Multilevel Simulation of Mean Exit Times

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Outline

- Feynman-Kac formula
- prior work – Gobet & Menozzi
- multilevel Monte Carlo
- prior work – Higham *et al*
- new idea – approximating a conditional expectation
- outline analysis
- numerical results

Feynman-Kac formula

Suppose that $u(x,t)$ satisfies the parabolic PDE

$$\frac{\partial u}{\partial t} + \sum_j a_j \frac{\partial u}{\partial x_j} + \tfrac{1}{2} \sum_{j,k,l} b_{jk} b_{kl} \frac{\partial^2 u}{\partial x_j \partial x_l} - V(x,t)\, u(x,t) + f(x,t) = 0$$

in bounded domain D, subject to $u(x,t) = g(x,t)$ on the boundary ∂D.

It will be assumed that $f(x,t), g(x,t), V(x,t), a(x,t), b(x,t)$ are all Lipschitz continuous.

Feynman-Kac formula

Feynman and Kac proved that $u(x,t)$ can also be expressed as

$$u(x,t) = \mathbb{E}\left[\int_t^\tau E(t,s)\, f(X_s,s)\,\mathrm{d}s + E(t,\tau)\, g(X_\tau,\tau) \mid X_t = x \right]$$

where X_t satisfies the SDE

$$\mathrm{d}X_t = a(X_t,t)\,\mathrm{d}t + b(X_t,t)\,\mathrm{d}W_t,$$

with W_t being a Brownian motion with independent components, τ is the first time at which X_t leaves D, and

$$E(t_0,t_1) = \exp\left(-\int_{t_0}^{t_1} V(X_t,t)\,\mathrm{d}t \right).$$

Note: in the special case in which $f(x,t)\!=\!0$, $g(x,t)\!=\!t$, $V(x,t)\!=\!0$ $u(x,t)$ is the expected exit time.

Feynman-Kac formula

Why is this alternative form useful?

In high dimensions, approximating the parabolic PDE can be expensive because the cost increases exponentially – *curse of dimensionality*

The cost of Monte Carlo simulation for the SDE scales linearly with dimension

Numerical approximation

An Euler-Maruyama discretisation with uniform timestep h gives

$$\widehat{X}_{n+1} = \widehat{X}_n + a(\widehat{X}_n, t)\, h + b(\widehat{X}_n, t)\, \Delta W_n,$$

with initial data $\widehat{X}_0 = x$ at time t.

If $\widehat{X}(t)$ is the piecewise-constant interpolant, we then have

$$\widehat{u}(x,t) = \mathbb{E}\left[\int_t^{\widehat{\tau}} \widehat{E}(t,s)\, f(\widehat{X}(s), s)\, \mathrm{d}s + \widehat{E}(t,\widehat{\tau})\, g(\widehat{X}(\tau), \widehat{\tau}) \right].$$

with $\widehat{\tau}$ being the exit time, and

$$\widehat{E}(t_0, t_1) = \exp\left(- \int_{t_0}^{t_1} V(\widehat{X}_t, t)\, \mathrm{d}t \right).$$

Prior work – Gobet & Menozzi

The Euler-Maruyama method has strong accuracy $O(h^{1/2})$,

$$\left(\mathbb{E} \left[\sup_{[0,\min(\tau,\widehat{\tau})]} \|X_t - \widehat{X}(t)\|^2 \right] \right)^{1/2} = O(h^{1/2}),$$

and Gobet & Menozzi (2007) proved that the weak error is also $O(h^{1/2})$,

$$u(x,t) - \widehat{u}(x,t) = O(h^{1/2}).$$

For standard Monte Carlo method, ε RMS accuracy needs $O(\varepsilon^{-2})$ paths, each with $h = O(\varepsilon^2)$, so total cost is $O(\varepsilon^{-4})$

Gobet & Menozzi (2010) reduced this to $O(\varepsilon^{-3})$ by shifting the boundary by $O(h^{1/2})$ to improve the weak error to $O(h)$.

Multilevel Monte Carlo

Introduced in 2006 for SDE simulations, this uses the identity

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$$

where $\widehat{P}_\ell$ represents the approximation using timestep $h_\ell = 2^{-\ell}\, h_0$, and independently estimates each of the expectations on the r.h.s. using the same Brownian path for the differences $\widehat{P}_\ell^{(\ell,n)} - \widehat{P}_{\ell-1}^{(\ell,n)}$:

$$N_0^{-1} \sum_{n=1}^{N_0} \widehat{P}_0^{(0,n)} \; + \; \sum_{\ell=1}^{L} \left\{ N_\ell^{-1} \sum_{n=1}^{N_\ell} \left(\widehat{P}_\ell^{(\ell,n)} - \widehat{P}_{\ell-1}^{(\ell,n)} \right) \right\}$$

Small variance as $h_\ell \to 0$ means few samples used on finer levels.

Finest level L depends on weak error, as before.

MLMC Theorem

(Slight generalisation of original 2006 version.)

If there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, each costing C_ℓ, and positive constants $\alpha, \beta, \gamma, c_1, c_2, c_3$ such that $\alpha \geq \frac{1}{2}\min(\beta, \gamma)$ and

i) $\left|\mathbb{E}[\widehat{P}_\ell - P]\right| \leq c_1\, 2^{-\alpha\, \ell}$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & \ell = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2\, N_\ell^{-1} 2^{-\beta\, \ell}$

iv) $\mathbb{E}[C_\ell] \leq c_3\, 2^{\gamma\, \ell}$

MLMC Theorem

then there exists a positive constant c_4 such that for any $\varepsilon < 1$ there exist L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has a mean-square-error with bound $\mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with an expected computational cost C with bound

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > \gamma, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = \gamma, \\ c_4\, \varepsilon^{-2-(\gamma-\beta)/\alpha}, & 0 < \beta < \gamma. \end{cases}$$

Prior work – Higham

Higham *et al* (2013) developed a MLMC treatment of the exit time problem:

- Euler-Maruyama discretisation
- $O(h_\ell^{1/2})$ weak convergence $\Longrightarrow \alpha = 1/2$
- $\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] \approx O(h_\ell^{1/2})$ (ignoring log terms) $\Longrightarrow \beta \approx 1/2$
- $O(h_\ell^{-1})$ cost per path $\Longrightarrow \gamma = 1$

Hence, overall cost is approximately $O(\varepsilon^{-3})$.

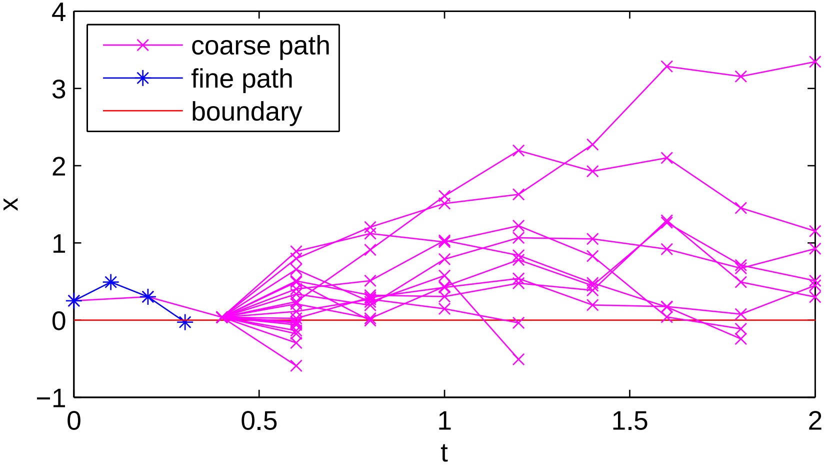
Gobet & Menozzi's boundary treatment would improve this to $O(\varepsilon^{-2.5})$.

G & Primozic (2011) developed $O(\varepsilon^{-2})$ treatment using Milstein discretisation for SDEs with special commutativity property.

MLMC challenge

When coarse or fine path exits the domain, the other is within $O(h^{1/2})$.

However, there is a $O(h^{1/2})$ probability that it will not exit the domain until much later $\Longrightarrow V_\ell = O(h^{1/2})$.



MLMC challenge

How can we do better?

Similar to previous work on digital options, split second path into multiple copies, and average their outputs to approximate the conditional expectation.

$O(h^{1/2})$ expected time to exit for second path, so can afford to use $O(h^{-1/2})$ copies of second path.

This gives an approximation to the conditional expectation resulting in $\widehat{P}_\ell - \widehat{P}_{\ell-1} \approx O(h^{1/2})$, so $V_\ell \approx O(h)$.

Numerical results confirm this – numerical analysis is underway.

Numerical results

The test case comes from Gobet & Menozzi (2009)

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$$

in the domain $\|x\| \leq 2,\ 0 \leq t \leq 1$ with

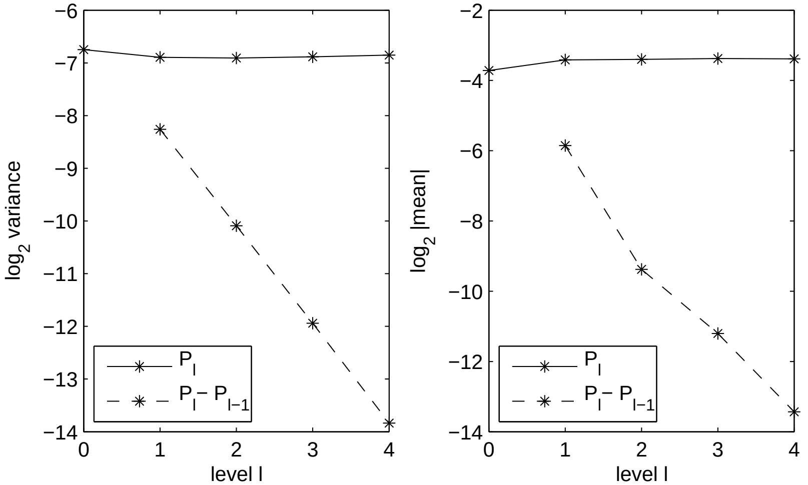
$$b(x) = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} (1+|x_3|)^{\frac{1}{2}} & 0 & 0 \\ \frac{1}{2}(1+|x_1|)^{\frac{1}{2}} & (\frac{3}{4})^{\frac{1}{2}}(1+|x_1|)^{\frac{1}{2}} & 0 \\ 0 & \frac{1}{2}(1+|x_2|)^{\frac{1}{2}} & (\frac{3}{4})^{\frac{1}{2}}(1+|x_2|)^{\frac{1}{2}} \end{pmatrix}$$

$V(x,t) \equiv 0$, and $f(x,t), g(x,t)$ are chosen so that the PDE solution is $u(x,t) = x_1 x_2 x_3$.

$X_0 = (0.56, 0.52, 0.33)^T$, so we are estimating $u(X_0, 0)$.

Timestep comes down by factor 4 on each level – better than factor 2 when $V_\ell = O(h_\ell)$. Gobet-Menozzi boundary shift used on each level.

Numerical results



$$V_\ell = O(h_\ell)$$

$$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = O(h_\ell)$$

Conclusions

- multilevel Monte Carlo method is very simple
- key in Feynmac-Kac application is use of splitting to approximate a conditional expectation – greatly reduces the variance
- resulting computational complexity is approximately $O(\varepsilon^{-2})$

Webpages:
people.maths.ox.ac.uk/gilesm/mlmc.html
people.maths.ox.ac.uk/gilesm/mlmc_community.html
people.maths.ox.ac.uk/gilesm/acta/