Multilevel Monte Carlo for computing mean exit times

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joint research with L. Szpruch (Edinburgh)
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Outline

- formulation and prior work on Euler schemes for exit times
- prior work on multilevel Milstein for 1D barrier options
- extension to exit times for 1D problems
- extension to exit times for multi-dimensional problems
- numerical results

Exit times

Given initial data $S_0 \in \mathbb{R}^d$ and the SDE

$$\mathrm{d}S = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W$$

for d-dimensional uncorrelated W_t we are interested in estimating $\mathbb{E}[f(\tau)]$ where τ is the first exit time from some domain V_t:

$$\tau = \min(\inf_{S_t \notin V_t} t,\ T\,)$$

One engineering application is the exit of contaminated groundwater from a nuclear waste repository – research with Rob Scheichl and Andrew Cliffe

Another is the modelling of a particle separator – Tigran Nagapetyan discussed the estimation of the PDF of the exit times (joint research with Klaus Ritter and Oleg Iliev)

Exit times

Also interested in

$$\mathbb{E}[f(S_\tau,\tau)\,\mathbf{1}_{\tau<T} + g(S_T)\,\mathbf{1}_{\tau\geq T}]$$

due to Feynman-Kac link to solution of a parabolic PDE.

Assumptions:

- $a(S,t), b(S,t)$ satisfy the usual conditions for first order strong convergence using the Milstein discretisation
- $b(S,t)$ satisfies commutativity condition so Lévy areas are not needed
- $\exists c>0$ such that $\xi^T(b\,b^T)\,\xi \;\geq\; c\|\xi\|^2,\ \forall\xi$
- boundary ∂V_t is smooth

Prior work

Higham, Mao, Roj, Song, Yin (2013) developed a multilevel method of estimating $\mathbb{E}[\tau]$ using an Euler-Maruyama discretisation.

The fine path approximation with timestep h uses

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

with approximate exit time

$$\widehat{\tau} = \min(\min_{\widehat{S}_n \notin V_{t_n}} t_n, \;\; T\,)$$

The coarse path approximation is essentially the same, with the Brownian increments obtained by summing the fine path increments in pairs

Prior work

Gobet & Menozzi (2010) proved that

$$\mathbb{E}[\,\tau - \widehat{\tau}\,] = O(h^{1/2})$$

so a single level Monte Carlo method achieves an RMS accuracy of ε with $O(\varepsilon^{-4})$ cost.

Higham, Mao, Roj, Song, Yin proved that

$$\mathbb{E}[\,|\tau - \widehat{\tau}|^p\,] = O(\,|h \log h|^{1/2}), \quad \forall p \geq 1$$

and hence the multilevel complexity is $O(\varepsilon^{-3}|\log \varepsilon|^{1/2})$.

(New 2013 paper by Bouchard, Geiss & Gobet removes log term)

Note: the errors are due to

- $O(h^{1/2})$ path variation within each timestep
- $O(h^{1/2})$ strong error in Euler-Maruyama discretisation

Prior work

By addressing the first of these, Gobet (2000, 2001, 2009, 2010) has developed and analysed 3 ways of improving the weak convergence:

- move the barrier inwards by a distance proportional to $h^{1/2}$
- use a local half-space approximation to the boundary, and sample from the minimum of the continuous Euler approximation (Brownian Bridge construction)
- instead of sampling from the minimum, compute the probability of Brownian Bridge crossing the half-space boundary

I prefer the third approach, because it allows pathwise sensitivities for barrier options – it is also the basis of our multilevel approach

Prior work

These approaches improve the weak convergence to

$$\mathbb{E}[\,\tau - \widehat{\tau}\,] = O(h)$$

giving a single level method with complexity $O(\varepsilon^{-3})$.

Conjecture: we still get

$$\mathbb{E}[\,|\tau - \widehat{\tau}|^p\,] = O(\,|h \log h|^{1/2}), \quad \forall p \geq 1$$

due to the $O(h^{1/2})$ strong convergence, so a multilevel version will have complexity $O(\varepsilon^{-2.5}|\log\varepsilon|^{1/2})$.

(New paper by Bouchard, Geiss & Gobet proves this, without the log term)

In this new work, we aim to achieve $O(\varepsilon^{-2})$ complexity, by using a Milstein discretisation and a Brownian Bridge interpolation within each timestep.

1D barrier options

Our new work builds on an multilevel approximation and numerical analysis developed for down-and-out barrier options with payoff

$$P = f(S_T)\ \mathbf{1}_{\inf_{(0,T)} S_t > B}$$

so the option only pays out if S_t doesn't drop below the barrier B.

A Milstein discretisation is used to compute $\widehat{S}_n$, and approximating the drift and volatility as being constant within each timestep leads to a Brownian Bridge interpolation for $\widehat{S}(t)$

$$\begin{aligned}\widehat{S}(t) \;&=\; \widehat{S}_n + \lambda(t)\,(\widehat{S}_{n+1} - \widehat{S}_n) \\ &\quad + \; b(\widehat{S}_n, t_n)\,\Big(W(t) - W_n - \lambda(t)\,(W_{n+1} - W_n)\Big)\end{aligned}$$

where $\lambda(t) \equiv (t - t_n)/h$.

1D barrier options

For the fine path, a standard Brownian Bridge result gives the probability of having crossed the barrier during the timestep:

$$\widehat{p}_n \;=\; \exp\left(\frac{-2\,(\widehat{S}_n - B)^+\,(\widehat{S}_{n+1} - B)^+}{b_n^2\,h} \right)$$

$$\;=\; \exp\left(-2\;\frac{(\widehat{S}_n - B)^+}{b_n\,h^{1/2}}\;\;\frac{(\widehat{S}_{n+1} - B)^+}{b_n\,h^{1/2}} \right)$$

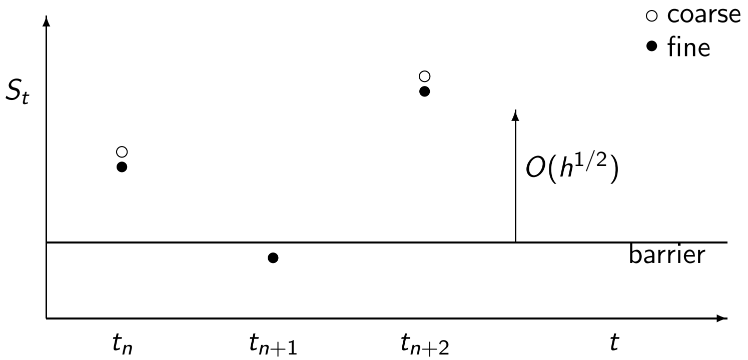
and then the payoff approximation is

$$\widehat{P} = f(\widehat{S}_N)\;\prod_{n=0}^{N-1}(1 - \widehat{p}_n).$$

This gives $O(h)$ weak convergence, but using exactly the same treatment for the coarse path does not improve the multilevel variance.

1D barrier options

Why no improvement in the multilevel variance?



For paths near the barrier, can still get an $O(1)$ difference in the crossing probability for the fine and coarse paths – need to make the coarse path simulation more tightly coupled to the fine path

Multilevel formulation

Multilevel is based on the telescoping sum

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$$

with each of the expectations on the r.h.s. being estimated independently, usually by a standard Monte Carlo estimator of the form

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell(\omega^{(i)}) - \widehat{P}_{\ell-1}(\omega^{(i)}) \right)$$

Note that $\widehat{P}_\ell$ appears twice, in $\mathbb{E}[\widehat{P}_{\ell+1} - \widehat{P}_\ell]$ and $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$, and use of the same $\widehat{P}_\ell(\omega)$ naturally leads to cancellation and the telescoping sum.

Multilevel formulation

However, there is freedom to use a different formulation depending on whether it is the coarser or finer of the two levels:

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^f(\omega^{(i)}) - \widehat{P}_{\ell-1}^c(\omega^{(i)}) \right)$$

<u>provided</u> $\mathbb{E}[\widehat{P}_\ell^f] = \mathbb{E}[\widehat{P}_\ell^c]$ so that the telescoping sum is still valid.

This freedom has been used by

- MBG for barrier and digital options with conditional expectations
- Chen for antithetic estimator in nested simulation
- MBG & Szpruch for antithetic Milstein estimator
- Park for elliptic SPDEs

In each case, the objective is to reduce the variance while respecting the telescoping sum.

1D barrier options

Considering a coarse timestep $[t_n, t_{n+2}]$ corresponding to two fine timesteps, we can evaluate the Brownian Bridge interpolant at the midpoint t_{n+1} to get

$$\widehat{S}^c_{n+1} = \tfrac{1}{2}(\widehat{S}^c_n + \widehat{S}^c_{n+2}) + b(\widehat{S}^c_n, t_n)\ \left(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2})\right)$$

and then compute the probability of crossing within each of the fine timesteps.

The telescoping sum is respected because

$$\mathbb{P}[\text{ not crossing } \mid \widehat{S}^c_n, \widehat{S}^c_{n+2}] = \mathbb{E}\left[\mathbb{P}[\text{ not crossing } \mid \widehat{S}^c_n, \widehat{S}^c_{n+1}, \widehat{S}^c_{n+2}]\right]$$

or, more precisely,

$$\mathbb{E}[\mathbf{1}_{\inf_{[t_n,t_{n+2}]} \widehat{S}^c_t > B} \mid \widehat{S}^c_n, \widehat{S}^c_{n+2}] = \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\inf_{[t_n,t_{n+2}]} \widehat{S}^c_t > B} \mid \widehat{S}^c_n, \widehat{S}^c_{n+1}, \widehat{S}^c_{n+2}]\right]$$

1D barrier options

Hence, for the coarse path we have

$$\prod_{\text{even } n} (1 - p_n^c) = \mathbb{E}\left[\prod_{\text{all } n} (1 - p_n^*)\right]$$

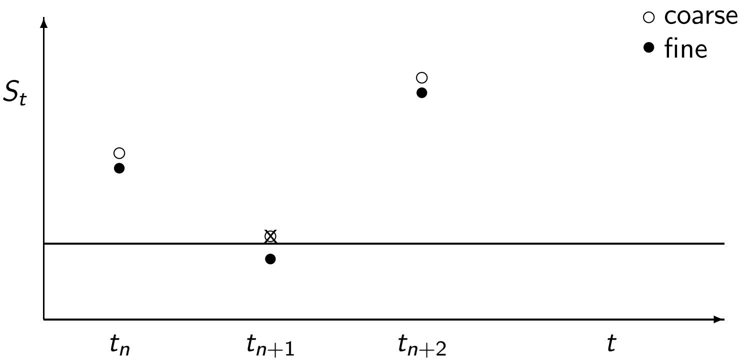
where p_n^c is the standard crossing probability for the coarse timesteps, and p_n^* is the crossing probability for each of the fine timesteps within the coarse path calculation.

There is an $O(h^{1/2})$ fraction of paths with a minimum within $O(h^{1/2})$ of the barrier. MBG, Debrabant, Roessler (2013) prove, roughly speaking, that for these paths $\widehat{P}_\ell - \widehat{P}_{\ell-1} = O(h^{1/2})$, and for the others it is $O(h)$.

Hence, $\mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2] = O(h^{3/2-\delta})$ for any $\delta > 0$, and the multilevel complexity is $O(\varepsilon^{-2})$.

1D barrier options

Why the improvement in the multilevel variance?



We now have first order convergence at all of the fine timesteps, not just the coarse timesteps, so for each timestep

$$\Delta p \;\approx\; \Delta \widehat{S}\, \frac{\partial p}{\partial S} \;=\; O(h) \times O(h^{-1/2}) \;=\; O(h^{1/2})$$

1D barrier options

This construction and numerical analysis is for constant barriers, but it could be easily extended to barriers $B(t)$ which vary in time.

The numerical approximation would treat the barrier as being linear within each timestep so it can compute the crossing probability.

If $B(t)$ is Lipschitz, the difference between the fine and coarse path piecewise linear approximations is $O(h)$, and the existing analysis extends very naturally.

1D exit times

The extension to 1D exit times for a fixed barrier value is very natural.

On the fine path, if we define

$$\widehat{q}^f_n = \prod_{m<n} (1 - \widehat{p}^f_m)$$

to be the computed probability at time t_n that the path has not yet crossed the boundary, then the fine path estimate for the expected exit time is given by

$$\widehat{P}^f = \widehat{q}^f_N \, T + \sum_{n=0}^{N-1} (\widehat{q}^f_n - \widehat{q}^f_{n+1}) \, t_n$$

since $\widehat{q}^f_n - \widehat{q}^f_{n+1}$ is the probability it crosses during timestep n.

1D exit times

Along the coarse path, we first construct the midpoint values as before, define

$$\widehat{q}_n^* = \prod_{m<n} (1 - \widehat{p}_m^*)$$

and then approximate the expected exit time by

$$\widehat{P}^c = \widehat{q}_N^* \, T + \sum_{\text{even } n=0}^{N-2} (\widehat{q}_n^* - \widehat{q}_{n+2}^*) \, t_n$$

Note the use of t_n for the whole coarse timestep $[t_n, t_{n+2}]$ – this is needed to ensure the telescoping sum is respected.

Primozic (2011) implemented this algorithm and obtained $O(h^{3/2})$ multilevel variance convergence, and hence $O(\varepsilon^{-2})$ complexity.

1D exit times

Generalising this to $\mathbb{E}[f(\tau)]$, we have

$$\begin{aligned}
\widehat{P}^f &= \widehat{q}^f_N\, f(T) + \sum_{n=0}^{N-1} (\widehat{q}^f_n - \widehat{q}^f_{n+1})\, f(t_n) \\
&= f(0) + \sum_{n=1}^{N} \widehat{q}^f_n\, (f(t_n) - f(t_{n-1}))
\end{aligned}$$

and

$$\begin{aligned}
\widehat{P}^c &= \widehat{q}^*_N\, f(T) + \sum_{\text{even } n=0}^{N-2} (\widehat{q}^*_n - \widehat{q}^*_{n+2})\, f(t_n) \\
&= f(0) + \sum_{\text{even } n=2}^{N} \widehat{q}^*_n\, (f(t_n) - f(t_{n-2})).
\end{aligned}$$

1D exit times

Theorem: if $f(t)$ is Lipschitz with constant L, then

$$\mathbb{E}[(\widehat{P}^f_\ell - \widehat{P}^c_{\ell-1})^2] = O(h^{3/2-\delta}), \quad \forall \delta > 0$$

Proof: After some slight rearrangement, we obtain

$$\begin{aligned} \widehat{P}^f - \widehat{P}^c &= \sum_{\text{even } n=0}^{N-2} (\widehat{q}^f_n - \widehat{q}^*_n)\,(f(t_n) - f(t_{n-2})) \\ &+ \sum_{\text{odd } n=1}^{N-1} (\widehat{q}^f_{n+1} - \widehat{q}^f_n)\,(f(t_n) - f(t_{n-1})) \end{aligned}$$

1D exit times

$$\left| \sum_{\text{odd } n=1}^{N-1} (\widehat{q}^f_{n+1} - \widehat{q}^f_n)\,(f(t_n) - f(t_{n-1})) \right| \;<\; L\,h \sum_{\text{odd } n=1}^{N-1} (\widehat{q}^f_n - \widehat{q}^f_{n+1}) \;<\; L\,h$$

Also,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{\text{even } n=0}^{N-2} (\widehat{q}^f_n - \widehat{q}^*_n)\,(f(t_n) - f(t_{n-2}) \right)^2 \right]
&\leq \frac{N}{2} \sum_{\text{even } n=0}^{N-2} \mathbb{E}\left[(\widehat{q}^f_n - \widehat{q}^*_n)^2\,(f(t_n) - f(t_{n-2})^2 \right] \\
&\leq L^2\,T^2 \max_n \mathbb{E}[(\widehat{q}^f_n - \widehat{q}^*_n)^2]
\end{aligned}$$

The barrier option error analysis gives $\mathbb{E}[(\widehat{q}^f_n - \widehat{q}^*_n)^2] = O(h^{3/2-\delta})$, $\forall \delta > 0$, and hence the result follows.

Multi-dimensional exit times

How can we extend the 1D approach to multiple dimensions?

If V_t is a half-space, so ∂V_t is planar with inward normal n and distance to the boundary $D = n^T S$, then

$$\mathrm{d}D = n^T a(S,t)\,\mathrm{d}t + n^T b(S,t)\,\mathrm{d}W$$

which is equivalent in distribution to

$$\mathrm{d}D = n^T a(S,t)\,\mathrm{d}t + \sigma(S,t)\,\mathrm{d}B$$

where B is a scalar Brownian motion and

$$\sigma^2(S,t) \;=\; \left\| n^T b(S,t) \right\|_2^2 \;=\; n^T b(S,t)\, b^T(S,t)\, n$$

Multi-dimensional exit times

Can again use a Brownian Bridge interpolation

$$\widehat{S}(t) = \widehat{S}_n + \lambda\,(\widehat{S}_{n+1} - \widehat{S}_n) + b(\widehat{S}_n, t_n)\ (W(t) - W_n - \lambda\,(W_{n+1} - W_n))$$

where $\lambda \equiv (t - t_n)/h$.

Then

$$\widehat{D}(t) = \widehat{D}_n + \lambda\,(\widehat{D}_{n+1} - \widehat{D}_n) + n^T b(\widehat{S}_n, t_n)\ (W(t) - W_n - \lambda\,(W_{n+1} - W_n))$$

which is equivalent in distribution to

$$\widehat{D}(t) = \widehat{D}_n + \lambda\,(\widehat{D}_{n+1} - \widehat{D}_n) + \sigma(\widehat{S}_n, t_n)\ (B(t) - B_n - \lambda\,(B_{n+1} - B_n))$$

Multi-dimensional exit times

The conditional probability of the fine path crossing the boundary is

$$\widehat{p}_n = \exp\left(\frac{-2\,(\widehat{D}^f_n)^+\,(\widehat{D}^f_{n+1})^+}{\sigma_n^2\, h} \right)$$

The coarse path midpoints can be defined from the Brownian Bridge as

$$\widehat{S}^c_{n+1} = \tfrac{1}{2}(\widehat{S}^c_n + \widehat{S}^c_{n+2}) + b(\widehat{S}^c_n, t_n)\,\left(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2})\right)$$

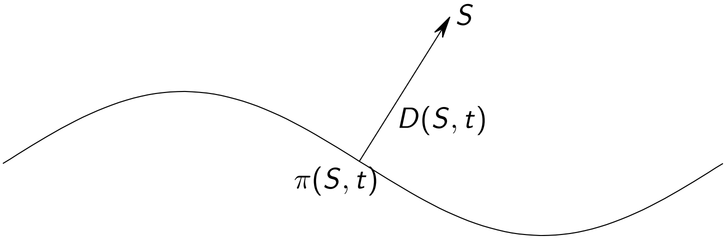
which gives

$$\widehat{D}^c_{n+1} = \tfrac{1}{2}(\widehat{D}^c_n + \widehat{D}^c_{n+2}) + n^T b(\widehat{S}^c_n, t_n)\,\left(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2})\right)$$

and then $\widehat{p}^*_n$ can be defined accordingly, and everything works as in the 1D case – bigger problem is how to extend the treatment to curved boundaries.

Multi-dimensional exit times

Following Gobet, we assume that in a neighbourhood of the boundary we have computable, $C^{2,1}$ signed distance $D(S,t)$ to boundary ∂V_t:



Then we have

$$S = \pi(S,t) + D(S,t)\, n(S,t)$$

where

- $\pi(S,t)$ is projection onto the boundary ∂V
- $n(S,t) = \nabla D$ is the inward normal at that point

Multi-dimensional exit times

We need to respect the telescoping sum by ensuring that $\mathbb{E}[\widehat{P}^f_\ell] = \mathbb{E}[\widehat{P}^c_\ell]$.

The only way we have found to do this is based on the 1D distance SDE:

$$\mathrm{d}D = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

The fine path conditional crossing probability

$$\widehat{p}_n = \exp\left(\frac{-2\,(\widehat{D}^f_n)^+\,(\widehat{D}^f_{n+1})^+}{(\sigma^f_n)^2\,h}\right)$$

corresponds to the Brownian Bridge interpolant

$$\widehat{D}^f(t) = \widehat{D}^f_n + \lambda\,(\widehat{D}^f_{n+1} - \widehat{D}^f_n) + \sigma^f_n\,\left(B(t) - B_n - \lambda\,(B_{n+1} - B_n)\right)$$

with

$$\sigma^f_n = \left\| n^T(S^f_n, t_n)\; b(S^f_n, t_n)\right\|_2$$

Multi-dimensional exit times

For the coarse path midpoint, we directly define

$$\widehat{D}^c_{n+1} = \tfrac{1}{2}(\widehat{D}^c_n + \widehat{D}^c_{n+2}) + n(\widehat{S}^c_n, t_n)^T b(\widehat{S}^c_n, t_n)\ \left(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2})\right)$$

and then use

$$\begin{aligned} \widehat{p}^*_n &= \exp\left(\frac{-2\,(\widehat{D}^c_n)^+\,(\widehat{D}^c_{n+1})^+}{(\sigma^c_n)^2\,h} \right) \\ \widehat{p}^*_{n+1} &= \exp\left(\frac{-2\,(\widehat{D}^c_{n+1})^+\,(\widehat{D}^c_{n+2})^+}{(\sigma^c_n)^2\,h} \right) \end{aligned}$$

Because

$$n(\widehat{S}^c_n, t_n)^T b(\widehat{S}^c_n, t_n)\ \left(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2})\right)$$

is equivalent in distribution to

$$\sigma(\widehat{S}^c_n, t_n)\ \left(B_{n+1} - \tfrac{1}{2}(B_n + B_{n+2})\right)$$

the condition for the telescoping sum is respected.

Multi-dimensional exit times

Theorem:

$$\max_n \mathbb{E}[(\widehat{q}_n^f - \widehat{q}_n^*)^2] = O(h^{3/2-\delta}), \quad \forall \delta > 0$$

Proof: generalisation of the 1D proof by MBG, Debrabant, Roessler (2013).

Theorem: if $f(t)$ is Lipschitz with constant L, then

$$\mathbb{E}[(\widehat{P}_\ell^f - \widehat{P}_{\ell-1}^c)^2] = O(h^{3/2-\delta}), \quad \forall \delta > 0$$

Proof: same as for 1D.

Multi-dimensional exit times

Outline of key steps in extension of 1D proof:

- Since $D \in C^{2,1}$,

$$D(S+\Delta S, t+h) = D(S,t) + n^T(S,t)\,\Delta S + O(\|\Delta S\|^2, h) \qquad (1)$$

- Applying (1) to the coarse path for even n gives

$$\widehat{D}^c_{n+2} = \widehat{D}^c_n + n^T(\widehat{S}^c_n, t_n)\, b(\widehat{S}^c_n, t_n)\,(W_{n+2}-W_n) + O(h)$$

  and hence

$$\widehat{D}^c_{n+1} = \widehat{D}^c_n + n^T(\widehat{S}^c_n, t_n)\, b(\widehat{S}^c_n, t_n)\,(W_{n+1}-W_n) + O(h) \qquad (2)$$

  while applying (1) to the fine path gives

$$\widehat{D}^f_{n+1} = \widehat{D}^f_n + n^T(\widehat{S}^f_n, t_n)\, b(\widehat{S}^f_n, t_n)\,(W_{n+1}-W_n) + O(h) \qquad (3)$$

- strong convergence gives $\widehat{D}^f_n - \widehat{D}^c_n = O(h)$, for even n, and then comparing (2) and (3) gives $\widehat{D}^f_{n+1} - \widehat{D}^c_{n+1} = O(h)$.

Multi-dimensional exit times

Conjecture: if we extend the problem to $\mathbb{E}[f(S_\tau,\tau)]$ for some function f which is Lipschitz with respect to both S_τ and τ, then

$$\mathbb{E}[(\widehat{P}^f_\ell - \widehat{P}^c_{\ell-1})^2] = O(h^{1-\delta}), \quad \forall \delta > 0$$

so the multilevel complexity is $\varepsilon^{-2-\delta}, \;\; \forall \delta > 0$.

Why is this worse than for $\mathbb{E}[f(\tau)]$?

The problem is the $O(h^{1/2})$ difference between $\widehat{S}^f$ and $\widehat{S}^c$ at the mid-point of each coarse timestep – currently can't see how to fix this while still respecting the telescoping sum

Numerical Results

Test case:

- 5-dimensional Geometric Brownian Motion

$$\mathrm{d}S_i = r\, S_i\, \mathrm{d}t + \sigma_i\, S_i\, \mathrm{d}W_i, \quad 0 < t < T$$

  with $T=1$ and $r=0.05$, $\sigma_i = (0.3,\, 0.35,\, 0.4,\, 0.45,\, 0.5)$

- Brownian motions have correlation $\mathbb{E}[\mathrm{d}W_i\, \mathrm{d}W_j] = 0.2\, \mathrm{d}t, \quad i \neq j$

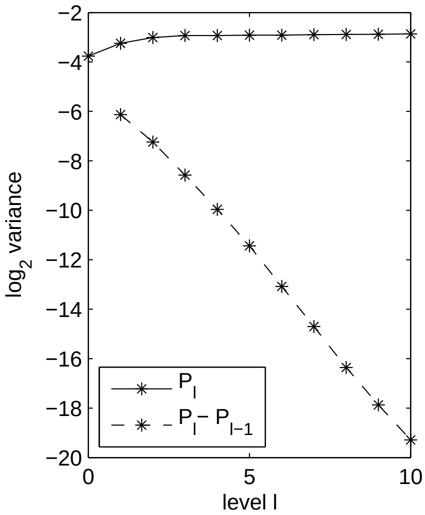
  If $\Omega = L\, L^T$ is the correlation matrix, then

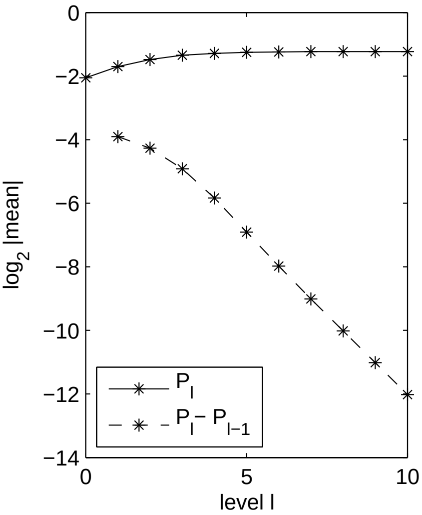
$$b = \Sigma\ L \quad \Longrightarrow \quad n^T b\ b^T n = n^T \Sigma\, \Omega\, \Sigma\, n$$

  where Σ is the diagonal matrix with eigenvalues $\sigma_i\, S_i$

- Initial data is $S_i = 100$ and boundary is set at $\|S\|_2 = 110\sqrt{5}$

Numerical results





Details of MATLAB MLMC code

If a, b, c are estimates for $\mathbb{E}[P^f_{\ell-1}]$, $\mathbb{E}[P^f_\ell]$, $\mathbb{E}[P^f_\ell - P^c_{\ell-1}]$, respectively, then it should be true that $a - b + c \approx 0$

A consistency check verifies that this is true, to within the accuracy one would expect due to sampling error.

Since

$$\sqrt{\mathbb{V}[a-b+c]} \;\leq\; \sqrt{\mathbb{V}[a]} + \sqrt{\mathbb{V}[b]} + \sqrt{\mathbb{V}[c]}$$

the code computes and plots the ratio

$$\frac{|a-b+c|}{3(\sqrt{\mathbb{V}[a]} + \sqrt{\mathbb{V}[b]} + \sqrt{\mathbb{V}[c]})}$$

The probability of this ratio being greater than 1 based on random sampling errors is extremely small – if it is, it indicates a likely error.

Details of MATLAB MLMC code

Optimal MLMC needs a good estimate for $V_\ell = \mathbb{V}[P^f_\ell - P^c_{\ell-1}]$, but how many samples are needed for this?

10 is often sufficient, but more are needed when there are rare outliers.

The standard deviation of the sample variance for a random variable X with zero mean is approximately

$$\sqrt{\frac{\kappa - 1}{N}}\, \mathbb{E}[X^2] \quad \text{where kurtosis } \kappa = \frac{\mathbb{E}[X^4]}{(\mathbb{E}[X^2])^2}$$

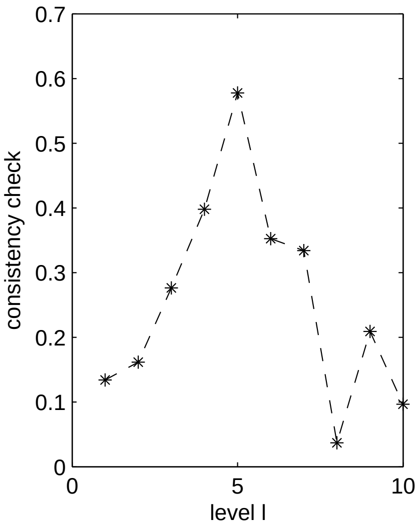
(see `http://mathworld.wolfram.com/SampleVarianceDistribution.html`)

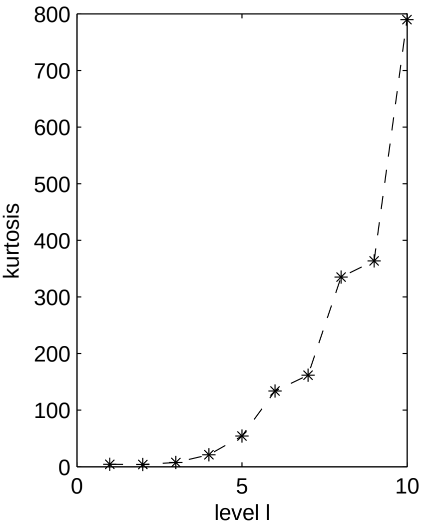
In this case, $P^f_\ell - P^c_{\ell-1} = O(h^{1/2})$ with probability $O(h^{1/2})$ so

$$\mathbb{E}[X^2] = O(h^{3/2}), \quad \mathbb{E}[X^4] = O(h^{5/2}) \quad \Longrightarrow \quad \kappa = O(h^{-1/2})$$

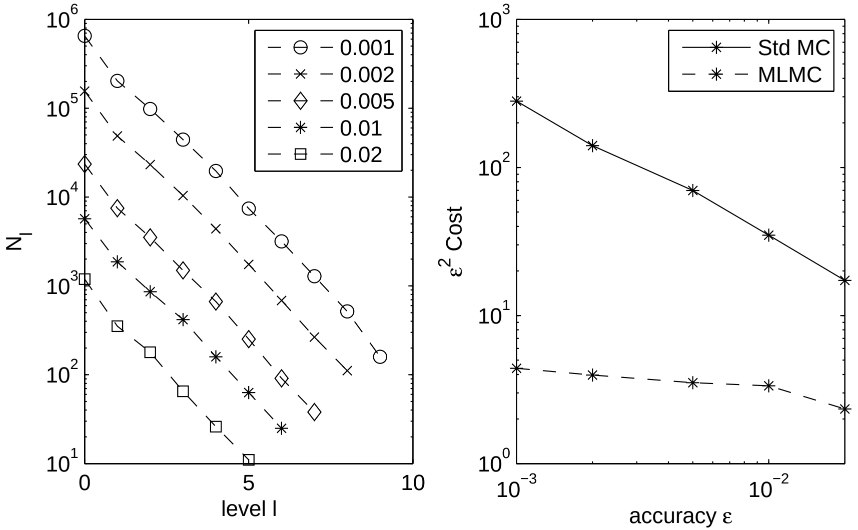
More precisely, we obtain $\kappa \sim a\, h^{-1/2} \, \dfrac{1 + b\, h^{3/2}}{(1 + c\, h^{1/2})^2}$

Numerical results





Numerical results



Conclusions

We have developed and analysed a multilevel method for computing the expected value of functionals of the exit time.

Using the Milstein discretisation, and a probabilistic treatment of boundary crossing, an r.m.s. error of ε is achievable with an $O(\varepsilon^{-2})$ complexity.

The primary restrictions are:

- the SDE must satisfy the commutativity condition so that there is no need to simulate the Lévy areas
- the signed distance to the boundary must be $C^{2,1}$ in a neighbourhood of the boundary

Future work will address the first of these restrictions by approximately simulating the Lévy areas.

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