$, stagnation enthalpy H and stagnation pressure p_0 all remain constant along the duct. Therefore, solutions to the linear homogeneous equations must introduce uniform perturbations to these three quantities. The general solution to the linear homogeneous equations may then be written in the form

$$u(x) = \frac{a}{h(x)} \frac{\partial U}{\partial m}(x)\bigg|_{H,p_0} + b\, \frac{\partial U}{\partial H}(x)\bigg|_{p_0,M} + c\, \frac{\partial U}{\partial p_0}(x)\bigg|_{H,M},$$

where the three vectors are linearly independent and a, b and c represent the uniform perturbations to mh, H and p_0. To simplify the analysis, perturbations to stagnation enthalpy and pressure are introduced at fixed Mach number rather than at fixed mass flux, so that non-zero values for b and c both imply an additional uniform perturbation to mh. By contrast, a non-zero value for a does not perturb either H or p_0.

If we now consider the inhomogeneous equations with source terms $f_j(\xi)\delta(x-\xi)$, the corresponding solutions

$$u_j(x,\xi) = a(x,\xi) \frac{1}{h(x)} \frac{\partial U}{\partial m}(x)\bigg|_{H,p_0} + b(x,\xi)\, \frac{\partial U}{\partial H}(x)\bigg|_{p_0,M} + c(x,\xi)\, \frac{\partial U}{\partial p_0}(x)\bigg|_{H,M}$$

must satisfy the homogeneous equations on either side of ξ, and therefore a,b,c will have uniform values a_1,b_1,c_1 for $x<\xi$ and a_2,b_2,c_2 for $x>\xi$. The jump conditions for the constants are obtained by integrating the dominant terms in (3.1) from $x=\xi^-$ to $x=\xi^+$, giving

$$h(\xi)\left((a_2-a_1) \frac{1}{h(\xi)} \frac{\partial F}{\partial m}(\xi)\bigg|_{H,p_0} + (b_2-b_1)\, \frac{\partial F}{\partial H}(\xi)\bigg|_{p_0,M} + (c_2-c_1)\, \frac{\partial F}{\partial p_0}(\xi)\bigg|_{H,M} \right) = f_j(\xi).$$

Hence, by choosing the three linearly independent source vectors

$$
\begin{aligned}
f_1(\xi) &= \left.\frac{\partial F}{\partial m}(\xi)\right|_{H,p_0} = \begin{pmatrix} 1 \\ q \\ H \end{pmatrix}, \\
f_2(\xi) &= h(\xi)\left.\frac{\partial F}{\partial H}(\xi)\right|_{p_0,M} = \frac{h(\xi)}{2H}\begin{pmatrix} -\rho q \\ 0 \\ \rho q H \end{pmatrix}, \\
f_3(\xi) &= h(\xi)\left.\frac{\partial F}{\partial p_0}(\xi)\right|_{H,M} = \frac{h(\xi)}{p_0}\begin{pmatrix} \rho q \\ \rho q^2 + p \\ \rho q H \end{pmatrix},
\end{aligned}
$$

the perturbations will have the simple properties

$$
\begin{aligned}
f_1(\xi) &\Rightarrow a_2 - a_1 = 1, \quad b_2 = b_1, \quad c_2 = c_1, \\
f_2(\xi) &\Rightarrow b_2 - b_1 = 1, \quad c_2 = c_1, \quad a_2 = a_1, \\
f_3(\xi) &\Rightarrow c_2 - c_1 = 1, \quad a_2 = a_1, \quad b_2 = b_1.
\end{aligned}
\tag{3.3}
$$

For each source vector $f_j(\xi)$, the three remaining unknowns in the corresponding solution $u_j(x,\xi)$ are determined by the three homogeneous boundary conditions appropriate to the Mach regime under consideration. These homogeneous boundary conditions are equivalent to demanding that there is no perturbation to the boundary conditions for the original nonlinear problem.

4 Supersonic Flow

For supersonic flow, M, H and p_0 are fixed at the supersonic inlet and there are no boundary conditions at the supersonic exit. Hence, for all three source vectors we require

$$a_1 = b_1 = c_1 = 0$$

to prevent perturbations to the inlet boundary conditions. Making reference to the jump relations (3.3), we then obtain

$$
\begin{aligned}
f_1(\xi) &\Rightarrow a = \mathcal{H}(x-\xi), \quad b = 0, \quad c = 0, \\
f_2(\xi) &\Rightarrow b = \mathcal{H}(x-\xi), \quad c = 0, \quad a = 0, \\
f_3(\xi) &\Rightarrow c = \mathcal{H}(x-\xi), \quad a = 0, \quad b = 0,
\end{aligned}
$$

corresponding to the solutions

$$
u_1(x,\xi) = \mathcal{H}(x-\xi)\frac{1}{h(x)}\left.\frac{\partial U}{\partial m}(x)\right|_{H,p_0},
$$

$$
\begin{aligned}
u_2(x,\xi) &= \mathcal{H}(x-\xi)\left.\frac{\partial U}{\partial H}(x)\right|_{p_0,M}, \\
u_3(x,\xi) &= \mathcal{H}(x-\xi)\left.\frac{\partial U}{\partial p_0}(x)\right|_{H,M}.
\end{aligned}
$$

The objective functions are then

$$
\begin{aligned}
I_1(\xi) &= \int_\xi^1 \frac{1}{h(x)}\left.\frac{\partial p}{\partial m}(x)\right|_{H,p_0} dx, \\
I_2(\xi) &= \int_\xi^1 \left.\frac{\partial p}{\partial H}(x)\right|_{p_0,M} dx, \\
I_3(\xi) &= \int_\xi^1 \left.\frac{\partial p}{\partial p_0}(x)\right|_{H,M} dx,
\end{aligned}
$$

with

$$
\left.\frac{\partial p}{\partial m}(x)\right|_{H,p_0} = \frac{-q}{1-M^2}, \qquad \left.\frac{\partial p}{\partial H}(x)\right|_{p_0,M} = 0, \qquad \left.\frac{\partial p}{\partial p_0}(x)\right|_{H,M} = \frac{p}{p_0}.
$$

The objective function $I_2(\xi)$ is zero because the pressure is constant at fixed M and p_0.

5 Subsonic Flow

For subsonic flow, there are two boundary conditions on H and p_0 at the subsonic inlet and one boundary condition on static pressure p at the subsonic exit.

5.1 Change in mh at fixed H, p_0

For f_1, the inlet boundary conditions require $b = c = 0$ and the exit condition requires $a_2 = 0$, corresponding to the solution and objective function

$$
u_1(x,\xi) = -\mathcal{H}(\xi - x)\frac{1}{h(x)}\left.\frac{\partial U}{\partial m}(x)\right|_{H,p_0}, \qquad I_1(\xi) = \int_{-1}^{\xi} \frac{1}{h(x)}\left.\frac{\partial p}{\partial m}(x)\right|_{H,p_0} dx.
$$

5.2 Change in H at fixed p_0, M

In this case, the inlet conditions give $b_1 = c = 0$ and the exit condition gives $a = 0$, yielding a solution and objective function that are identical to the supersonic case

$$
u_2(x,\xi) = \mathcal{H}(x-\xi)\left.\frac{\partial U}{\partial H}(x)\right|_{p_0,M}, \qquad I_2(\xi) = 0.
$$

5.3 Change in p_0 at fixed H, M

The inlet conditions now give $b = c_1 = 0$. Also, to ensure zero perturbation to the exit pressure, we require

$$\left(\frac{a}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} + c_2 \frac{\partial p}{\partial p_0}(x)\bigg|_{H,M} \right)\Bigg|_{x=1} = 0,$$

where $c_2 = 1$. The solution then becomes

$$u_3(x,\xi) = \mathcal{H}(x-\xi)\, \frac{\partial U}{\partial p_0}(x)\bigg|_{H,M} + \frac{a}{h(x)} \frac{\partial U}{\partial m}(x)\bigg|_{H,p_0},$$

with corresponding objective function

$$I_3(\xi) = \int_\xi^1 \frac{\partial p}{\partial p_0}(x)\bigg|_{H,M} dx + \int_{-1}^1 \frac{a}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} dx.$$

6 Isentropic transonic flow

For isentropic transonic flow, H and p_0 are fixed at the subsonic inlet and there are no boundary conditions at the supersonic exit. The third requirement is that the Mach number remains unity at the throat.

6.1 Change in mh at fixed H, p_0

For f_1, the inlet boundary conditions ensure that $b = c = 0$ and the throat condition requires that a equals zero at the throat. Therefore, $a_2 = 0$ for $\xi < 0$ and $a_1 = 0$ for $\xi > 0$, leading to the solution

$$u_1(x,\xi) = \begin{cases} -\mathcal{H}(\xi - x)\dfrac{1}{h(x)} \dfrac{\partial U}{\partial m}(x)\bigg|_{H,p_0}, & \xi < 0\,, \\[2ex] \mathcal{H}(x-\xi)\dfrac{1}{h(x)} \dfrac{\partial U}{\partial m}(x)\bigg|_{H,p_0}, & \xi > 0\,. \end{cases}$$

Hence, if $\xi < 0$, the mass flux upstream of $x = \xi$ is reduced by a unit amount, whereas if $\xi > 0$, the mass flux downstream of $x = \xi$ is increased by a unit amount.

The objective function is

$$I_1(\xi) = \begin{cases} -\displaystyle\int_{-1}^{\xi} \frac{1}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} dx\,, & \xi < 0\,, \\[2ex] \displaystyle\int_{\xi}^{1} \frac{1}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} dx\,, & \xi > 0\,. \end{cases} \tag{6.1}$$

Since

$$\left.\frac{\partial p}{\partial m}(x)\right|_{H,p_0} = \frac{-q}{1-M^2},$$

and M varies approximately linearly through a choked throat, then

$$\left.\frac{\partial p}{\partial m}(x)\right|_{H,p_0} \sim \frac{1}{x}, \qquad \text{as} \;\; x \to 0.$$

It follows that

$$I_1(\xi) \sim \log(\xi), \qquad \text{as} \;\; \xi \to 0,$$

so there is a logarithmic singularity in the adjoint variables at a sonic throat.

6.2 Change in H at fixed p_0, M

In this case, the inlet conditions on H and p_0 require $b_1 = c = 0$ and the throat condition gives $a = 0$. The solution is then

$$u_2(x,\xi) = \mathcal{H}(x-\xi)\left.\frac{\partial U}{\partial H}(x)\right|_{p_0,M},$$

and the corresponding objective function, $I_2(\xi)$, is zero because $\left.\frac{\partial p}{\partial H}(x)\right|_{p_0,M} = 0$.

6.3 Change in p_0 at fixed H, M

Now, the inlet conditions on H and p_0 yield $b = c_1 = 0$, and the Mach number is fixed at the throat, so again $a = 0$. The solution and linear functional thus become

$$u_3(x,\xi) = \mathcal{H}(x-\xi)\left.\frac{\partial U}{\partial p_0}(x)\right|_{H,M}, \qquad I_3(\xi) = \int_\xi^1 \left.\frac{\partial p}{\partial p_0}(x)\right|_{H,M} dx.$$

7 Shocked flow

For shocked flow, there are two boundary conditions on H and p_0 at the subsonic inlet, the throat is again sonic, there is a shock downstream of the throat and there is one boundary condition on p at the subsonic exit. The nonlinear equations once again ensure uniform mass flux and stagnation enthalpy throughout the duct, but the stagnation pressure now has different values on either side of the shock. Consequently, solutions to the linearized equations must now admit different but uniform stagnation pressure perturbations on either side of the shock. To account for the shock, the form of the solution must be generalised to

$$u_j(x,x_s,\xi) = a(x,x_s,\xi)\frac{1}{h(x)}\left.\frac{\partial U}{\partial m}(x)\right|_{H,p_0} + b(x,x_s,\xi)\left.\frac{\partial U}{\partial H}(x)\right|_{p_0,M} + c(x,x_s,\xi)\left.\frac{\partial U}{\partial p_0}(x)\right|_{H,M}$$

where the perturbations a, b, and c may now be discontinuous at the shock location x_s as well as at ξ.

7.1 Shock movement

The displacement in the shock can be calculated from the normal shock relation

$$p_{02} = p_{01} f(M_1), \qquad f(M_1) = \left(\frac{p_2}{p_1}\right)\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\gamma/\gamma-1},$$

with shock jump conditions

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2-1), \qquad M_2^2 = \frac{1+[(\gamma-1)/2]M_1^2}{\gamma M_1^2-(\gamma-1)/2},$$

where the subscripts 1 and 2 represent quantities upstream and downstream of the shock, respectively. The perturbations to the stagnation pressure then satisfy

$$c_2 = c_1 f(M_1) + p_{01} f'(M_1)\left(\frac{dM}{dx}\delta + \frac{a_1}{h(x)}\frac{\partial M}{\partial m}(x)\bigg|_{H,p_0}\right)\Bigg|_{x=x_s^-}, \tag{7.1}$$

where δ is the resulting displacement of the shock and

$$\frac{\partial M}{\partial m}(x)\bigg|_{H,p_0} = \frac{M}{m}\left(\frac{1+[(\gamma-1)/2]M^2}{1-M^2}\right).$$

If $h(x)$ is a piecewise differentiable function, then dM/dx may be evaluated analytically using the area Mach number relation

$$\left(\frac{h}{h^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}.$$

The throat is sonic so the sonic area h^* is identically equal to the throat area h_t.

7.2 Change in mh at fixed H, p_0

Since the throat is choked and H and p_0 are fixed at the inlet, the form of the solution and objective function will be the same as for the isentropic transonic case when $\xi < 0$. The two new scenarios to consider are when ξ is between the throat and the shock, and between the shock and the exit. In either case, the mass flux perturbation will cause the shock to move and the solution will need to ensure that the perturbations to mass flux and stagnation enthalpy remain constant across the shock, in addition to satisfying the exit boundary condition on pressure.

7.2.1 Perturbation between the throat and the shock $(0 < \xi < x_s)$

The choked condition at the throat requires that all perturbations are zero for $x < \xi$. For consistency with the shock jump subscripts, perturbations between ξ and the shock are denoted by a_1, b_1, c_1 and perturbations between the shock and the exit are denoted by a_2, b_2, c_2. At ξ, there is a unit mass flux perturbation at constant H and p_0, so

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 0.$$

Furthermore, H remains constant for any shock location so $b_2 = 0$. The perturbation to mass flux across the shock must be constant, so

$$a_1 = a_2 + c_2 \left(h(x) \left. \frac{\partial m}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=x_s^+}.$$

Also, to avoid perturbing the exit pressure, we require

$$\left(\frac{a_2}{h(x)} \left. \frac{\partial p}{\partial m}(x) \right|_{H,p_0} + c_2 \left. \frac{\partial p}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=1} = 0.$$

These two equations determine the two unknowns a_2 and c_2 and equation (7.1) then determines the shock movement δ. The perturbed solution is then

$$u_1(x, x_s, \xi) = \frac{1}{h(x)} \left[\mathcal{H}(x-\xi) + (a_2 - 1)\mathcal{H}(x - x_s) \right] \left. \frac{\partial U}{\partial m}(x) \right|_{H,p_0} + c_2\, \mathcal{H}(x - x_s) \left. \frac{\partial U}{\partial p_0}(x) \right|_{H,M},$$

and the corresponding objective function is

$$I_1(\xi) = \int_\xi^{x_s} \frac{1}{h(x)} \left. \frac{\partial p}{\partial m}(x) \right|_{H,p_0} dx + \int_{x_s}^1 \left(\frac{a_2}{h(x)} \left. \frac{\partial p}{\partial m}(x) \right|_{H,p_0} + c_2 \left. \frac{\partial p}{\partial p_0}(x) \right|_{H,M} \right) dx - (p_2 - p_1)\, \delta.$$

7.2.2 Perturbation between the shock and the exit ($x_s < \xi < 1$)

All perturbations are now zero for $x < x_s$, so

$$a_1 = b_1 = c_1 = 0,$$

since perturbations introduced in the subsonic region following the shock cannot affect the supersonic zone. Perturbations between the shock and ξ are now denoted by a_2, b_2, c_2 and perturbations between ξ and the exit are denoted by a_3, b_3, c_3.

For compatibility with the upstream flow, there must be no perturbation to H across the shock, so $b_2 = b_3 = 0$. The perturbation to the stagnation pressure must be uniform throughout the subsonic region, so $c_2 = c_3 \equiv c$. At ξ, the source term produces a unit perturbation in mass flux so

$$a_3 - a_2 = 1.$$

To match the flow upstream of the shock, there must be no mass flux perturbation on the downstream side of the shock

$$a_2 + c \left(h(x) \left. \frac{\partial m}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=x_s^+} = 0.$$

Also, to ensure zero perturbation of the exit static pressure we require,

$$\left(\frac{a_3}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} + c\,\frac{\partial p}{\partial p_0}(x)\bigg|_{H,M} \right)\Bigg|_{x=1} = 0,$$

giving three equations for the three unknowns. The perturbed solution then has the form

$$u_1(x, x_s, \xi) = \frac{1}{h(x)}\left[a_2 \mathcal{H}(x - x_s) + \mathcal{H}(x - \xi)\right] \frac{\partial U}{\partial m}(x)\bigg|_{H,p_0} + c\,\mathcal{H}(x - x_s)\,\frac{\partial U}{\partial p_0}(x)\bigg|_{H,M},$$

with objective function

$$I_1(\xi) = \int_{x_s}^{\xi} \frac{a_2}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} dx + \int_{\xi}^{1} \frac{a_3}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} dx + \int_{x_s}^{1} c\,\frac{\partial p}{\partial p_0}(x)\bigg|_{H,M} dx - (p_2 - p_1)\,\delta.$$

7.3 Change in H at fixed p_0, M

Ahead of the shock, the perturbation to stagnation pressure c must be zero due to the inlet boundary condition, and the mass flux perturbation a must be zero due to the choked throat. The inlet condition on H ensures the perturbation to stagnation enthalpy is zero for $x < \xi$, and the unit jump in b at ξ will produce a uniform perturbation in H across the shock, without affecting the exit condition on pressure.

There still exists the possibility that a and c are non-zero constants following the shock, balancing to produce zero mass flux perturbation at the shock

$$\left(a + c\,h(x)\,\frac{\partial m}{\partial p_0}(x)\bigg|_{H,M} \right)\Bigg|_{x=x_s^+} = 0,$$

and zero pressure perturbation at the exit

$$\left(\frac{a}{h(x)} \frac{\partial p}{\partial m}(x)\bigg|_{H,p_0} + c\,\frac{\partial p}{\partial p_0}(x)\bigg|_{H,M} \right)\Bigg|_{x=1} = 0.$$

However, the determinant of this system is nonzero, so there is only the trivial solution $a = c = 0$. Hence, the solution and objective function in the shocked case have the form

$$u_2(x, x_s, \xi) = \mathcal{H}(x - \xi)\,\frac{\partial U}{\partial H}(x)\bigg|_{p_0,M}, \qquad I_2(\xi) = 0,$$

and there is no displacement of the shock.

7.4 Change in p_0 at fixed H, M

For shocked flow with a unit jump in stagnation pressure, the presence of the shock affects the perturbed solution for all locations of ξ. This is in contrast to the shocked case with a jump in mass flux, where the solution remained unchanged from the isentropic transonic case for $\xi < 0$. The two scenarios to consider in the present case are when ξ is between the inlet and the shock, and between the shock and the exit.

7.4.1 Perturbation between the inlet and the shock $(-1 < \xi < x_s)$

As in the shock-free case, there is no perturbation for $x < \xi$. Denoting the perturbations between ξ and the shock by a_1, b_1, c_1 and those after the shock by a_2, b_2, c_2, we have by definition

$$a_1 = 0, \quad b_1 = 0, \quad c_1 = 1.$$

The perturbation to H must be constant across the shock so $b_2 = 0$. Constant mass flux perturbation at the shock requires

$$c_1 \left(h(x) \left. \frac{\partial m}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=x_s^-} = a_2 + c_2 \left(h(x) \left. \frac{\partial m}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=x_s^+},$$

and zero perturbation to the exit pressure is ensured by setting

$$\left(\frac{a_2}{h(x)} \left. \frac{\partial p}{\partial m}(x) \right|_{H,p_0} + c_2 \left. \frac{\partial p}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=1} = 0\,,$$

providing two equations for the two unknowns. The solution then has the form

$$u_3(x, x_s, \xi) = [\mathcal{H}(x-\xi) + (c_2 - 1)\mathcal{H}(x - x_s)] \left. \frac{\partial U}{\partial p_0}(x) \right|_{H,M} + \frac{a_2}{h(x)} \mathcal{H}(x - x_s) \left. \frac{\partial U}{\partial m}(x) \right|_{H,p_0},$$

with corresponding objective function

$$I_3(x, x_s, \xi) = \int_\xi^{x_s} \left. \frac{\partial p}{\partial p_0}(x) \right|_{H,M} dx + \int_{x_s}^1 \left(\frac{a_2}{h(x)} \left. \frac{\partial p}{\partial m}(x) \right|_{H,p_0} + c_2 \left. \frac{\partial p}{\partial p_0}(x) \right|_{H,M} \right) dx - (p_2 - p_1)\,\delta\,.$$

7.4.2 Perturbation between the shock and the exit $(x_s < \xi < 1)$

There are now no perturbations upstream of the shock, so

$$a_1 = b_1 = c_1 = 0.$$

Perturbations in the region between the shock and ξ are denoted by a_2, b_2, c_2 and those between ξ and the exit are denoted by a_3, b_3, c_3.

Compatibility at the shock and the fact that mh and p_0 are perturbed at constant H, together imply that there are no perturbations to stagnation enthalpy following the shock, so $b_2 = b_3 = 0$. Perturbations to the mass flux must be constant throughout the subsonic region ($a_2 = a_3 \equiv a$) since the jump condition at ξ corresponds solely to a unit perturbation in stagnation pressure

$$c_3 - c_2 = 1.$$

Zero mass flux perturbation at the shock then gives

$$a + c_2 \left(\left. \frac{\partial m}{\partial p_0}(x) \right|_{H,M} \right) \Bigg|_{x=x_s^+} = 0\,,$$

and zero perturbation to the exit pressure requires

$$\left(\frac{a}{h(x)} \frac{\partial p}{\partial m}(x) \bigg|_{H,p_0} + c_3 \frac{\partial p}{\partial p_0}(x) \bigg|_{H,M} \right) \Bigg|_{x=1} = 0\,,$$

providing three equations for three unknowns. The solution has the form

$$u_3(x, x_s, \xi) = [c_2 \mathcal{H}(x-x_s) + \mathcal{H}(x-\xi)] \frac{\partial U}{\partial p_0}(x) \bigg|_{H,M} + \frac{a}{h(x)} \mathcal{H}(x-x_s) \frac{\partial U}{\partial m}(x) \bigg|_{H,p_0},$$

with corresponding objective function

$$I_3(\xi) = \int_{x_s}^{1} \frac{a}{h(x)} \frac{\partial p}{\partial m}(x) \bigg|_{H,p_0} dx + \int_{x_s}^{\xi} c_2 \frac{\partial p}{\partial p_0}(x) \bigg|_{H,M} dx + \int_{\xi}^{1} c_3 \frac{\partial p}{\partial p_0}(x) \bigg|_{H,M} dx - (p_2 - p_1)\delta.$$

8 Sample solutions

The analytic objective functions $I(\xi)$ and adjoint solutions $v(\xi)$ corresponding to supersonic, subsonic, isentropic and shocked transonic flows are shown in figures 1 to 4. The boundary conditions for these test cases are defined in the figure captions and the geometric definition of the duct is given by

$$h(x) = \begin{cases} 2, & -1 \le x \le -\frac{1}{2}, \\ 1 + \sin^2(\pi x), & -\frac{1}{2} < x < \frac{1}{2}, \\ 2, & \frac{1}{2} \le x \le 1. \end{cases}$$

The analytic results have been verified using numerical solutions obtained by discretising the adjoint equation (2.4) directly (Giles & Pierce 1998). For the supersonic case of figure 1, the adjoint variables are all zero at the exit, as required to eliminate the dependence on u of the boundary term (2.6) in the adjoint derivation. For the isentropic transonic case of figure 3, the logarithmic singularity in I_1 at the sonic throat is reflected in the singularities of all three adjoint variables. For the shocked case of figure 4, the objective functions are discontinuous at the shock, but the adjoint variables are continuous with zero gradient, as proved earlier.

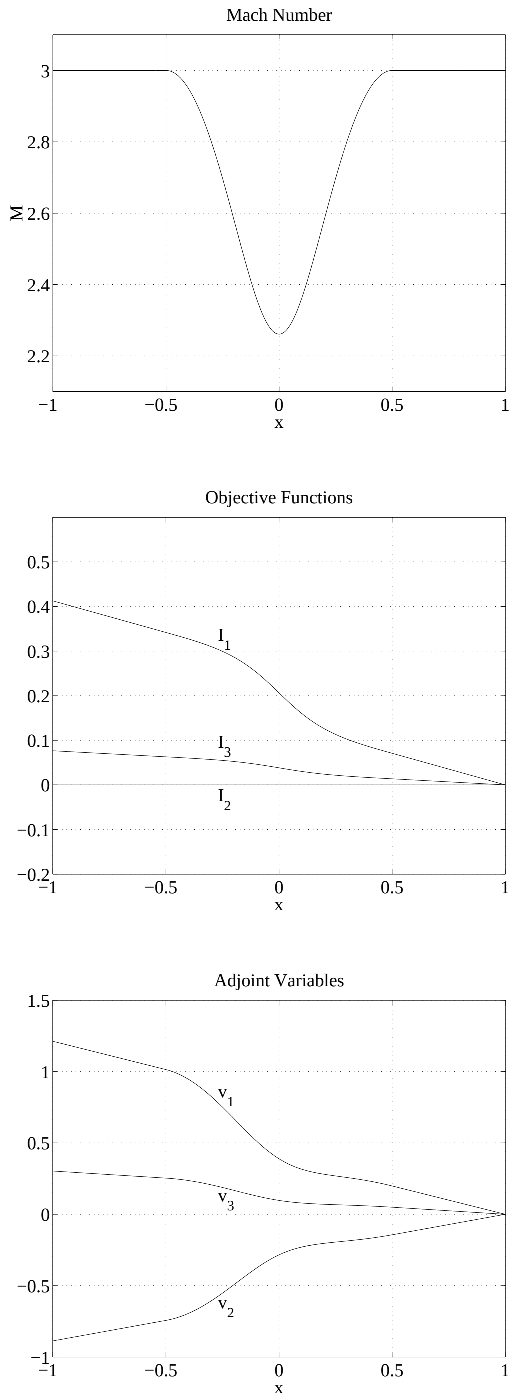

Figure 1: Mach number, objective functions and adjoint variables for supersonic flow conditions. $M_{\text{in}} = 3, H_{\text{in}} = 4, p_{0\text{in}} = 2$.

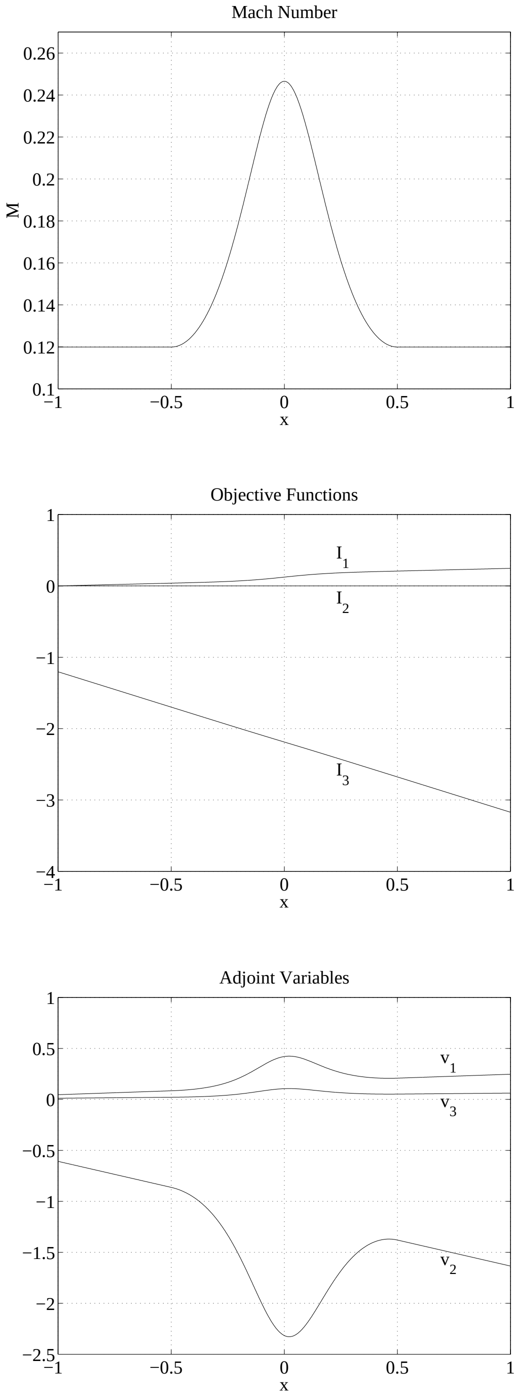

Figure 2: Mach number, objective functions and adjoint variables for subsonic flow conditions. $H_{\text{in}} = 4, p_{0_{\text{in}}} = 2, p_{\text{ex}} = 1.98$.

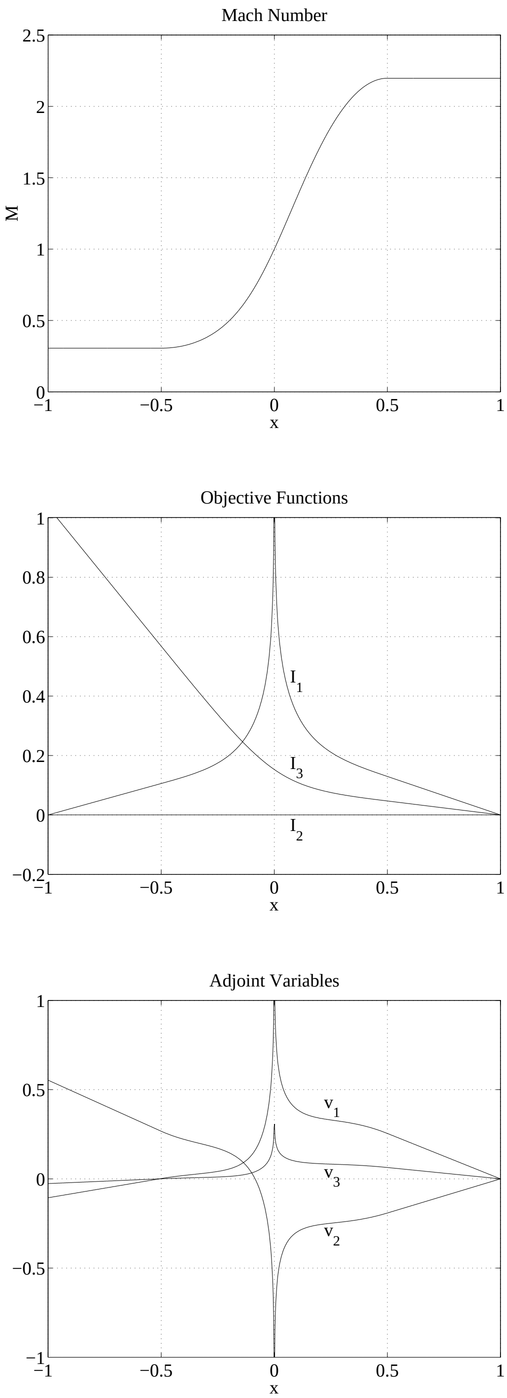

Figure 3: Mach number, objective functions and adjoint variables for isentropic transonic flow conditions. $H_{\text{in}} = 4, p_{0_{\text{in}}} = 2$.

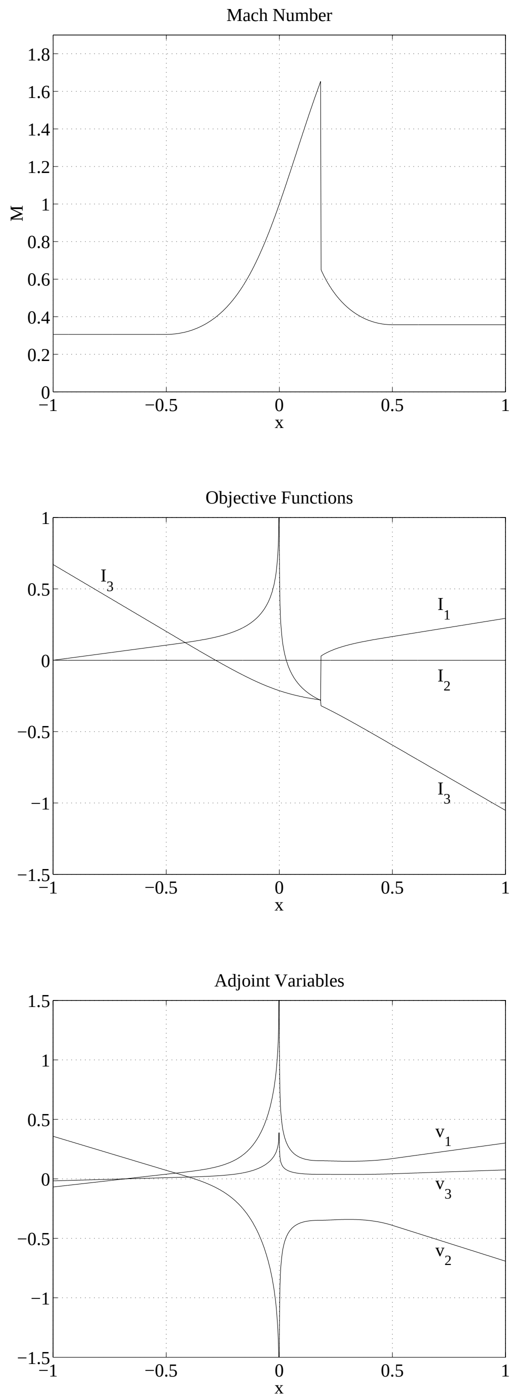

Figure 4: Mach number, objective functions and adjoint variables for shocked flow conditions. $H_{\rm in} = 4, p_{0\rm in} = 2, p_{\rm ex} = 1.6$.

9 Conclusions

In this paper we have undertaken a detailed investigation of adjoint solutions for the quasi-1D Euler equations, focusing in particular on the solution behaviour at a shock or a sonic point where there is a change in sign of one of the hyperbolic characteristics.

Formulating the adjoint equations using Lagrange multipliers to enforce the Rankine-Hugoniot shock jump conditions proves that, contrary to previous literature, the adjoint variables are continuous at the shock. This result is supported by the derivation of a closed form solution to the adjoint equations using a Green's function approach. In addition to proving the existence of a $\log(x)$ singularity at the sonic point, this closed form solution should be very helpful as a test case for others developing numerical methods for the adjoint equations.

Future research will attempt to extend this analysis to two dimensions. Preliminary analysis, supported by the results of numerical computations (Giles & Pierce 1997), shows that the adjoint variables are again continuous at a shock, and that an adjoint boundary condition is required along the length of the shock. However, since adjoint computations currently employed for transonic aerofoil optimisation do not enforce this internal boundary condition, it remains an open question as to whether there is a consistency error in the limit of increasing grid resolution. In two dimensions, numerical evidence suggests that there is no longer a singularity at a sonic line if (as is usually the case) it is not orthogonal to the flow. This can be explained qualitatively by considering the region of influence of points in the neighbourhood of the sonic line (Giles & Pierce 1997). An important new feature that must be considered for two-dimensional flows is the behavior of the adjoint solution at stagnation points. Here, the analysis indicates an inverse square-root singularity along the incoming stagnation streamline, but further numerical experiments are required to confirm this behavior.

An improved understanding of the behaviour of adjoint solutions is necessary both to rigorously establish the theoretical basis for engineering optimal design methods and to illuminate the role of the adjoint solution in numerical error analysis. In this latter setting, the adjoint solution reveals the sensitivity of a functional, such as lift, to the truncation errors associated with the numerical discretisation. Where there are singularities in the adjoint variables it is desirable to greatly increase the grid resolution so as to reduce the contribution of the local truncation error to the error in the functional. Thus, adjoint analysis offers a rigorous basis for optimal grid adaptation (Venditti & Darmofal 1999). Furthermore, by estimating the truncation error in the original nonlinear numerical solution, and using the adjoint solution to estimate the consequential error in the functional of interest, one can obtain an improved estimate with twice the order of accuracy (Giles & Pierce 1998, 1999; Pierce & Giles 1998, 2000). Future developments along these lines will lead to great improvements in accuracy for key engineering quantities such as lift and drag.

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