A Hybrid FE/Spectral Formulation of Turbofan Noise Radiation

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1 Introduction

This paper introduces a novel method for analysing the tonal noise radiated from non-axisymmetric turbofan inlets. It combines a standard finite element (FE) discretisation of the acoustic field in the axial and radial direction with a Fourier spectral representation [6] in the circumferential direction. Because relatively few Fourier modes need to be retained for an accurate field representation, the method involves many fewer discrete unknowns than a conventional 3D finite element analysis.

The noise source due to the fan blades is characterised by a single circumferential mode number, which in the case of an axisymmetric inlet would produce a radiated acoustic field with the same circumferential mode number. However, real engine inlets are not axisymmetric, so in practice the circumferential modes of the acoustic field are coupled and all are excited. However, the level of asymmetry is quite moderate and this coupling is relatively weak. This feature is used to construct a very efficient preconditioner for the iterative solution of the asymmetric aeroacoustic problem.

2 Aeroacoustic equations

Given the standard assumption of irrotational flow with uniform entropy, the velocity field is represented as the gradient of a potential function ϕ and the density ρ and speed of sound c are given by

$$(\,\rho\,/\,\rho_\infty)^{\gamma-1} = \;(\,c\,/\,c_\infty)^2 = \;1-(\gamma{-}1)\;(\,q-q_\infty)\,/\,c_\infty^2,$$

where $q=\frac{1}{2}|\nabla\phi|^2$ and $\rho_\infty, c_\infty, q_\infty$ are the freestream values. Integrating by parts the steady mass equation gives the weak form of the steady equation,

$$\int_V \rho\,\nabla\phi\cdot\nabla w\;\mathrm{d}V - \int_{\partial V} \beta\; w\;\mathrm{d}S = 0, \qquad (1)$$

for all smooth test functions w which are zero on the far-field boundary where Dirichlet boundary conditions are specified for ϕ. The other two boundaries have the Neumann boundary condition $\partial\phi/\partial n=\beta$, with $\beta=0$ on the inlet surface, and a prescribed massflow $\beta\neq 0$ on the fan face – see Fig.1(a).

The propagation of acoustic waves of a single frequency ω is modelled as the real part of a harmonic perturbation $\widehat{\phi}\,\exp(\mathrm{i}\omega t)$ superimposed on the steady mean flow given by (1). The weak form of the acoustic equation comes from the linearisation of the unsteady mass conservation equation,

$$\int_V \rho\,\nabla\widehat{\phi}\cdot\nabla w - \frac{\rho}{c^2}\left(\nabla\phi\cdot\nabla\widehat{\phi}+\mathrm{i}\,\omega\,\widehat{\phi}\right)\left(\nabla\phi\cdot\nabla w-\mathrm{i}\,\omega\,w\right)\,\mathrm{d}V - \int_{\partial V}\hat{\beta}\;w\;\mathrm{d}S = 0, \qquad (2)$$

where

$$\hat{\beta} = \rho\,\frac{\partial\widehat{\phi}}{\partial n} - \frac{\rho}{c^2}\left(\nabla\phi\cdot\nabla\widehat{\phi}+\mathrm{i}\,\omega\widehat{\phi}\right)\frac{\partial\phi}{\partial n}.$$

All the boundary conditions of practical interest in engine inlet aeroacoustics are represented by $\hat{\beta}$. At the fan boundary, the field $\widehat{\phi}$ is decomposed into a sum of incident and radiated eigenmodes [5]. $\hat{\beta}$ is in that case a function of the incident mode prescribed to model the presence of the downstream fan as the source of noise. At the far-field, ray theory is used to determine the angle at which the acoustic waves cross the boundary and establish an expression for $\hat{\beta}$ which minimises the reflection of acoustic waves back into the computational domain. Finally, $\hat{\beta} = 0$ at the solid inlet surface but if acoustic liners are present then additional modelling yields a modified boundary integral [2].

3 Axisymmetric discretisation and solution

The standard approach for axisymmetric inlets is based on cylindrical coordinates (x, r). Using quadrilateral elements with bi-quadratic shape functions $N_n(\xi,\eta)$, the iso-parametric finite element representation for the coordinates and the steady potential flow field is

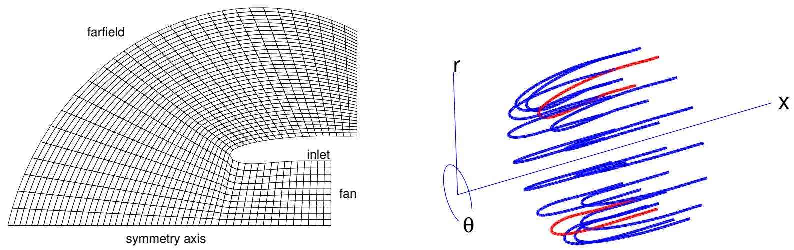
$$\begin{pmatrix} x(\xi,\eta) \\ r(\xi,\eta) \\ \phi(\xi,\eta) \end{pmatrix} = \sum_n N_n(\xi,\eta) \begin{pmatrix} x_n \\ r_n \\ \phi_n \end{pmatrix}.$$

Given this representation, a Galerkin discretisation of equation (1) yields a system of discrete nonlinear steady flow equations $\mathbf{R}(\mathbf{\Phi}) = 0$, where $\mathbf{\Phi}$ is the vector of unknown potentials at the nodes of the computational mesh. These nonlinear equations can be solved very efficiently using Newton iteration, with the linear system of equations at each Newton step being solved by direct Gaussian elimination.

The acoustic solution has the circumferential variation $\exp(\mathrm{i}\kappa\theta)$, where κ is the circumferential mode number of the specified incoming mode due to the rotating fan. Using the finite element approximation

$$\widehat{\phi}(\xi,\eta,\theta) = \sum_n \exp\left(\mathrm{i}\kappa\theta\right) N_n(\xi,\eta)\,\widehat{\phi}_n,$$

equation (2) yields a linear system of equations $\widehat{\mathbf{L}}\,\widehat{\mathbf{\Phi}} = \widehat{\mathbf{f}}_0$ which can again be efficiently solved by direct means.



(a) (b)

Fig. 1. (a) Coarse quadrilateral mesh around an axisymmetric inlet (A much finer grid is used for acoustic calculations.) (b) Asymmetric inlet geometry described by a series of axial sections at a number of equally spaced θ stations.

4 Non-axisymmetric discretisation and solution

The key to the non-axisymmetric discretisation is a spectral Fourier decomposition in the circumferential direction,

$$\begin{pmatrix} x(\xi,\eta,\theta) \\ r(\xi,\eta,\theta) \\ \phi(\xi,\eta,\theta) \end{pmatrix} = \sum_n \exp\left(\mathrm{i}m\theta\right) N_n(\xi,\eta) \begin{pmatrix} x_{mn} \\ r_{mn} \\ \phi_{mn} \end{pmatrix}.$$

This uses the same shape functions as in the axisymmetric case to interpolate in the $(\xi,\, \eta)$ coordinates, but there is an additional summation over the different circumferential modes. The x_{mn}, r_{mn} coefficients are first obtained for the nodes on the inlet geometry, through an FFT of corresponding points on a number of axial sections, as shown in Fig.1(b). The axisymmetric component of the inlet geometry is then used to construct the axisymmetric grid shown in Fig.1(a) and the non-axisymmetric component is interpolated onto the interior grid nodes.

The unknowns are now the nodal values of each of the circumferential modes (including the axisymmetric mode $m = 0$) of the steady potential. Only a limited number of modes $-M \leq m \leq M$ need to be retained in the solution; in practice, this number is very small relative to the number of nodes in the circumferential direction which would be required by an accurate standard 3D FE solution. With the above spectral representation, the Galerkin discretisation of the steady equation (1) gives a discrete system of nonlinear equations of the form $\mathbf{R}(\mathbf{\Phi}) = 0$, with the steady potential values grouped by circumferential mode within the vector $\mathbf{\Phi} = (\mathbf{\Phi}_{-M},\, \ldots\,, \mathbf{\Phi}_0,\, \ldots\,, \mathbf{\Phi}_M)$.

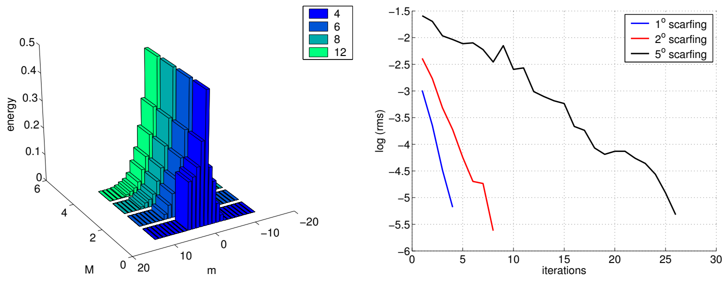
There are now two important questions, how to efficiently evaluate the residual $\mathbf{R}(\mathbf{\Phi})$, and how to solve the equations to obtain the steady solution. The finite element construction of $\mathbf{R}(\mathbf{\Phi})$ requires for each element the evaluation of a 3D integral over (ξ,η,θ). The integration over (ξ,η) is performed using Guass quadrature, while the integration over θ follows a pseudo-spectral approach. This starts with the circumferential modal values $\widehat{\phi}_{mn}$, and performs an FFT to obtain the values and circumferential derivatives at a number of equally spaced points circumferentially. These are used to form nodal residuals which are then combined through an inverse FFT to obtain the modal residuals. It is important that enough points are used circumferentially to avoid aliasing errors.

The equations are again solved by Newton iteration, but it is no longer efficient to use direct solution methods to solve the resulting linear Newton update equations, since the solution cost is approximately proportional to M^3. Instead, since the Jacobian matrix $\partial \mathbf{R}/\partial \mathbf{\Phi}$ is symmetric and positive definite, the linear equations are solved iteratively using the Conjugate Gradient (CG) method [1]. The iterative convergence rate is greatly improved through the use of a preconditioner. The preconditioner is based on the observation that if the geometry is axisymmetric, then the Jacobian matrix $\partial \mathbf{R}/\partial \mathbf{\Phi}$ becomes block-diagonal, with each of the circumferential modes becoming decoupled. It is this block-diagonal matrix, based on the axisymmetric component of the geometry and the steady flow field, and constructed in the standard way using Guass quadrature, which is therefore used as the preconditioner. Inverting the preconditioner as part of the preconditioned CG iteration requires the solution of a separate 2D system of equations for each circumferential mode, which is done efficiently by direct solution.

The acoustic field is represented by

$$\widehat{\phi}(\xi,\eta,\theta) = \sum_m \sum_n \exp\left(\mathrm{i}(\kappa{+}m)\theta\right) N_n(\xi,\eta)\ \widehat{\phi}_{mn}.$$

The Galerkin discretisation of equation (2) yields the discrete acoustic equations $\widehat{\mathbf{L}}\,\widehat{\mathbf{\Phi}} = \widehat{\mathbf{f}}$, with $\widehat{\mathbf{\Phi}} = (\widehat{\mathbf{\Phi}}_{-M}, \ldots, \widehat{\mathbf{\Phi}}_0, \ldots, \widehat{\mathbf{\Phi}}_M)$ comprising the complex amplitudes grouped by circumferential mode number, coupled together by the matrix $\widehat{\mathbf{L}}$. The forcing term is $\widehat{\mathbf{f}} = (0, \ldots, \widehat{\mathbf{f}}_0, \ldots, 0)$ and reflects the fact that the forcing duct mode ($m{=}0$) is prescribed at the fan face which is necessarily axisymmetric. The Quasi-Minimal Residual (QMR) [1, 3] method is used to solve the equations iteratively. Although not guaranteed to converge, it has been found to behave very well on this acoustic problem. As with the solution of the linear Newton equations for the steady problem, the spectral acoustic problem is preconditioned using the block-diagonal matrix given by the axisymmetric components of the geometry and the steady flow field. This effectively requires the direct solution of a separate 2D axisymmetric equation for each of the circumfernetial modes for each step of the QMR iteration. However, the CPU cost can be reduced by performing an LU factorisation of each of the axisymmetric matrices, so that then each QMR iteration requires only a back-solve for each circumferential mode.



(a) (b)

Fig. 2. (a) The variation of the modal "energy" distribution with the number of circumferential modes used to represent the solution. (b) The QMR iteration convergence histories for the inlet with 1°, 2° and 5° scarfing.

5 Example

The numerical results are for the real engine inlet shown in Fig.1. The inlet is asymmetric with a scarfing angle of 5° (the angle with which the plane of the inlet front is tilted with respect to the engine axis) and was defined by a series of axial sections equally spaced circumferentially at 22.5°, Fig.1(b).

The Mach number of the steady flow is specified to be 0.3 in freestream and 0.4 at the fan face. The source of acoustic excitation is the rotation of the shocks attached to the fan blades and considering 26 fan blades with a tip Mach number 1.2, the circumferential mode number is $\kappa = 26$ and the reduced frequency of the first blade passing frequency (based on engine radius and freestream speed of sound) is $\omega R/c_\infty = 30$.

Using bi-quadratic elements and a minimum of 8 nodes per wavelength, the axisymmetric mesh has 14,000 nodes. To determine the number of circumferential modes required to accurately represent the acoustic field, a series of calculations were performed with $M = 4, 6, 8$ and 12 modes. The distribution of "energy" (the RMS of the solution vector in each mode) across modes was computed, Fig.2(a). It was thus found that $M = 8$ is sufficient to represent the circumferential variation. This corresponds to 0.25 million unknowns and the advantage of the spectral discretisation becomes clear as the equivalent acoustic calculation using a standard 3D FE formulation with at least 8 nodes per wavelength in the circumferential direction would require approximately 3 million unknowns.

Finally, keeping the axisymmetric mean of the inlet geometry and scaling the asymmetry modes, configurations with scarfing angles of 1° (almost axisymmetric), 2° and 5° (original geometry) were obtained. Fig.2(b) depicts the QMR convergence histories for the acoustic problems in these three cases, demonstrating clearly how the convergence rate is fastest when the preconditioner is most effective, *i.e.* when the degree of asymmetry is smallest.

6 Conclusions

This paper has introduced a new method for the analysis of the tonal noise radiated from non-axisymmetric turbofan inlets. It combines a standard FE discretisation of the acoustic field in the axial and radial coordinates with a Fourier spectral representation in the circumferential direction. As a relatively small number of Fourier modes are enough for an accurate field representation, the method is far less costly than the conventional 3D FE approach. The most novel feature of the work is the iterative solution technique using a preconditioner based on an axisymmetric gemoetry and steady flow field. A numerical example illustrates how few Fourier modes need to be retained, and demonstrates the effectiveness of the preconditioner.

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