

# Different MLMC treatments of the stochastic heat equation

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# Motivation

New research interest – MLMC for parabolic SPDEs:

- 1D stochastic heat equation is the simplest example driven by space-time white noise (cylindrical Wiener process)
- focus on
  - three different noise representations: spectral, mass-lumped finite element, finite volume
  - three different quantities of interest (QoI): squared amplitude of a single Fourier mode, energy  $\|u\|_2^2$ , and  $\langle \varphi, u \rangle^2$

Key messages:

- finite volume treatment has worst MLMC variance
- finite element is as good as spectral, and both can benefit from Richardson extrapolation to overcome MLMC oddity ( $\beta > 2\alpha$ )

This talk concentrates on ideas and results – see longer presentation online for more details of Fourier analysis, with arXiv paper due shortly

# Stochastic heat equation

Stochastic heat equation:

$$du - u_{xx} dt = dW, \quad 0 < x < 1, \quad t > 0,$$

subject to homogeneous initial data and b.c.'s.

$dW$  is the increment of space-time white noise (cylindrical Wiener process) so that for arbitrary  $f, g$ , the spatial inner product

$$\langle f, dW \rangle$$

is Normally-distributed with zero mean and variance  $\|f\|^2 dt$ , and

$$\mathbb{E}[\langle f, dW \rangle \langle g, dW \rangle] = \langle f, g \rangle dt$$

# Stochastic heat equation

Solution has Fourier mode expansion

$$u(x, t) = \sum_{k=1}^{\infty} \hat{u}_k(t) e_k(x),$$

where  $e_k(x) = \sqrt{2} \sin(k\pi x)$ , and substitution gives Ornstein-Uhlenbeck SDE for each mode:

$$d\hat{u}_k = -\lambda_k \hat{u}_k dt + d\widehat{W}_k, \quad \lambda_k = k^2\pi^2,$$

where  $\widehat{W}_k$  are independent Brownian motions, due to orthonormality.

As well as square amplitude of a single Fourier mode, other two QoIs are:

- energy  $P = \|u\|_2^2 = \sum_{k=1}^{\infty} \hat{u}_k^2$
- squared functional  $P = \langle \varphi, u \rangle^2 = \sum_{k=1}^{\infty} \hat{\varphi}_k^2 \hat{u}_k^2$

# Stochastic heat equation

Insight comes from truncating expansion to give:

$$u_K(x, t) = \sum_{k=1}^{K-1} \hat{u}_k(t) e_k(x),$$

where  $K$  is roughly equivalent to  $1/\Delta x$  in a numerical approximation.

For energy,

$$P - P_K = \sum_{k=K}^{\infty} \hat{u}_k^2,$$

so  $\mathbb{E}[P - P_K] = O(K^{-1})$  and  $\mathbb{V}[P - P_K] = O(K^{-3})$ .

If  $\hat{\varphi}_k = O(k^{-p})$  then for squared functional  $\mathbb{E}[P - P_K]$  and  $\mathbb{V}[P - P_K]$  are both  $O(K^{-2p-1})$ .

# Spectral noise representation

Finite difference approximation to PDE, and spectral (K-L) representation of noise gives semi-discretisation

$$dU_j - \frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) dt = \sum_{k=1}^{J-1} d\widehat{W}_k e_k(x_j).$$

The semi-discrete solution is then

$$U_j(t) = \sum_{k=1}^{J-1} \widehat{U}_k(t) e_k(x_j),$$

in which  $\widehat{U}_k$  satisfy the modified O-U SDE

$$d\widehat{U}_k = -\tilde{\lambda}_k \widehat{U}_k dt + d\widehat{W}_k,$$

with  $\tilde{\lambda}_k = (4/\Delta x^2) \sin^2(k\pi\Delta x/2) = \lambda_k + O(k^4\Delta x^2)$ .

# Spectral noise representation

The fully-discrete equations use Euler-Maruyama time discretisation

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x^2} (U_{j+1}^n - 2U_j^n + U_{j-1}^n) + \sum_{k=1}^{J-1} \Delta \widehat{W}_k^n e_k(x_j).$$

For MLMC use  $\Delta x_\ell \propto 2^{-\ell}$ ,  $\Delta t_\ell \propto 4^{-\ell}$  with fixed  $\Delta t/\Delta x^2 < 1/2$  for stability.

If the fine grid uses

$$\sum_{k=1}^{J-1} \Delta \widehat{W}_k^n e_k(x_j),$$

then the coarse grid uses

$$\sum_{m=n}^{n+3} \sum_{k=1}^{J/2-1} \Delta \widehat{W}_k^m e_k(x_j).$$

## Finite element noise

Using the usual “hat” piecewise linear basis functions  $\phi_j$ , and mass-lumping, the Galerkin semi-discrete approximation is

$$dU_j - \frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) dt = \frac{1}{\Delta x} \langle dW, \phi_j \rangle, \quad j = 1, \dots, J-1.$$

Note that  $\mathbb{E}[\langle dW, \phi_{j_1} \rangle \langle dW, \phi_{j_2} \rangle] = 0$  when  $|j_1 - j_2| > 1$ , and

$$\mathbb{E}[\langle dW, \phi_j \rangle^2] = \frac{2}{3} \Delta x dt, \quad \mathbb{E}[\langle dW, \phi_j \rangle \langle dW, \phi_{j\pm 1} \rangle] = \frac{1}{6} \Delta x dt,$$

The Euler-Maruyama discretisation gives

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n) + \Delta W_j^n,$$

where  $\Delta W_j^n$  can be simulated as

$$\Delta W_j^n = \sqrt{\Delta t / \Delta x} \left( Z_j^n / \sqrt{3} + Z_{j-1/2}^n / \sqrt{6} + Z_{j+1/2}^n / \sqrt{6} \right)$$

using iid standard Normals  $Z_j^n, Z_{j\pm 1/2}^n$ .

## Finite element noise

The coarse grid basis functions can be expressed in terms of fine grid ones:

$$\phi_{\ell-1,j}(x) = \frac{1}{2}\phi_{\ell,j-1}(x) + \phi_{\ell,j}(x) + \frac{1}{2}\phi_{\ell,j+1}(x).$$

so there is a natural MLMC coupling with

$$\Delta W_{\ell-1,j}^n = \sum_{m=n}^{n+3} \left( \frac{1}{4} \Delta W_{\ell,j-1}^m + \frac{1}{2} \Delta W_{\ell,j}^m + \frac{1}{4} \Delta W_{\ell,j+1}^m \right).$$

One of the takeaways from this talk is that this is easy to implement, very natural, and comparable to the spectral treatment in accuracy.

## Finite volume noise

For the finite volume treatment

$$\phi_j(x) = \mathbf{1}_{[x_j - \Delta x/2, x_j + \Delta x/2]}(x)$$

so

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x^2} (U_{j+1}^n - 2U_j^n + U_{j-1}^n) + \Delta W_j^n,$$

where  $\Delta W_j^n = \sqrt{\Delta t / \Delta x} Z_j^n$ , the  $Z_j^n$  again iid standard Normal.

However, this doesn't give a natural MLMC coupling.

## Finite volume noise

Instead split the interval into two halves,

$$\phi_j(x) = \mathbf{1}_{[x_j - \Delta x/2, x_j]}(x) + \mathbf{1}_{[x_j, x_j + \Delta x/2]}(x),$$

$$\Delta W_j^n = \sqrt{\frac{\Delta t}{2\Delta x}} (Z_{j-1/4}^n + Z_{j+1/4}^n)$$

The MLMC coupling is then

$$\Delta W_{\ell-1,j}^n = \sqrt{\frac{\Delta t}{8\Delta x}} \sum_{m=n}^{n+3} \left( Z_{j-3/4}^m + Z_{j-1/4}^m + Z_{j+1/4}^m + Z_{j+3/4}^m \right),$$

corresponding to integrating over the coarse grid interval  $[x_{j-1}, x_{j+1}]$ .

# MLMC parameters

Using  $\Delta x_\ell \sim 2^{-\ell}$ ,  $\Delta t_\ell \sim 4^{-\ell}$  for discrete stability gives  $\gamma = 3$ .

Fourier analysis gives the following values for  $\alpha$ ,  $\beta$ :

	$\ u\ _2^2$		$\langle \varphi, u \rangle^2, p=1$		$\langle \varphi, u \rangle^2, p=2$	
	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
Spectral	1	3	2	3	2	4
Finite element	1	3	2	3	2	4
Finite volume	1	2	2	2	2	2

For details of analysis, see recent presentation at University of Bath:  
<https://people.maths.ox.ac.uk/gilesm/slides.html>

# Richardson extrapolation

If

$$P_\ell = P + a2^{-\ell} + O(2^{-2\ell}),$$

then

$$P_\ell^{\text{ex}} = 2P_\ell - P_{\ell-1} = P + O(2^{-2\ell}).$$

Doubling weak order  $\alpha$  eliminates anomalous  $\beta > 2\alpha$  situation.

With extrapolation, MLMC estimator becomes

$$\begin{aligned} 2P_\ell - P_{\ell-1} - (2P_{\ell-1} - P_{\ell-2}) &= 2(P_\ell - P_{\ell-1}) - (P_{\ell-1} - P_{\ell-2}) \\ &= 2P_\ell - 3P_{\ell-1} + P_{\ell-2} \end{aligned}$$

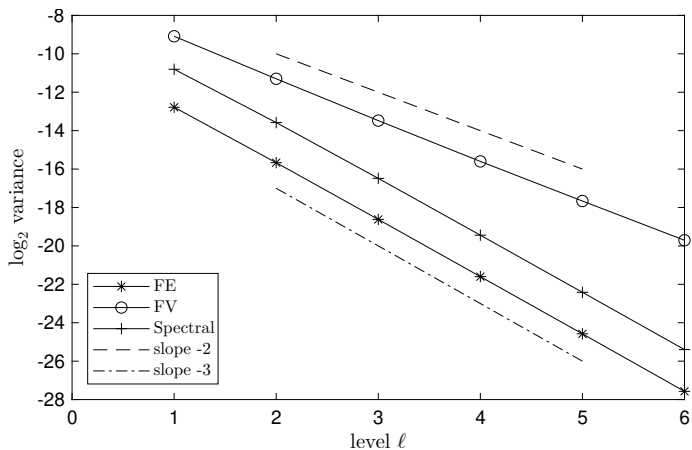
Extrapolation increases the variance, but it greatly reduces the finest level  $L$  required for weak convergence, so overall gives big savings.

# Numerical results

- $T = 1/8$
- $\Delta x_\ell = 2^{-\ell-2}$ ,  $\Delta t_\ell = 2^{-2\ell-6} \implies \Delta t_\ell / \Delta x_\ell^2 = 1/4$ .
- $10^5$  samples for spectral and finite element (FE),  
 $4 \times 10^5$  for finite volume (FV) due to poorer variance
- $\log_2 \mathbb{V}[P_\ell - P_{\ell-1}]$  and  $\log_2 |\mathbb{E}[P_\ell - P_{\ell-1}]|$  plotted versus level  $\ell$
- functional weighting is  $\varphi(x) = 2\sqrt{3} \min(x, 1-x)$  so  $p=2$

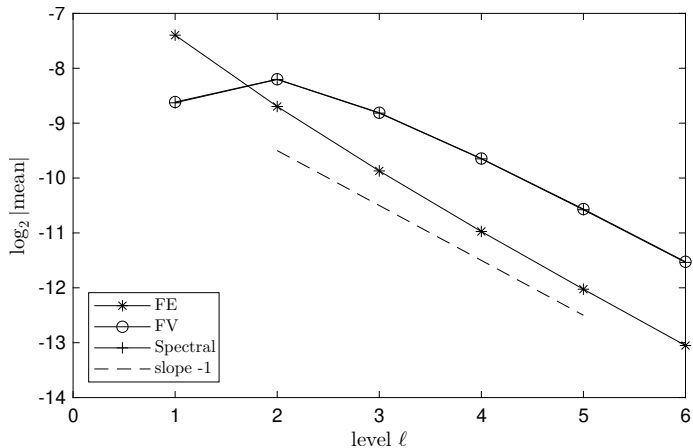
# Numerical results

$\mathbb{V}[P_\ell - P_{\ell-1}]$  for energy QOI:



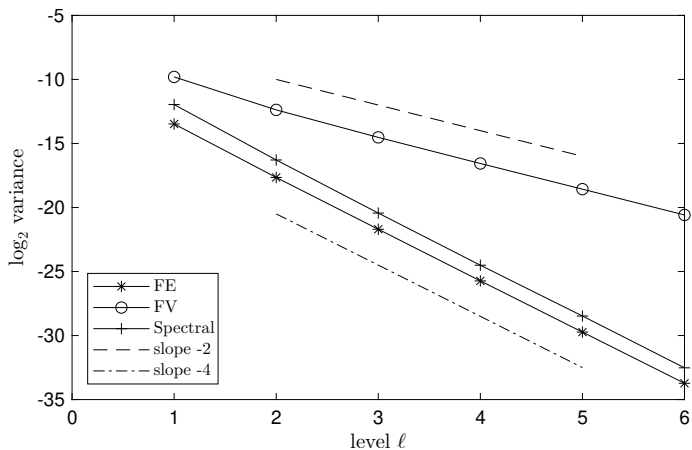
# Numerical results

$\mathbb{E}[P_\ell - P_{\ell-1}]$  for energy QOI:



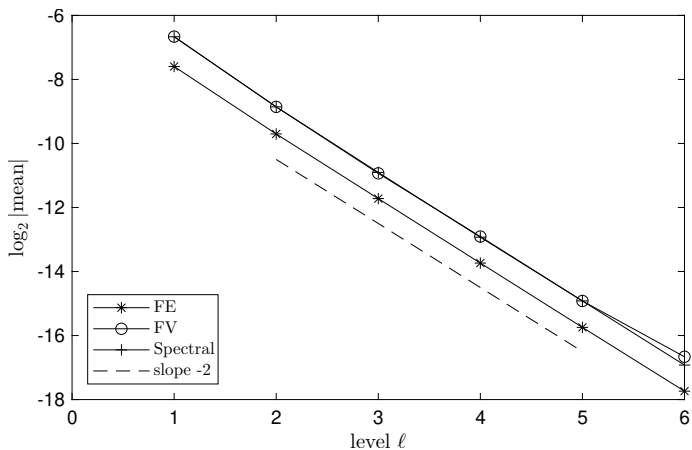
# Numerical results

$\mathbb{V}[P_\ell - P_{\ell-1}]$  for functional QOI:



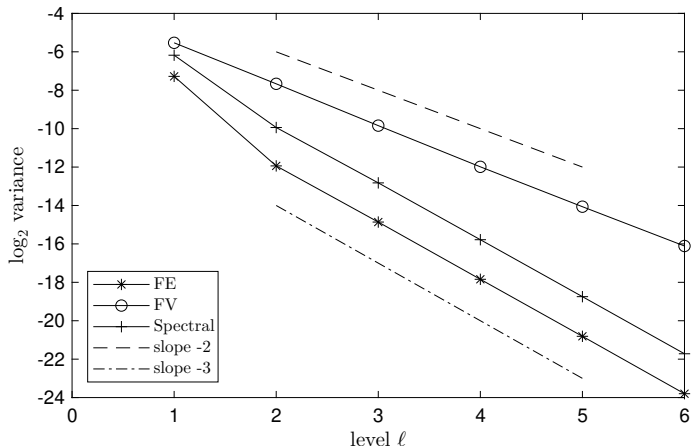
# Numerical results

$\mathbb{E}[P_\ell - P_{\ell-1}]$  for functional QOI:



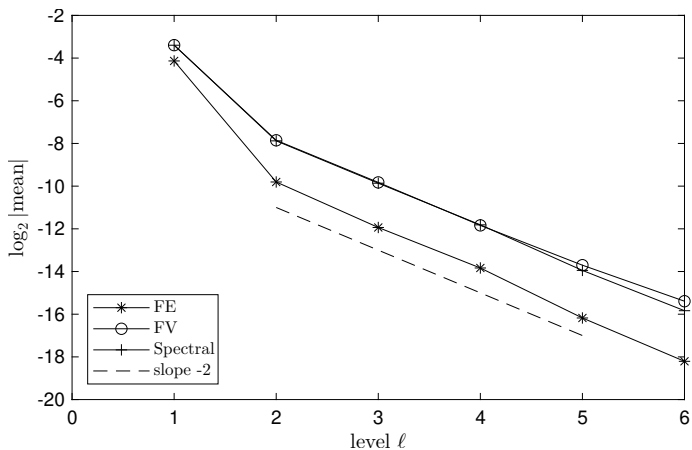
# Numerical results

$\mathbb{V}[P_\ell - P_{\ell-1}]$  for energy QOI with extrapolation:



# Numerical results

$\mathbb{E}[P_\ell - P_{\ell-1}]$  for energy QOI with extrapolation:



# Conclusions

- Fourier analysis provides complete analysis of MLMC variance for stochastic heat equation
- 3 different white noise treatments, 3 different output QoI's
- finite volume treatment is clearly the worst due to aliasing errors
- finite element treatment as good as spectral treatment – both need Richardson extrapolation to get full benefits
  
- now ready to move on to more interesting SPDEs

# References



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